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NONLOCAL EIGENVALUE PROBLEMS IN VARIABLE EXPONENT SOBOLEV SPACES

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Abstract. In this paper, we discuss a nonlinear eigenvalue problem driven by the $p(x)$-Laplacian in a variable exponent space. The differentiability and the energy functional using Ekeland's principle are investigated. The existence of infinitely many eigenvalues is also established.

Keywords. Variable exponent Sobolev space; $p(x)$-Laplacian; Eigenvalue; Critical point; Ekeland's principle.

1. Introduction

Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary of finite measure $\partial\Omega$. The aim of this paper is to investigate the eigenvalues of a nonstandard partial differential equation subject to a nonlocal boundary condition. We deal with a problem of the form

$$(1) \qquad \begin{cases} -\Delta_{p(x)} u = \lambda V(x)|u|^{q(x)-2}u, \ \ x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v}(\xi) = \int\limits_{\Omega} k'_u(\xi, x, u(x))dx, \ \ \xi \in \partial\Omega, \end{cases}$$

where $p, q \in C(\overline{\Omega})$, the function V is positive on Ω and λ is real. The sign $\frac{\partial}{\partial v}$ is the outer unit derivative and k'_u is the first derivative with respect to the third variable of a nonlinear kernel $k(.,.,.) : \partial\Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$. The operator $-\triangle_{p(x)} u := -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is the so-called $p(x)$-Laplacian which is intimately related to variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and

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variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. These spaces have appeared for the first time in the thirties of the last century and had ripe for applications to partial differential equations (see [1] for an overview on the field). Much attention has given lately to problems involving $p(x)$-growth conditions as a consequence of the fact that such equations describe phenomena in Electrorheological fluids dynamics, Elastic mechanics and image processing, etc. (see [2, 3, 4]). Problems like (1) with Dirichlet boundary conditions have been widely considered [5, 6]. We refer to [7] and [8] for problems with Neumann boundary conditions and to [9, 10, 11, 6] for recent papers involving $p(x)$-Laplacian problems.

One of physical motivations for studying problems with nonlocal boundary conditions arises in diffusion problems. In [12], the authors described a diffusion processes in which the integral over the domain of the diffused chemical concentration in a straight glass tube is proportional to the electrical signal produced on the boundary by a light beam. In such diffusion processes, the boundary integral condition is used to determine the unknown concentration.

Motivated by the above, our purpose is to prove, under appropriate assumptions, the existence of nontrivial solutions to the Problem (1). Especially, we use the Ekeland's principle to show the existence of a Palais-Smale sequence. We would like to mention that we found no literature on this field for the p-Laplacian operator, i.e. when $p(x) \equiv p$ constant.

This paper is divided into four sections organized as follows. In Section 2, we recall the definition of variable exponent Lebesgue and Sobolev spaces. Some important properties are also exhibited. The main assumptions and definitions of the weak solution of problem (1) are stated in Section 3 besides some auxiliary results. The last section is concerned with the main result of the present paper as well as its proof.

2. Variable Lebesgue and Sobolev spaces

In this section, we recall definitions and preliminary properties of variable exponent Lebesgue and sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For fundamental properties and more details in this context, we refer to [13, 14, 15, 16, 17].

Let Ω be a bounded domain in $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}), p(x) > 1, \forall x \in \Omega \right\}.$$

We write

$$p^+ := \max_{x\in\Omega} p(x), \;\; p^- := \min_{x\in\Omega} p(x).$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$, for $p \in C_+(\overline{\Omega})$, is defined by

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

On this space, we define a norm, the so-called Luxemburg norm, by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

If $p^- > 1$, $L^{p(x)}(\Omega)$ is a separable reflexive Banach space [**?**].

Proposition 1. [15, 16]

(i) The conjugate space of $L^{p(x)}(\Omega)$ *is* $L^{p'(x)}(\Omega)$*, where* $p'(x) = \frac{p(x)}{p(x)-1}$.

(ii) If $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \leq p_2(x)$ *for all* $x \in \overline{\Omega}$*, then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

(iii) For any $f \in L^{p(x)}(\Omega)$ *and* $g \in L^{p'(x)}(\Omega)$*, we have the Hölder inequality*

$$\left| \int_\Omega fg dx \right| \leq (\frac{1}{p^-} + \frac{1}{q^-}) |f|_{p(x)} |g|_{q(x)} \leq 2 |f|_{p(x)} |g|_{q(x)}. \tag{2}$$

(iv) For any $f \in L^{p(x)}(\Omega)$*,* $g \in L^{q(x)}(\Omega)$ *and* $k \in L^{r(x)}(\Omega)$ *such that* $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$*, we have*

$$\left| \int_\Omega fgk dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-} \right) |f|_{p(x)} |g|_{q(x)} |k|_{r(x)} \leq 3 |f|_{p(x)} |g|_{q(x)} |k|_{r(x)}. \tag{3}$$

The modular in $L^{p(x)}(\Omega)$ is given by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$$

and it has closer relations to the norm $|.|_{p(x)}$ that are illuminated in the next proposition.

