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AN ITERATIVE METHOD FOR ZEROS OF ACCRETIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, for finding a zero of an accretive mapping in uniformly smooth Banach spaces, a modification of the method, introduced by Ceng, Xu and Yao in [5], is presented. The strong convergence of the modified method is proved under weaker conditions than those in [5]. We also show that some iterative methods in literature are special cases of our method.

Keywords. Accretive mapping; Nonexpansive mapping; Fixed point; Variational inequality.

1. Introduction

Let E be a Banach space with the dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We use $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. A mapping J from E into E^*, satisfying the following condition

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\| \quad \text{and} \quad \|x^*\| = \|x\|\},$$

is called a normalized duality mapping of E. It is well known that if $x \neq 0$, then $J(tx) = tJ(x)$, for all $t > 0$ and $x \in E$, and $J(-x) = -J(x)$. Let $F : E \to E$ be an η-strongly accretive and

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γ-strictly pseudocontractive mapping, i.e., F satisfies, respectively, the following conditions:

$$\langle Fx - Fy, j(x-y)\rangle \geq \eta \|x-y\|^2,$$

and

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \gamma\|(I-F)x-(I-F)y\|^2,$$

for all $x, y \in E$ and some element $j(x-y) \in J(x-y)$, where I denotes the identity mapping of E, η and $\gamma \in (0,1)$ are some positive constants. Clearly, if F is γ-strictly pseudocontractive mapping, then $\|Fx - Fy\| \leq L\|x-y\|$ with $L = 1 + 1/\gamma$ and, in this case, F is called L-Lipschitz continuous. In addition, if $L \in [0,1)$ or $L = 1$, then F is called, respectively, contractive or nonexpansive. Further, a mapping A from E into itself is said to be accretive, if $\langle u - v, j(x-y)\rangle \geq 0$, $u \in A(x), v \in A(y)$ $\forall x, y \in \mathscr{D}(A)$, the domain of A, and m-accretive, if, in addition, $\mathscr{R}(I + rA) = E$ for each $r > 0$, where $\mathscr{R}(A)$ denotes the range of A.

For the problem of finding a point $p_* \in E$ such that, for a fixed element u in E,

$$p_* \in C: \quad \langle u - p_*, j(p - p_*)\rangle \leq 0 \quad \forall p \in C = ZerA, \tag{1.1}$$

the set of zeros for an m-accretive mapping A in E, we consider the following iterative method,

$$\begin{aligned} &x^0 \in E, \text{ any element,} \\ &x^{k+1} = \alpha_k u + (1-\alpha_k)(I - \lambda_k F)T^k x^k, \\ &T^k = \beta_k I + (1-\beta_k)J^A_{r_k},\ k \geq 1, \end{aligned} \tag{1.2}$$

where $J^A_{r_k} = (I + r_k A)^{-1}$ is the resolvent operator of A and the parameters $r_k > \varepsilon > 0$, $\alpha_k \in (0,1)$ $\lambda_k \in [0,1)$ and $\beta_k \in (a,b) \subset (0,1)$ satisfy some additional conditions. When $\lambda_k \equiv 0$ for all $k \geq 1$, (1.2) has the form

$$x^{k+1} = \alpha_k u + (1-\alpha_k)T^k x^k,\ k \geq 1, \tag{1.3}$$

introduced by Qin and Su [1]. They proved its strong convergence under that E is reflexive and has a weak continuous duality mapping; $\alpha_k \in (0,1), \beta_k \in [0,1]$ with r_k satisfy the following conditions:

(i) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(ii) $r_k > \varepsilon > 0$ for all $k \geq 1$; and

(iii)' $\beta_k \in [0,a)$ for all $k \geq 1$ and some $a \in (0,1)$.

