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EXISTENCE AND UNIQUENESS FOR SOLUTIONS OF A CLASS OF SECOND ORDER DISTRIBUTED PARAMETER SYSTEMS

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Abstract. In this paper, we consider a class of second order elliptic distributed parameter systems, the bilinearity appears in the form $u_i \frac{\partial y}{\partial x_i}$. The existence and uniqueness of solutions of the state equations describing the system are given. Next we derive a priori estimates for solutions of the state equations are presented. Finally, we obtain the existence and uniqueness of the optimal control $\tilde{u}$..

Keywords: Distributed parameter systems; A priori estimate; Existence and uniqueness.

1. Introduction

Distributed parameter systems is an important problem in engineering applications. The research of distributed parameter systems are intensively developed. The literature in this aspect is huge. Systematic introduction of distributed parameter systems can be found in, for example, [2, 6, 5, 11, 13, 16, 15, 7]. In [3], the authors gave a wide list of distributed parameter systems. Tiba investigated many properties of nonsmooth distributed parameter systems in [17]. Addou and Benbrik studied a fourth-order parabolic bilinear distributed parameter systems and derived the existence and uniqueness of the optimal control in [1]. In [14], the author discussed existence and uniqueness for solution of bilinear elliptic optimal control problems. The purpose of this paper is to prove the existence and uniqueness for solution of a class of second order elliptic distributed parameter systems, the bilinearity appears in the form $u_i \frac{\partial y}{\partial x_i}$.

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In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^{p} = \sum_{|\alpha|\le m} \|D^{\alpha}v\|_{L^{p}(\Omega)}^{p},$$

a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^{p} = \sum\limits_{|\alpha|=m} \|D^{\alpha}v\|_{L^{p}(\Omega)}^{p}$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$.

For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega),\ H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2},\ \|\cdot\| = \|\cdot\|_{0,2}$. We consider the following second order elliptic distributed parameter systems:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|y - z_0\|^2 + \frac{1}{2}\|u - u_d\|^2 \right\}$$

subject to the state equation

$$\Delta y = f + \sum_{i=1}^{N} u_i \frac{\partial y}{\partial x_i}, \qquad x \in \Omega, \tag{0.1}$$

$$y = 0, \qquad x \in \partial\Omega, \tag{0.2}$$

where the bounded open set $\Omega \subset \mathrm{R}^2$ with the smooth boundary $\partial\Omega$. We shall assume that $U = L^2(\Omega)$, $f \in H^{-1}(\Omega)$, and $f \not\equiv 0$.

Now, we denote $V = H_0^1(\Omega)$, let $V' = H^{-1}(\Omega)$ be dual space. Then we have the Gelfand triple

$$V \hookrightarrow L^2(\Omega) \hookrightarrow V';$$

all the embeddings are continuous, dense, and compact [8]. Now let $A :\ V \to V', \varphi \mapsto \Delta\varphi$ is a linear continuous operator and $\|\varphi\| = a(\varphi)$, where $a(\varphi) =< A\varphi, \varphi >_{V',V}$ is equivalent to the norm defined in $H_0^1(\Omega)$. We assume that there exists a positive constant c_1 such that

$$\|A\varphi\|_{V'} \le c_1 \|\varphi\|_V. \tag{0.3}$$

Let $B = (B_1, ..., B_N)$, where $B_i :\ V \to L^2(\Omega),\ \varphi \mapsto \frac{\partial \varphi}{\partial x_i}$, is linear and continuous. Then we have

$$uBy = \sum_{i=1}^{N} u_i \frac{\partial y}{\partial x_i}.$$

So we can verify the equation (0.1) as follows:

$$Ay = f + uBy. \tag{0.4}$$

The outline of this paper is as follows. In Section 2, we briefly prove existence and uniqueness of the solutions for the state equations. Furthermore, we construct a priori estimates for the state solutions of the state equations. In Section 3, we prove the existence of an optimal control $\tilde{u}$. Finally, the uniqueness of the optimal control $\tilde{u}$ are presented in Section 4.