Proposition 2. [15, 16]

(i) For any $f \in L^{p(x)}(\Omega)$ *we have*

$$
|f|_{p(x)}^{p^-} \leq \rho_{p(x)}(f) \leq |f|_{p(x)}^{p^+} \quad if \ \ |f|_{p(x)} > 1,
\tag{4}
$$
$$
|f|_{p(x)}^{p^+} \leq \rho_{p(x)}(f) \leq |f|_{p(x)}^{p^-} \quad if \ \ |f|_{p(x)} \leq 1.
$$

(ii) For $f_n, f \in L^{p(x)}(\Omega)$*, we have*

$$
f_n \to f \ \ in \ \ L^{p(x)}(\Omega) \text{ if and only if } \rho_{p(x)}(f_n - f) \to 0, \ as \ n \to +\infty.
\tag{5}
$$

Proposition 3. [14] *Let* $p, q \in C_+(\overline{\Omega})$ *and* $1 \leq p(x)q(x) < \infty$ *for a.e. in* Ω*. Let* $f \in L^{q(x)}(\Omega)$*. Then*

$$
|f|_{p(x)q(x)}^{p^-} \leq \left| |f|^{p(x)} \right|_{q(x)} \leq |f|_{p(x)q(x)}^{p^+} \quad if \ \ |f|_{p(x)q(x)} > 1,
\tag{6}
$$
$$
|f|_{p(x)q(x)}^{p^+} \leq \left| |f|^{p(x)} \right|_{q(x)} \leq |f|_{p(x)q(x)}^{p^-} \quad if \ \ |f|_{p(x)q(x)} \leq 1.
$$

In particular, if $p(x) = p$ *is constant, then*

$$
|\, | \, u \, |^p \, |_{q(x)} = |u|_{q(x)}^p.
$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$
W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}
$$

and it is equipped with the norm

$$
\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.
$$

Assuming $p^- > 1$, $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space [**?**].

An essential tool in our study is the following embedding result regarding variable exponent Sobolev spaces. Let us define the critical Sobolev exponent of p

$$
p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}
\tag{7}
$$

We have the following proposition.

Proposition 4. [15] *Let $p,q \in C_+(\overline{\Omega})$ such that $q(x) < p^*(x)$ for all $x \in \Omega$. Then there is a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Notations. In what follows, for simplicity we will denote $X := W^{1,p(x)}(\Omega)$. The sign $\int$ designs $\int\limits_{\Omega}$ otherwise it will be specified. The letter c will indiscriminately used to denote various constants when the exact values are irrelevant.

3. Definition of weak solutions, assumptions and auxiliary results

Definition 5. *We say that $\lambda \in \mathbb{R}$ is an eigenvalue of Problem (1) if there exists $u \in X, u \neq 0$ that satisfies*

$$\int |\nabla u|^{p(x)} \nabla u.\nabla v dx - \lambda \int V(x)|u|^{q(x)-2} uv dx - \int\limits_{\partial\Omega} \int k'_u(\xi, x, u(x)) dx d\xi = 0 \tag{8}$$

for all $v \in X$. Such solution is called weak solution of Problem (1). The symbol $d\xi$ is the surface measure on $\partial\Omega$.

Next, we state the assumptions. The domain Ω is bounded in $\mathbb{R}^N, N \geq 2$ with smooth boundary $\partial\Omega$ of finite measure, i.e. $\text{meas}(\partial\Omega) < \infty$.

$(H_{p,q})$ $p,q \in C_+(\overline{\Omega}), q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $q^- < q^+ < p^- < p^+$.

(H_V) $V \in L^{w(x)}(\Omega)$ where $w \in C_+(\overline{\Omega})$ with $w(x) \geq \frac{p^*(x)}{p^*(x)-q(x)}$ for all $x \in \overline{\Omega}$.