In addition, with $\beta_k = 0$ for all $k \geq 1$, we obtain the following algorithm,

$$x^{k+1} = \alpha_k u + (1-\alpha_k) J_{r_k}^A x^k,\ k \geq 1, \tag{1.4}$$

given by Kamimura and Takahashi in [2]. They proved that the sequence $\{x^k\}$, defined by (1.4), converges strongly to a point in $ZerA$ when E is a reflexive Banach space, whose norm is uniformly Gâteaux differentiable and whose every weakly compact convex subset has the fixed point property for nonexpansive mapping, under conditions (i) and (i)': $r_k > 0$ for all $k \geq 1$ and $r_k \to \infty$ as $k \to \infty$. Note that, (1.4) is a modification of the Halpern method (see, [3]), for finding a fixed point for a nonexpansive mapping on a closed convex subset C in Hilbert spaces. Next, Xu [4] considered the method in uniformly smooth Banach spaces. Recently, Ceng et al. [5] presented the iterative method,

$$x^{k+1} = \alpha_k u + (1-\alpha_k)(I - \lambda_k F) J_{r_k}^A x^k,\ k \geq 1, \tag{1.5}$$

and proved its strong convergence to an element of $ZerA$ in uniformly smooth Banach space E under conditions: (i), (ii),

(iii) $\lambda_k \in [0,1), \lim_{k\to\infty}(\lambda_k/\alpha_k) = 0$, and
(iv)' $\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$, $\sum_{k=1}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty$, and $\sum_{k=1}^{\infty} |r_{k+1} - r_k| < \infty$.

In this paper, we prove strong convergence of algorithm (1.2) under conditions (i), (ii), (iii) and (iv) $r_k > \varepsilon > 0$ for all $k \geq 1$ and $\lim_{k\to\infty}(r_k/r_{k+1}) = 1$ or $\lim_{k\to\infty} |r_{k+1} - r_k| = 0$. We also showed that the last requirement is overcomed, when E is a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

The paper is organized as follows. In Section 2, we list some related facts, that will be used in the proof of our result. In Section 3, we prove strong convergence results for implicit variant of (1.2) and (1.2). In the remark of the section, we show that some iterative methods in literature are special cases of our method.

2. Preliminaries

Let $S_1(0) := \{x \in E : \|x\| = 1\}$. The space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_1(0)$. Such an E is called a smooth Banach space. The space E is said to have a uniformly Gâteaux differentiable norm if the limit is attained uniformly for $x \in S_1(0)$. The norm of E is called Fréchet differentiable, if for all $x \in S_1(0)$, the limit is attained uniformly for $y \in S_1(0)$. The norm of E is called uniformly Fréchet differentiable (and E is called uniformly smooth) if the limit is attained uniformly for all $x, y \in S_1(0)$. It is well known that every uniformly smooth real Banach space is reflexive and has a uniformly Gâteaux differentiable norm (see, [6]).

Recall that a Banach space E is said to be

(i) uniformly convex, if for any $\varepsilon, 0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \geq 0$ such that $\|(x+y)/2\| \leq 1 - \delta$;

(ii) strictly convex, if for $x, y \in S_1(0)$ with $x \neq y$, then

$$\|(1-\lambda)x + \lambda y\| < 1 \quad \forall \lambda \in (0,1).$$

It is well known that each uniformly convex Banach space E is reflexive and strictly convex; If the norm of E is uniformly Gâteaux differentiable, then J is norm to weak star uniformly continuous on each bounded subset of E; and if E is smooth, then duality mapping is single valued. In the sequel, we shall denote the single valued normalized duality mapping by j.

Lemma 2.1. [5] *Let E be a Banach space and $F : E \to E$ be a mapping.*

(i) If F is γ-strictly pseudocontractive, then F is Lipschitz continuous with constant $L \geq 1 + (1/\gamma)$.

(ii) If E is smooth and F is η-strongly accretive and γ-strictly pseudocontractive with $\eta + \gamma \geq 1$, then $I - F$ is nonexpansive.

Lemma 2.2. (Lemma 2.2 in [7]) *Let E be a real smooth Banach space. Then, the following inequalities hold*

$$\|x + y\|^2 \leq \|x\|^2 + \, 2\langle y, j(x+y)\rangle, \; \forall x, y \in E.$$

Lemma 2.3. [4] *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following conditions $a_{k+1} \leq (1-b_k)a_k + b_k c_k + d_k$, where $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$ are sequences of real numbers such that*

(i) $b_k \in [0,1]$ and $\sum_{k=1}^{\infty} b_k = \infty$;

(ii) $\limsup_{k\to\infty} c_k \leq 0$;

(iii) $d_k \geq 0$ for all $k \geq 1$ and $\sum_{k=1}^{\infty} d_k < \infty$.

Then, $lim_{k\to\infty} a_k = 0$.