2. A priori estimates of the state solution

Let $A_u = A - uB$, obviously $A_u \neq 0$, we denote the pair consisting of equations with the operator A_u and f by $\theta(u,f)$. We recall that by a solution of the equations $\theta(u,f)$, we mean a function y satisfies that

$$Ay = f + uBy. \tag{0.5}$$

Lemma 2.1. *Assume that $u \in U$. The equations $\theta(u,f)$ admit a unique solution y in V.*

Proof. The existence of the solution y of equations $\theta(u,f)$ follows from the standard application of the Galerkin method (see [4]). Now we prove the uniqueness of the solution. Assume that $y_1,\ y_2$ are solutions of $\theta(u,f)$, then $y = y_1 - y_2$ satisfies

$$Ay = uBy. \tag{0.6}$$

By using $A_u \neq 0$, we can obtain that

$$y = 0. \tag{0.7}$$

It implies

$$y_1 = y_2. \tag{0.8}$$

Now we can obtain the following results:

Lemma 2.2. *Assume that $u \in L^2(\Omega)$, $f \in V'$. Then if y is a solution of $\theta(u,f)$, we have*

$$\|y\|_V \leq \|f\|_{V'}. \tag{0.9}$$

Proof. We multiply equations $\theta(u,f)$ by y:

$$< Ay, y >=< f + uBy, y > . \tag{0.10}$$

Since $< uBy, y >_{V,V'}= 0$ and $\|y\|_V^2 =< Ay, y >$, we have

$$\|y\|_V^2 \le \|y\|_V \|f\|_{V'}. \tag{0.11}$$

Thus

$$\|y\|_V \le \|f\|_{V'}. \tag{0.12}$$

This implies (0.9).

Lemma 2.3. *The mapping $\Psi : U \to L^2(\Omega)$, $u \mapsto y$ is differentiable, and $\Psi'(\tilde{u})(v)$ satisfies the equation*

$$Az = \tilde{u}Bz + vB\hat{y}, \tag{0.13}$$

where $\hat{y} = \Psi(\tilde{u})$.

Proof. From the equation (0.13), we can find that the mapping $S :\ v \to z$ is linear and z is a solution of $\theta(\tilde{u}, vB\hat{y})$, then from (0.9) yield

$$\|z\|_V \le \|vB\hat{y}\|_{V'} \le C\|v\|_U \|\hat{y}\|_{L^2(\Omega)}. \tag{0.14}$$

It follows that

$$\|z\|_{L^2(\Omega)} \le C\|v\|_U. \tag{0.15}$$

Obviously, the mapping $S :\ v \in U \to z \in L^2(\Omega)$ is continuous. Set $y_v = \Psi(\tilde{u}+v)$ and $y = y_v - \hat{y}$. Then y satisfies

$$Ay = \tilde{u}By + vBy_v. \tag{0.16}$$

Since vBy_v is in V', the a priori estimates (0.9) gives

$$\|y\|_{L^2(\Omega)} \le \|vBy_v\|_{V'} \le C\|v\|_U \|y_v\|_{L^2(\Omega)}, \tag{0.17}$$

consequently,

$$\|y\|_{L^2(\Omega)} \le C\|v\|_U. \tag{0.18}$$

Let $w = y - z$, then w satisfies

$$Aw = \tilde{u}Bw + vBy. \tag{0.19}$$

Since $w \in L^2(\Omega)$, we deduce that

$$\|w\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}, \tag{0.20}$$

which implies

$$\|\Psi(\tilde{u}+v) - \Psi(\tilde{u}) - \Psi'(\tilde{u})(v)\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}. \tag{0.21}$$

This concludes the proof.

3. Existence of optimal control

Now, we show the existence of the optimal control $\tilde{u}$ in this section.