(H_k) $k(.,.,.) : \partial\Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $|k(\xi,.,0)|_{p(x)} \in L^1(\partial\Omega), \forall \xi \in \partial\Omega$ and there exist $\Gamma \subset \partial\Omega, |\Gamma| > 0$ and $\Omega_1 \subset \Omega, |\Omega_1| > 0$ such that

$$\int_\Gamma \int_{\Omega_1} k(\xi, x, 0) dx d\xi > 0.$$

We suppose that $k'_s(.,.,s)$ exits and is a Carathéodory function that satisfies

$$|k'_s(\xi, x, s)| \leq a(\xi, x) + b(\xi, x)|s|^{\alpha(x)-1}$$

for all $\xi \in \partial\Omega$ and $x \in \Omega$. The exponent $\alpha \in C_+(\overline{\Omega})$ satisfies $\alpha(x) < p(x)$ for all $x \in \overline{\Omega}$ with

$$\alpha^- < \alpha^+ < p^- < p^+.$$

The coefficients $a(.,.), b(.,.) : \partial\Omega \times \Omega \to \mathbb{R}$ are positive measurable functions that verify

$$|a(\xi,.)|_{p'(x)}, |b(\xi,.)|_{r(x)} \in L^1(\partial\Omega), \forall \xi \in \partial\Omega,$$

where $r \in C_+(\overline{\Omega})$ is given by $r(x) = \frac{p(x)}{p(x)-\alpha(x)}, \forall x \in \Omega$.

Example 6. *Let*

- $p, \alpha, r \in C_+(\overline{\Omega})$ *with* $\alpha(x) < p(x)$ *and* $r(x) = \frac{p(x)}{p(x)-\alpha(x)}$,
- $M > 0$ *and* $f \in C(\mathbb{R},\mathbb{R})$ *such that* $|f(t)| \le M$ *for all* $t \in \mathbb{R}$,
- $a \in L^{p'(x)}(\Omega)$ *and* $b \in L^{r(x)}(\Omega)$ *positive functions and* $c \in \mathbb{R}^+$.

Then, the kernel defined by

$$k(\xi,x,t) = a(x)t + b(x)\int_0^t f(s)s^{\alpha(x)-1}ds + c$$

satisfies assumption (H_k).

The energy functional corresponding to Problem (1) is defined by $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(u) = \int \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda\int \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_{\partial\Omega}\int k(\xi,x,u(x))dxd\xi.$$

Lemma 7. *There exists* $c > 0$ *such that for all* $u \in X$ *we have*

$$\int \frac{V(x)}{q(x)}|u|^{q(x)}dx \le \frac{2c}{q^-}|V|_{w(x)}\left(\|u\|_{1,p(x)}^{q^+} + \|u\|_{1,p(x)}^{q^-}\right). \tag{9}$$

Proof. For all $u \in X$, we have by the Hölder inequality and relation (6)

$$\int \frac{V(x)}{q(x)}|u|^{q(x)}dx \le \frac{2}{q^-}|V|_{w(x)}\left||u|^{q(x)}\right|_{w'(x)} \le \frac{2}{q^-}|V|_{w(x)}\left(|u|^{q^+}_{w'(x)q(x)} + |u|^{q^-}_{w'(x)q(x)}\right).$$

By assumption (H_V), we have that $w'(x)q(x) \le p^*(x)$ for all $x \in \Omega$ so there holds and a continuous embedding of X into $L^{w'(x)q(x)}(\Omega)$. Hence there exits $c > 0$ such that (9) is valid.

Lemma 8. *There exist* $c', c'' > 0$ *such that for all* $u, v \in X$*, the following inequalities hold*

$$\int |k'_u(\xi,x,u)||v|dx \le c'\left(|a(\xi,.)|_{p'(x)} + |b(\xi,.)|_{r(x)}\left(\|u\|_{1,p(x)}^{\alpha^+-1} + \|u\|_{1,p(x)}^{\alpha^--1}\right)\right)\|v\|_{1,p(x)}, \tag{10}$$

(11)

$$\int |k(\xi,x,u)|dx \le c''\left(|k(\xi,.,0)|_{p(x)} + |a(\xi,.)|_{p'(x)}\|u\|_{1,p(x)} + |b(\xi,.)|_{r(x)}\left(\|u\|_{1,p(x)}^{\alpha^+} + \|u\|_{1,p(x)}^{\alpha^-}\right)\right)$$

for all $\xi \in \partial\Omega$.