Lemma 2.4. [8] *For any $\lambda, \mu > 0$, there holds the indentity,*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right), \ \forall x \in E.$$

Lemma 2.5. [8] *Assume that $c_2 \geq c_1 > 0$. Then, $\|x - J_{c_1}^A x\| \leq 2\|x - J_{c_2}^A x\|$ for all $x \in E$.*

Let μ be a continuous linear functional on l^∞ and let $(a_1, a_2, ...) \in l^\infty$. We write $\mu_k(a_k)$ instead of $\mu((a_1, a_2, ...))$. We recall that μ is a Banach limit when μ satisfies $\|\mu\| = \mu_k(1) = 1$ and $\mu_k(a_{k+1}) = \mu_k(a_k)$ for each $(a_1, a_2, ...) \in l^\infty$. For a Banach limit μ, we know that

$$\liminf_{k\to\infty} a_k \leq \mu_k(a_k) \leq \limsup_{k\to\infty} a_k,$$

for all $(a_1, a_2, ...) \in l^\infty$. If $a = (a_1, a_2, ...) \in l^\infty$, $b = (b_1, b_2, ...) \in l^\infty$ and $a_k \to c$ (respectively, $a_k - b_k \to 0$), as $k \to \infty$, we have $\mu_k(a_k) = \mu(a) = c$ (respectively, $\mu_k(a_k) = \mu_k(b_k)$).

Lemma 2.6. [9] *Let C be a nonempty, closed and convex subset of a Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{x^k\}$ be a bounded subset of E, let z be an element of C and μ be a Banach limit. Then,*

$$\mu_k \|x^k - z\|^2 = \min_{x \in C} \mu_k \|x^k - x\|^2$$

if and only if $\mu_k \langle u - z, j(x^k - z) \rangle \leq 0$ for all $u \in C$.

Lemma 2.7. [10] *Let $\{x^k\}$ and $\{z^k\}$ be bounded sequences in a Banach space E such that $x^{k+1} = h_k x^k + (1-h_k) z^k$ for $k \geq 1$, where $\{h_k\}$ be a sequence in $[0,1]$ and satisfies the condition*

$$0 < \liminf_{k\to\infty} h_k \leq \limsup_{k\to\infty} h_k < 1.$$

Asume that

$$\limsup_{k\to\infty}\left(\|z^{k+1}-z^k\|-\|x^{k+1}-x^k\|\right)\le 0.$$

Then, $\lim_{k\to\infty}\|x^k-z^k\|=0$.

Proposition 2.8. [11] *Let E be a Banach space and A be an accretive mapping in E such that $\mathscr{D}(A)\subset\mathscr{R}(I+tA)$ for all $t>0$. Then,*

$$\frac{1}{r}\|J_t^A x-J_r^A J_t^A x\|\le\frac{1}{t}\|x-J_t^A x\|\quad\text{for all } x\in\mathscr{R}(I+tA) \text{ and } r,t>0.$$

3. Main results

First, we prove the following proposition.

Proposition 3.1. *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta+\gamma\ge 1$ and let A be an m-accretive mapping in a uniformly smooth Banach space E. Let $J_{r_t}^A=(I+r_tA)^{-1}$ with $r_t>\varepsilon>0$ for all $t>0$. For each $t>0$, choose three numbers $\alpha_t\in(0,1)$, $\lambda_t\in[0,1)$ and $\beta_t\in(a,b)\subset(0,1)$ arbitrarily with fixed numbers a and b such that $\alpha_t,\lambda_t/\alpha_t\to 0$ as $t\to 0$ and let $\{y^t\}$ be defined by*

$$y^t=\alpha_t u+(1-\alpha_t)(I-\lambda_tF)T^ty^t,\ T^t=\beta_tI+(1-\beta_t)J_{r_t}^A, \tag{3.1}$$

with a fixed element $u\in E$. Then, the net $\{y^t\}$ converges strongly to p_*, solving (1.1) *with $C=ZerA$, assumed to be nonempty, as $t\to 0$.*