Theorem 3.1. *Let J be defined as*

$$J = \frac{1}{2}\|y - z_0\|^2 + \frac{1}{2}\|u - u_d\|^2,$$

where y is the unique solution of the equation

$$Ay = uBy. \tag{0.22}$$

Then there exists a pair $(\hat{y}, \tilde{u})$ such that $\hat{y}$ is a solution of $\theta(\tilde{u}, 0)$ and $\tilde{u}$ minimizes J in U.

Proof. Let

$$I = \inf J(u),$$

and let u_n be a minimizing sequence. Then $J(u_n)$ is bounded and it follows that

$$\|u_n\|_{L^2(\Omega)} \leq C.$$

We deduce from the a priori estimates that

$$\|y_n\|_{L^2(\Omega)} \leq C, \quad \|y_n\|_V \leq C, \tag{0.23}$$

$$\|A(y_n)\|_{V'} \leq C, \quad \|u_n B y_n\|_{V'} \leq C,' \tag{0.24}$$

where C is a general positive constant. By passing to subsequences if necessary, we assume that

$$u_n \rightharpoonup \tilde{u} \text{ weakly in } L^2(\Omega);$$

$$y_n \rightharpoonup \hat{y} \text{ weakly}^* \text{ in } L^2(\Omega);$$

$$y_n \rightharpoonup \hat{y} \text{ weakly in } V;$$

$$A(y_n) \rightharpoonup \zeta \text{ weakly in } V';$$

$$u_n B(y_n) \rightharpoonup \eta \text{ weakly in } V'.$$

We now pass to the limit in the state equation

$$Ay_n = u_n B y_n.$$

The operator $A : V \to V', y \to Ay$ is a linear strongly continuous, therefore, weakly continuous. It follows that $A(\hat{y}) = \zeta$. The embedding $V \hookrightarrow L^2(\Omega)$ is compact, therefore, the sequence y_n admits a subsequence which converges strongly in $L^2(\Omega)$. Furthermore, the operator $L^2(\Omega) \to V' : y \mapsto \frac{\partial y}{\partial x_i}, 1 \leq i \leq N$, is linear and continuous, thus

$$\frac{\partial y_n}{\partial x_i} \to \frac{\partial \hat{y}}{\partial x_i} \text{ strongly in } V'.$$

Since $u_n \rightharpoonup \tilde{u}$ weakly in $L^2(\Omega)$, it follows that

$$\tilde{u} B \hat{y} = \eta.$$

We can conclude that

$$A\hat{y} = \tilde{u} B \hat{y}. \tag{0.25}$$

By using the lower semicontinuity of the norms, we deduce that

$$J(\tilde{u}) \leq \liminf_n J(u_n) = \liminf J(u).$$

This shows that $(\hat{y}, \tilde{u})$ is an optimal pair.

4. Uniqueness of optimal control

In this section, we will show that the optimal control $\tilde{u}$ is unique. First, we prove the following result.

Lemma 4.1. *Let p is a solution of the adjoint equation*

$$Ap + uBp = y. \tag{0.26}$$

Then the function $J(u)$ is differentiable and we have

$$J'(u)(v) =< u + pBy, v >_U . \tag{0.27}$$

Proof. By Lemma 2.1, it admits a unique solution p. From the differentiability of Ψ, we can deduce the differentiability of J, the expression for the derivative is

$$J'(u)(v) =< \Psi'(u)(v), y >_{L^2(\Omega)} + < u, v >_U .$$

From the adjoint state equation and $\Psi'(u)(v) = z$, we have

$$\begin{aligned} < \Psi'(u)(v), y >_{L^2(\Omega)} &= < z, y > =< z, Ap + uBp > \\ &= < Az - uBz, p > =< pBy, v >, \end{aligned} \tag{0.28}$$