Proof. Let $\xi \in \partial\Omega$ be fixed. For all $u, v \in X$ we have by assumption (H_k)

$$\int |k'_u(\xi,x,u)||v|dx \leq \int a(\xi,x)|v|dx + \int b(\xi,x)|u|^{\alpha(x)-1}|v|dx.$$

The Hölder inequalities (2) and (3) give

$$\int |k'_u(\xi,x,u)||v|dx \leq 2|a(\xi,.)|_{p'(x)}|v|_{p(x)} + 3|b(\xi,.)|_{r(x)} \left||u|^{\alpha(x)-1}\right|_{\frac{p(x)}{\alpha(x)-1}} |v|_{p(x)}$$

since $\frac{1}{r(x)} + \frac{\alpha(x)-1}{p(x)} + \frac{1}{p(x)} = 1$. Using Proposition 3, we get

$$\int |k'_u(\xi,x,u)||v|dx \leq \left(2|a(\xi,.)|_{p'(x)} + 3|b(\xi,.)|_{r(x)}|u|_{p(x)}^{\alpha^i-1}\right)|v|_{p(x)}; i = -,+.$$

The continuous embedding of X into $L^{p(x)}(\Omega)$ ensures the existence of a positive constant c for what we have

$$\int |k'_u(\xi,x,u)||v|dx \leq c\left(2|a(\xi,.)|_{p'(x)} + 3c|b(\xi,.)|_{r(x)}\left(\|u\|_{1,p(x)}^{\alpha^+-1} + \|u\|_{1,p(x)}^{\alpha^--1}\right)\right)\|v\|_{1,p(x)}$$

$$\leq c\max(2,3c)\left(|a(\xi,.)|_{p'(x)} + |b(\xi,.)|_{r(x)}\left(\|u\|_{1,p(x)}^{\alpha^+-1} + \|u\|_{1,p(x)}^{\alpha^--1}\right)\right)\|v\|_{1,p(x)}.$$

To establish the second inequality we write for all $\xi \in \partial\Omega, x \in \Omega$ and $u \in \mathbb{R}$

$$k(\xi,x,u) = k(\xi,x,0) + k'_u(\xi,x,\theta(\xi,x)u)u; 0 < \theta(\xi,x) < 1.$$

Integrating the last equation over Ω and using the Hölder inequality, we get

$$\int |k(\xi,x,u)|dx \leq 2|1|_{p'(x)}|k(\xi,.,0)|_{p(x)} + \int |k'_u(\xi,x,\theta u)||u|dx.$$

Hence, inequality (11) is immediate from the last inequality and inequality (10).

Proposition 9. *For all* $\lambda \in \mathbb{R}$*, the functional* $I_\lambda \in C^1(X,\mathbb{R})$ *and we have*

$$I'_\lambda(u)v = \int |\nabla u|^{p(x)-2}\nabla u.\nabla v dx - \lambda\int V(x)|u|^{q(x)-2}uvdx - \int_{\partial\Omega}\int k'_u(\xi,x,u)vdxd\xi$$

for all $v \in X$.

Proof. It is well known that the functional defined by

$$\phi_1(u) = \int \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int \frac{V(x)}{q(x)}|u|^{q(x)}dx.$$

is $C^1(X,\mathbb{R})$ for all $\lambda \in \mathbb{R}$ and that

$$\phi_1'(u)v = \int |\nabla u|^{p(x)-2}\nabla u.\nabla v dx - \lambda \int V(x)|u|^{q(x)-2}uvdx.$$

We denote by $\phi_2 : X \to \mathbb{R}$ the functional defined by $\phi_2(u) = I_\lambda(u) - \phi_1(u)$. Next, we show $\phi_2 \in C^1(X,\mathbb{R})$ and

$$\phi_2'(u)v = \int_{\partial\Omega}\int k_u'(\xi,x,u(x))vdxd\xi.$$

For all $u,h \in X$, we put

$$F(u,h) = \phi_2(u+h) - \phi_2(u) - G(u,h),$$

where $G(u,h) = \int_{\partial\Omega}\int k_u'(\xi,x,u(x))hdxd\xi$. By the Mean Value Theorem, we have for all $\xi \in \partial\Omega$ and $x \in \Omega$

$$k(\xi,x,u+h) - k(\xi,x,u) = k_u'(\xi,x,u+\theta h)h, 0 < \theta = \theta(\xi,x) < 1.$$

So

$$\begin{aligned}
|F(u,h)| &= \left| \int_{\partial\Omega}\int \big(k(\xi,x,u+h) - k(\xi,x,u) - k_u'(\xi,x,u)h\big)\, dxd\xi \right| \\
&= \left| \int_{\partial\Omega}\int \big(k_u'(\xi,x,u+\theta h) - k_u'(\xi,x,u)\big)\, hdxd\xi \right| \\
&\le \int_{\partial\Omega}\int \big|k_u'(\xi,x,u+\theta h) - k_u'(\xi,x,u)\big|\, |h|dxd\xi.
\end{aligned}$$

The Hölder inequality and Sobolev embedding give

$$\begin{aligned}
|F(u,h)| &\le |h|_{p(x)} \int_{\partial\Omega} |k_u'(\xi,.,u+\theta h) - k_u'(\xi,.,u)|_{p'(x)}d\xi \\
&\le c\|h\|_{1,p(x)} \int_{\partial\Omega} |k_u'(\xi,.,u+\theta h) - k_u'(\xi,.,u)|_{p'(x)}d\xi
\end{aligned}$$

$$\frac{1}{\|h\|_{1,p(x)}}|F(u,h)| \le c\int_{\partial\Omega} |k_u'(\xi,.,u+\theta h) - k_u'(\xi,.,u)|_{p'(x)}d\xi, \tag{12}$$

where c denoted the embedding constant. The following result is needed.