Proof. Consider the mapping $U_t=\alpha_tu+(1-\alpha_t)(I-\lambda_tF)T^t$. From Lemma 2.1 and the non-expansive property of $J_{r_t}^A$ (see, [5]), and hence, T^t, it follows that

$$\begin{aligned}\|U_tx-U_ty\| &= \|(1-\alpha_t)(I-\lambda_tF)T^tx-(1-\alpha_t)(I-\lambda_tF)T^ty\|\\ &= (1-\alpha_t)\|[(1-\lambda_t)T^t+\lambda_t(I-F)T^t]x-[(1-\lambda_t)T^t+\lambda_t(I-F)T^t]y\|\\ &\le (1-\alpha_t)[(1-\lambda_t)\|T^tx-T^ty\|+\lambda_t\|T^tx-T^ty\|]\\ &\le (1-\alpha_t)\|x-y\|\quad \forall x,y\in E,\ \alpha_t\in(0,1)\ \forall t>0.\end{aligned}$$

Thus, U_t is a contraction in E. By Banach's Contraction Principle, there exists a unique element $y^t\in E$, satisfying (3.1). Next, we show that $\{y^t\}$ is bounded. Indeed, for any point $p\in ZerA$, we have $p = T^t p$. Then, by virtue of Lemma 2.1,

$$\begin{aligned}
\|y^t - p\|^2 &= \langle y^t - p, j(y^t - p)\rangle \\
&= \alpha_t\langle u - p, j(y^t - p)\rangle + (1-\alpha_t)\langle (I - \lambda_t F)T^t y^t - p, j(y^t - p)\rangle \\
&= \alpha_t\langle u - p, j(y^t - p)\rangle + (1-\alpha_t)(1-\lambda_t)\langle T^t y^t - T^t p, j(y^t - p)\rangle \\
&\quad + (1-\alpha_t)\lambda_t\langle (I - F)T^t y^t - p, j(y^t - p)\rangle \\
&\le \alpha_t\langle u - p, j(y^t - p)\rangle + (1-\alpha_t)(1-\lambda_t)\|y^t - p\|^2 \\
&\quad + (1-\alpha_t)\lambda_t\left[\|y^t - p\|^2 + \|Fp\|\|y^t - p\|\right] \\
&= \alpha_t\langle u - p, j(y^t - p)\rangle + (1-\alpha_t)\|y^t - p\|^2 + (1-\alpha_t)\lambda_t\|Fp\|\|y^t - p\|.
\end{aligned}$$

Therefore,

$$\|y^t - p\|^2 \le \langle u - p, j(y^t - p)\rangle + \frac{\lambda_t}{\alpha_t}\|Fp\|\|y^t - p\|. \tag{3.2}$$

Consequently, $\{y^t\}$ is bounded. So, are the nets $\{J^A_{r_t} y^t\}$, $\{T^t y^t\}$ and $\{FT^t y^t\}$. Suppose that $y^m = y^{t_m}$ and $\varphi(x) = \mu_m\|y^m - x\|^2$ for all $x \in E$, where $\{t_m\}$ is a sequence in $(0, t_0)$ that converges to 0 as $m \to \infty$. We see that $\varphi(x) \to \infty$ as $\|x\| \to \infty$ and φ is continuous and convex, so as E is reflexive, there exists an element $\tilde{y} \in E$ such that $\varphi(\tilde{y}) = \min_{x\in E}\varphi(x)$, i.e., the set

$$C^* = \{\tilde{y} \in E:\ \varphi(\tilde{y}) = \min_{y\in E}\ \varphi(y)\} \neq \emptyset.$$

It is easy to see that C^* is a bounded, closed and convex subset in E (see, [12]). Further, from (3.1), the conditions on λ_m, α_m and the bounded property of the sequences $\{FT^m y^m\}$ and $\{T^m y^m\}$ with $T^m = \beta_m I + (1-\beta_m)J^A_{r_m}$, $\alpha_m = \alpha_{t_m}$, $\beta_m = \beta_{t_m}$ and $r_m = r_{t_m}$, it follows that

$$\|y^m - T^m y^m\| \le \alpha_m(\|u\| + \|T^m y^m\|) + (1-\alpha_m)\lambda_m\|FT^m y^m\| \to 0 \quad \text{as } m \to \infty. \tag{3.3}$$