where $A^* = A$, $B^* = -B$ have been used. So we have

$$J'(u)(v) =< u, v >_U + < pBy, v >_U =< u + pBy, v >_U . \tag{0.29}$$

Theorem 4.2. *The optimal control $\tilde{u}$ is unique.*

Proof. Assume that $\tilde{u}_1$ and $\tilde{u}_2$ are two optimal controls, let $\hat{y}_1$ and $\hat{y}_2$ be the states corresponding to the optimal controls satisfy the equations $\theta(\tilde{u}_i, 0)$ $(i = 1, 2)$. Using the a priori estimates (0.9) applied to $\hat{y}_i$ $(i = 1, 2)$ yield

$$\|\hat{y}_i\|_V = 0, \tag{0.30}$$

$$\|\hat{y}_i\|_{L^2(\Omega)} = 0. \tag{0.31}$$

Now, let $\hat{p}_1$ and $\hat{p}_2$ be the corresponding adjoint states to $\hat{y}_1$ and $\hat{y}_2$, then we can apply the estimates (0.9) to it, we have for $i = 1, 2$:

$$\|\hat{p}_i\|_V \leq \|\hat{y}_i\|_{V'}, \tag{0.32}$$

$$\|\hat{p}_i\|_{L^2(\Omega)} \leq \|\hat{y}_i\|_{V'}. \tag{0.33}$$

Thus,

$$\|\hat{p}_i\|_V = 0, \tag{0.34}$$

$$\|\hat{p}_i\|_{L^2(\Omega)} = 0. \tag{0.35}$$

On the other hand, we can see that $\hat{y} = \hat{y}_1 - \hat{y}_2$ satisfies the equations $\theta(u_1, (\tilde{u}_1 - \tilde{u}_2)B\hat{y}_2)$. The estimate (0.9) implies

$$\begin{aligned} \|\hat{y}\|_V &\leq \|(\tilde{u}_1 - \tilde{u}_2)B\hat{y}_2\|_{V'} \\ &\leq C\|\tilde{u}_1 - \tilde{u}_2\|_U \|\hat{y}_2\|_{L^2(\Omega)}. \end{aligned} \tag{0.36}$$

Combining (0.36) with (0.31), we have

$$\|\hat{y}\|_V = 0. \tag{0.37}$$

Let $\hat{p} = \hat{p}_1 - \hat{p}_2$. It is easy to see that $\hat{p}$ satisfies the equations $\theta(-\tilde{u}_1, \hat{y}_1 - \hat{y}_2 - (\tilde{u}_1 - \tilde{u}_2)B\hat{p}_2)$. The estimate (0.9) implies

$$\begin{aligned} \|\hat{p}\|_V \leq& \|\hat{y}_1 - \hat{y}_2 - (\tilde{u}_1 - \tilde{u}_2)B\hat{p}_2\|_{V'} \\ \leq& C\|\tilde{u}_1 - \tilde{u}_2\|_U \|\hat{p}_2\|_{L^2(\Omega)} + \|\hat{y}_1 - \hat{y}_2\|_{V'} = 0. \end{aligned}$$

Since $\tilde{u}_1$ and $\tilde{u}_2$ are two optimal controls, from the result (0.27), we have

$$\begin{aligned} \tilde{u}_1 - \tilde{u}_2 =& - \hat{p}_1 B\hat{y}_1 + \hat{p}_2 B\hat{y}_2 \\ =& - \hat{p}_1 B\hat{y} - \hat{p} B\hat{y}_2. \end{aligned}$$

Taking the norm in U and combining (0.31), (0.35), and (0.37), we have

$$\|\tilde{u}_1 - \tilde{u}_2\|_U \leq C\|\hat{p}_1\|_{L^2(\Omega)} \|\hat{y}\|_V + C\|\hat{p}\|_V \|\hat{y}_2\|_{L^2(\Omega)} = 0.$$

It follows that

$$\tilde{u}_1 = \tilde{u}_2.$$

This completes the proof.

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