Lemma 10. *For all $\xi \in \partial\Omega$, the operator*

$$(K_\xi u)(x) = k'_u(\xi, x, u(x)), x \in \Omega$$

is continuous from $L^{p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)$.

Proof. Let $\xi \in \partial\Omega$ be fixed and let $(u_n)_{n\in\mathbb{N}} \subset L^{p(x)}(\Omega)$ be a strongly convergent sequence to an element $u \in L^{p(x)}(\Omega)$. This is equivalent to say that $\lim_{n\to+\infty} \rho_{p(x)}(u_n - u) = 0$ or that $|u_n - u|^{p(x)} \to 0$ in $L^1(\Omega)$ as $n \to +\infty$. Hence, there exists by Theorem IV.9 in [18] a subsequence still denoted u_n and a function $g \in L^1(\Omega)$ such that

$$|u_n(x) - u(x)|^{p(x)} \to 0 \text{ a.e. in } \Omega \text{ as } n \to +\infty$$

and

$$\forall n \in \mathbb{N}, |u_n(x) - u(x)|^{p(x)} \leq g(x) \text{ a.e. in } \Omega,$$

which is equivalent to

$$u_n(x) \to u(x) \text{ a.e. in } \Omega \text{ as } n \to +\infty$$

and

$$\forall n \in \mathbb{N}, |u_n(x)| \leq g^{\frac{1}{p(x)}}(x) + |u(x)| \text{ a.e. in } \Omega.$$

The kernel $k'_u(.,.,.)$ is a Carathéodory function, consequently we have

$$\forall n \in \mathbb{N}, k'_u(\xi, x, u_n(x)) \to k'_u(\xi, x, u(x)) \text{ a.e. } x \in \Omega.$$

On the other hand, we have by assumption (H_k)

$$\begin{aligned} |k'_u(\xi, x, u_n(x))| &\leq a(\xi, x) + b(\xi, x)|u_n(x)|^{\alpha(x)-1} \\ &\leq a(\xi, x) + b(\xi, x)\left(g^{\frac{1}{p(x)}}(x) + |u(x)|\right)^{\alpha(x)-1}. \end{aligned}$$

We have by assumption $a(\xi, .) \in L^{p'(x)}(\Omega)$. The term $b(\xi, .)\left(g^{\frac{1}{p(x)}} + |u|\right)^{\alpha(x)-1}$ is also in $L^{p'(x)}(\Omega)$. Indeed, we get by the Hölder inequality the following

$$\int \left|b(\xi, x)\left(g^{\frac{1}{p(x)}} + |u|\right)^{\alpha(x)-1}\right|^{p'(x)} dx \leq 2\left||b(\xi, .)|^{p'(x)}\right|_{\beta(x)} \left|\left|g^{\frac{1}{p(x)}} + |u|\right|^{(\alpha(x)-1)p'(x)}\right|_{\beta'(x)}$$

with $\beta(x)=\frac{p(x)-1}{p(x)-\alpha(x)}$. Hence, by relation (2.5) there exist $\eta,\tau>0$ such that

$$\int\left|b(\xi,x)\left(g^{\frac{1}{p(x)}}+|u|\right)^{\alpha(x)-1}\right|^{p'(x)}dx\leq 2\,|b(\xi,.)|_{r(x)}^{\eta}\left|g^{\frac{1}{p(x)}}+|u|\right|_{p(x)}^{\tau}$$

$$\leq c\,|b(\xi,.)|_{r(x)}^{\eta}\left(|g|_{1}^{\delta}+|u|_{p(x)}^{\tau}\right)<\infty$$

for some $c,\delta>0$. This implies that $\rho_{p'(x)}\left(b(\xi,.)(g^{\frac{1}{p(x)}}+|u|)^{\alpha(x)-1}\right)<\infty$ and then $b(\xi,.)(g^{\frac{1}{p(x)}}+|u|)^{\alpha(x)-1}\in L^{p'(x)}(\Omega)$. Consequently, applying Lebesgue's dominated convergent Theorem (Lemma 2.3.16 [19]) it follows that $k'_u(\xi,.,u_n)$ converges strongly to $k'_u(\xi,.,u)$ in $L^{p'(x)}(\Omega)$ for all $\xi\in\partial\Omega$ and this conclusion completes the proof of Lemma 10.