The following inequality

$$\begin{aligned}
\|y^m - J^A_{r_m} y^m\| &\le \|y^m - T^m y^m\| + \|T^m y^m - J^A_{r_m} y^m\| \\
&= \|y^m - T^m y^m\| + \beta_m\|y^m - J^A_{r_m} y^m\|,
\end{aligned}$$

or

$$(1-\beta_m)\|y^m - J^A_{r_m} y^m\| \le \|y^m - T^m y^m\|,$$

which together with (3.3) and the condition on β_m implies that

$$\lim_{m\to\infty} \|y^m - J^A_{r_m} y^m\| = 0.$$

Using Lemma 2.5 with $c_1 = \varepsilon$, $c_2 = r_m$ and the last limit, we obtain

$$\lim_{m\to\infty} \|y^m - J^A_{\varepsilon} y^m\| = 0.$$

On the other hand, from the properties of a Banach limit it follows that

$$\varphi(J^A_\varepsilon \tilde{y}) = \mu_m \|y^m - J^A_\varepsilon \tilde{y}\|^2 = \mu_m \|J^A_\varepsilon y^m - J^A_\varepsilon \tilde{y}\|^2 \le \mu_m \|y^m - \tilde{y}\|^2 = \varphi(\tilde{y}),$$

that is $J^A_\varepsilon C^* \subseteq C^*$, i.e., C^* is invariant under nonexpansive mapping J^A_ε. Because a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, J^A_ε has a fixed point, say $\tilde{p}$, in C^*. It means that $\tilde{p} \in ZerA \cap C^*$. Now, from Lemma 2.6, we know that $\tilde{p}$ is a minimizer of $\varphi(x)$ on E, if and only if

$$\mu_m \langle x - \tilde{p}, j(y^m - \tilde{p})\rangle \le 0 \quad \forall x \in E. \tag{3.4}$$

Taking $x = u$ in (3.4) and replacing y^t with p in (3.2) by y^m with $\tilde{p}$, respectively, we obtain that $\mu_m \|y^m - \tilde{p}\|^2 = 0$. Thus, there exists a subsequence $\{y^{m_l}\}$ of $\{y^m\}$ which converges strongly to $\tilde{p}$ as $l \to \infty$. Again, by virtue of (3.2) and the norm to weak star continuous property of the normalized duality mapping j on bounded subsets of E, we obtain that

$$\langle u - p, j(p - \tilde{p})\rangle \le 0 \quad \forall p \in ZerA. \tag{3.5}$$

Since p and $\tilde{p}$ belong to $ZerA$, a closed and convex subset, replacing p in (3.5) by $sp + (1-s)\tilde{p}$ for $s \in (0,1)$, using the well-known property $j(s(\tilde{p} - p)) = sj(\tilde{p} - p)$ for $s > 0$, dividing by s and taking $s \to 0$, we obtain

$$\langle u - \tilde{p}, j(p - \tilde{p})\rangle \le 0 \quad \forall p \in ZerA.$$

The uniqueness of p_*, satisfying (1.1), guarantees that $\tilde{p} = p_*$ and all net $\{y^t\}$ converges strongly to p_* as $t \to \infty$. This completes the proof.

Lemma 3.2. *Let F, A and E be as in Proposition 3.1. Assume that $r_k > \varepsilon$, α_k, λ_k and β_k satisfy conditions: $\alpha_k \in (0,1)$ $\lambda_k \in [0,1)$ and $\beta_k \in (a,b) \subset (0,1)$ with $\alpha_k, \lambda_k/\alpha_k \to 0$ as $k \to \infty$. Then,* *for any bounded sequence $\{x^k\}$ in E such that $\lim_{k\to\infty}\|J_r^A x^k - x^k\| = 0$ for any $r > \varepsilon$, we have*

$$\limsup_{k\to\infty}\langle u - p_*, j(x^k - p_*)\rangle \le 0. \tag{3.6}$$

Proof. Let $\{y^m\}$ be the sequence in the proof of Proposition 3.1. Then,

$$\begin{aligned}
\|y^m - x^k\|^2 &= \langle \alpha_m u + (1-\alpha_m)(I-\lambda_m F)T^m y^m - x^k, j(y^m - x^k)\rangle \\
&= \alpha_m\langle u - (I-\lambda_m F)T^m y^m, j(y^m - x^k)\rangle + \langle T^m y^m - T^m x^k, j(y^m - x^k)\rangle \\
&\quad + \langle T^m x^k - x^k, j(y^m - x^k)\rangle - \lambda_m\langle FT^m y^m, j(y^m - x^k)\rangle \\
&\le \|y^m - x^k\|^2 - (1-\alpha_m)\lambda_m\langle FT^m y^m, j(y^m - x^k)\rangle \\
&\quad + \|T^m x^k - x^k\|\tilde{M} + \alpha_m\langle u - T^m y^m, j(y^m - x^k)\rangle
\end{aligned}$$