Next, we continue the proof of Proposition 9. In relation (12), for all $\xi\in\partial\Omega$ we have by Lemma 10

$$\lim_{\|h\|_{1,p(x)}\to 0}|k'_u(\xi,x,u+\theta h)-k'_u(\xi,x,u)|_{p'(x)}=0.$$

We deduce that there exists $M>0$ such that

$$|k'_u(\xi,x,u+\theta h)-k'_u(\xi,x,u)|_{p'(x)}\leq M$$

for all $\xi\in\partial\Omega$. Since $\partial\Omega$ is of finite measure, then for all $\varepsilon>0$ there exists by Egorov's Theorem (see [20]) $A_\varepsilon\subset\partial\Omega$ such that

$$\text{meas}(\partial\Omega-A_\varepsilon)<\frac{\varepsilon}{2cM}$$

and

$$\sup_{\xi\in A_\varepsilon}\left|k'_u(\xi,.,u+\theta h)-k'_u(\xi,.,u)\right|_{p'(x)}<\frac{\varepsilon}{2c\text{meas}(\partial\Omega)}.$$

For $\|h\|_{1,p(x)}$ small enough, we have by relation (12)

$$\frac{1}{\|h\|_{1,p(x)}}|F(u,h)|\leq c\left(\int\limits_{\partial\Omega-A_\varepsilon}+\int\limits_{A_\varepsilon}\right)|k'_u(\xi,x,u+\theta h)-k'_u(\xi,x,u)|_{p'(x)}d\xi$$

$$\leq c(M\text{meas}(\partial\Omega-A_\varepsilon)+\text{meas}(\partial\Omega)\sup_{\xi\in A_\varepsilon}|k'_u(\xi,x,u+\theta h)-k'_u(\xi,x,u)|_{p'(x)}d\xi)<\varepsilon.$$

This proves the existence of the Fréchet derivative of the functional ϕ_2 at any point $u\in X$. Furthermore, we have $\phi'_2(u)v=G(u)v$, $\forall v\in X$. The continuity of ϕ_2 is immediate from Lemma 10.

Lemmas 7 and 8 show that the functional is well-defined on X and by Proposition 9 we conclude that its critical points are exactly the weak solutions of Problem (1).

4. Main results

Theorem 11. *Under assumptions* $(H_{p,q}),(H_V)$ *and* (H_k)*, any real* λ *is an eigenvalue of Problem (1).*

Proof. We begin by showing that the functional I_λ is bounded from below in X. For all u in X and $\lambda \in \mathbb{R}$, we have by inequalities (9), (11) and relation (2.3)

(13)

$$I_\lambda(u) \geq \frac{1}{p^+}\|u\|_{1,p(x)}^{p^i} - |\lambda|A\left(\|u\|_{1,p(x)}^{q^+} + \|u\|_{1,p(x)}^{q^-}\right) - B\|u\|_{1,p(x)} - C\left(\|u\|_{1,p(x)}^{\alpha^+} + \|u\|_{1,p(x)}^{\alpha^-}\right) -$$

$$-c''\left|\int_{\partial\Omega}\int k(\xi,x,0)dxd\xi\right|, i = - \text{ or } +,$$

with $A = \frac{2c}{q^-}|V|_{w(x)}, B = c'' \int_{\partial\Omega} |a(\xi,.)|_{p'(x)}$ and $C = c'' \int_{\partial\Omega} |b(\xi,.)|_{r(x)}$. We recall the fundamental inequality (see [21])

$$\delta t^k - \gamma t^l \leq \delta\left(\frac{\delta}{\gamma}\right), \forall t \geq 0 \tag{14}$$

for any $\delta,\gamma > 0$ and $0 < k < l$. We rewrite inequality (13) as follows

$$I_\lambda(u) \geq \left(\frac{1}{5p^+}\|u\|_{1,p(x)}^{p^i} - |\lambda|A\|u\|_{1,p(x)}^{q^+}\right) + \left(\frac{1}{5p^+}\|u\|_{1,p(x)}^{p^i} - |\lambda|A\|u\|_{1,p(x)}^{q^-}\right) +$$

$$+\left(\frac{1}{5p^+}\|u\|_{1,p(x)}^{p^i} - B\|u\|_{1,p(x)}\right) + \left(\frac{1}{5p^+}\|u\|_{1,p(x)}^{p^i} - C\|u\|_{1,p(x)}^{\alpha^+}\right) +$$

$$+\left(\frac{1}{5p^+}\|u\|_{1,p(x)}^{p^i} - C\|u\|_{1,p(x)}^{\alpha^-}\right) - c''\left|\int_{\partial\Omega}\int k(\xi,x,0)dxd\xi\right|.$$

Applying now inequality (13) five times to get

$$I_\lambda(u) \geq -c^2 - c''\left|\int_{\partial\Omega}\int k(\xi,x,0)dxd\xi\right|$$

since we have by assumptions $q^- < q^+ < p^- < p^+$ and $\alpha^- < \alpha^+ < p^- < p^+$.

Next, we verify the Palais-Smale condition. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a Palais-Smale sequence at an arbitrary level $c \in \mathbb{R}$. This means that

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_u) \to 0 \text{ in } X'. \tag{15}$$

The sequence u_n is necessary bounded in X. Indeed, suppose that $\|u_n\|_{1,p(x)} \to +\infty$ as $n \to +\infty$. There exits, by (15), a constant $c' > 0$ such that inequality (13) gives

$$\frac{1}{p^+}\|u_n\|^{p^-}_{1,p(x)} - \lambda A\left(\|u_n\|^{q^+}_{1,p(x)} + \|u_n\|^{q^-}_{1,p(x)}\right) - B\|u_n\|_{1,p(x)} - C\left(\|u_n\|^{\alpha^+}_{1,p(x)} + \|u_n\|^{\alpha^-}_{1,p(x)}\right) \le c'.$$

Let $\delta > 0$ be such that $1 < \max(q^+, \alpha^+) < \delta < p^-$. Hence we get

$$\frac{1}{p^+}\|u_n\|^{p^- -\delta}_{1,p(x)} \le \lambda A\left(\|u_n\|^{q^+ -\delta}_{1,p(x)} + \|u_n\|^{q^- -\delta}_{1,p(x)}\right) + B\|u_n\|^{1-\delta}_{1,p(x)} +$$

$$+C\left(\|u_n\|^{\alpha^+ -\delta}_{1,p(x)} + \|u_n\|^{\alpha^- -\delta}_{1,p(x)}\right) + c\|u_n\|^{-\delta}_{1,p(x)} \to 0 \text{ as } n \to +\infty$$

and this is a contradiction. We get the same conclusion if $\lambda \le 0$.

Consequently, the sequence (u_n) is weakly convergent in X to an element $u \in X$. Since X is compactly embedded in $L^{p(x)}(\Omega)$ then u_n, up to a subsequence, is strongly convergent to u in $L^{p(x)}(\Omega)$. Regarding to (15), it is clear that

$$\left(I'_\lambda(u_n) - I_\lambda(u)\right)(u_n - u) \to 0 \text{ as } n \to +\infty. \tag{16}$$

Using the statement of Lemma 10, we get for all $\xi \in \partial\Omega$

$$k'_u(\xi, ., u_n) \to k'_u(\xi, ., u) \text{ in } L^{p'(x)}(\Omega) \text{ as } n \to +\infty.$$

The Hölder inequality gives

$$\left|\int \left(k'_u(\xi,x,u_n) - k'_u(\xi,x,u)\right)(u_n - u)dx\right| \le 2\int \left|k'_u(\xi,x,u_n) - k'_u(\xi,x,u)\right| |u_n - u| dx$$

$$\le \left|k'_u(\xi,.,u_n) - k'_u(\xi,.,u)\right|_{p'(x)} |u_n - u|_{p(x)} \to 0 \text{ as } n \to +\infty.$$

Using as previous Egorov's Theorem to get

$$\int\limits_{\partial\Omega}\int \left(k'_u(\xi,x,u_n) - k'_u(\xi,x,u)\right)(u_n - u)dx d\xi \to 0 \text{ as } n \to +\infty. \tag{17}$$

On the other hand, we have by Hölder inequality (3)

$$\begin{aligned}
&\left|\int V(x)\left(|u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\right)(u_n-u)dx\right| \\
&\leq \int V(x)|u_n|^{q(x)-1}|u_n-u|dx+\int V(x)|u|^{q(x)-1}|u_n-u|dx \\
&\leq 3|V|_{w(x)}|u_n-u|_{w'(x)q(x)}\left(\left||u_n|^{q(x)-1}\right|_{w'(x)q'(x)}+\left||u|^{q(x)-1}\right|_{w'(x)q'(x)}\right) \\
&\leq 3|V|_{w(x)}|u_n-u|_{w'(x)q(x)}\left(|u_n|^{q^--1}_{w'(x)q(x)}+|u|^{q^--1}_{w'(x)q(x)}\right).
\end{aligned}$$

Since by (H_V), we have $w(x)\geq \frac{p^*(x)}{p^*(x)-q(x)}$. Then $w'(x)q(x)<p^*(x)$ for all $x\in\Omega$. This implies that sequence u_n is strongly convergent in $L^{w'(x)q(x)}(\Omega)$ to u. Consequently, the sequence u_n is bounded in $L^{w'(x)q(x)}(\Omega)$. From the above conclusions, we get

$$\int V(x)\left(|u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\right)(u_n-u)dx\to 0 \text{ as } n\to+\infty. \tag{18}$$

Combining (16), (17) and (18), we deduce

$$\int\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right).(\nabla u_n-\nabla u)dx\to 0 \text{ as } n\to+\infty.$$

Next, we apply the following elementary inequality

$$(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta).(\zeta-\eta)\geq c|\zeta-\eta|^p, \text{ for all } p\geq 2 \text{ and all } \zeta,\eta\in\mathbb{R}^N$$

to get

$$c\int|\nabla u_n-\nabla u|^{p(x)}dx\leq\int\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right).(\nabla u_n-\nabla u)dx.$$

Hence

$$\int|\nabla u_n-\nabla u|^{p(x)}dx\to 0 \text{ as } n\to+\infty.$$

This fact together with relation (5) imply that $\|u_n-u\|_{1,p(x)}\to 0$ as $n\to+\infty$. This means that the Palais-Smale sequence is strongly convergent in X. For $\lambda\in\mathbb{R}$ fixed, we put

$$c_\lambda=\inf_{u\in X}I_\lambda(u). \tag{19}$$

Since I_λ is bounded from below then c_λ is finite. Let $(u_n)_{n\in\mathbb{N}}\subset X$ be a minimizing sequence of (19). The Ekeland's principle ensures the existence of a sequence $(v_n)_{n\in\mathbb{N}}\subset X$ such that

$$I_\lambda(v_n)\to c_\lambda, I'_\lambda(v_n)\to 0 \text{ in } X' \text{ and } \|u_n-v_n\|_{1,p(x)}\to 0 \text{ as } n\to+\infty.$$

We remark that v_n is a Palais-Smale sequence. Then, there exits $u_\lambda \in X$ such that v_n converges strongly to u_λ in X. We claim that the sequence u_n converge strongly to u_λ in X. Indeed, we have

$$\|u_n - u_\lambda\|_{1,p(x)} \leq \|v_n - u_\lambda\|_{1,p(x)} + \|u_n - v_n\|_{1,p(x)} \to 0 \text{ as } n \to +\infty.$$

Hence for any $\lambda \in \mathbb{R}$, u_λ is a global minimizer of the functional I_λ and thus a weak solution of Problem (1).

In order to show that any $\lambda \in \mathbb{R}$ is an eigenvalue of Problem (1) we must check the nontrivially of the solution u_λ. To this end, we distinguish two cases.

If $k'_u(.,.,0) \neq 0$ on $\partial\Omega \times \Omega$ then zero can not be a weak solution and than u_λ is not trivial.

Suppose now that $k'_u(.,.,0) \equiv 0$ on $\partial\Omega \times \Omega$. In this case we have $I_\lambda(0) = 0$. Let $\varphi \in C_0^\infty(\Omega), \varphi \neq 0$ be fixed. For $t > 0$ sufficiently small we have

$$I_\lambda(t\varphi) = \int \frac{t^{p(x)}}{p(x)} |\nabla\varphi|^{p(x)} dx - \lambda \int \frac{t^{q(x)}}{q(x)} V(x)|\varphi|^{q(x)} dx - \int\limits_{\partial\Omega} \int k(\xi, x, t\varphi) dx d\xi$$

$$\leq \frac{t^{p^-}}{p^-} \int |\nabla\varphi|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int V(x)|\varphi|^{q(x)} dx - \int_\Gamma \int_{\Omega_1} k(\xi, x, 0) dx d\xi$$

since for $t > 0$ sufficiently small we have $k(.,.,t\varphi) \sim k(.,.,,0)$. The real function

$$f(t) = \frac{t^{p^-}}{p^-} \int |\nabla\varphi|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int V(x)|\varphi|^{q(x)} dx - \int_\Gamma \int_{\Omega_1} k(\xi, x, 0) dx d\xi$$

is continuous and we have $f(0) = -\int_\Gamma \int_{\Omega_1} k(\xi, x, 0) dx d\xi < 0$. It follows that there exists $t_0 > 0$ such that $f(t_0) < 0$ which is negative by assumption.

Consequently, we have $I_\lambda(u_\lambda) < I_\lambda(t_0\varphi) < 0$ and this means that u_λ is not trivial because $I_\lambda(0) = 0$. Theorem 11 is completed.

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