where $\tilde{M} \ge \|y^m - x^k\|$. Therefore, one has

$$\begin{aligned}
-\langle u - T^m y^m, j(y^m - x^k)\rangle + (1-\alpha_m)\frac{\lambda_m}{\alpha_m}\langle FT^m y^m, j(y^m - x^k)\rangle &\le \frac{\|T^m x^k - x^k\|\tilde{M}}{\alpha_m} \\
&\le \frac{(1-a)\|J_{r_m}^A x^k - x^k\|\tilde{M}}{\alpha_m},
\end{aligned}$$

that together with the assumption implies

$$\limsup_{k\to\infty}\left[\langle u - T^m y^m, j(x^k - y^m)\rangle + (1-\alpha_m)\frac{\lambda_m}{\alpha_m}\langle FT^m y^m, j(y^m - x^k)\rangle\right] \le 0.$$

Passing $m \to \infty$ in the last inequality and using Proposition 3.1, the conditions on α_m, λ_m, and (3.3), we get (3.6). Lemma is proved.

Now, we are in position to prove our main results.

Theorem 3.3. *Let F, A and E be as in Proposition 3.1. Assume that, there hold conditions (i), (ii), (iii) and (iv). Then, the sequence $\{x^k\}$, defined by* (1.2), *converges strongly to the element p_*, solving* (1.1).

Proof. Since $T^k = \beta_k I + (1-\beta_k)T_k, T_k = J^A_{r_k}$ is nonexpansive and $T^k p = p$ for any point $p \in ZerA$. Then, by Lemma 2.1,

$$
\begin{aligned}
\|x^{k+1} - p\| &= \big\|\alpha_k(u-p) + (1-\alpha_k)\big[(1-\lambda_k F)T^k x^k - p\big]\big\| \\
&= \big\|\alpha_k(u-p) + (1-\alpha_k)\big[(1-\lambda_k)(T^k x^k - T^k p) + \lambda_k((I-F)T^k x^k - p)\big]\big\| \\
&\le \alpha_k\|u-p\| + (1-\alpha_k)\big[(1-\lambda_k)\|x^k - p\| + \lambda_k(\|x^k - p\| + \|Fp\|)\big] \\
&\le \alpha_k\|u-p\| + (1-\alpha_k)\|x^k - p\| + \lambda_k\|Fp\| \\
&= (1-\alpha_k)\|x^k - p\| + \alpha_k\big[\|u-p\| + (\lambda_k/\alpha_k)\|Fp\|\big] \\
&\le \max\ \{\|x^1 - p\|, \|u-p\|\} + \sup_{k\ge 1}(\lambda_k/\alpha_k)\|Fp\|.
\end{aligned}
$$

Therefore, $\{x^k\}$ is bounded. So, are the sequences $\{T_k x^k\}$, $\{T^k x^k\}$, $\{T^{k+1}x^k\}$, $\{FT^k x^k\}$ and $\{FT^{k+1}x^k\}$. Without any loss of generality, we assume that they are bounded by a positive constant M_1. Further, it is easy to see that

$$
\begin{aligned}
x^{k+1} &= \alpha_k u + (1-\alpha_k)\big[\lambda_k(I-F)T^k x^k + (1-\lambda_k)T^k x^k\big] \\
&= \alpha_k u + (1-\alpha_k)\big[\lambda_k(I-F)T^k x^k + (1-\lambda_k)\big(\beta_k x^k + (1-\beta_k)T_k x^k\big)\big] \\
&= h_k x^k + (1-h_k)z^k,
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
&h_k = (1-\alpha_k)(1-\lambda_k)\beta_k \text{ and} \\
&z^k = \frac{\alpha_k u + (1-\alpha_k)\big[\lambda_k(I-F)T^k x^k + (1-\lambda_k)(1-\beta_k)T_k x^k\big]}{1-h_k}.
\end{aligned}
$$

Clearly, $0 < \liminf_{k\to\infty} h_k \le \limsup_{k\to\infty} h_k < 1$. We write $z^k = z_1^k + z_2^k + z_3^k$ with

$$
z_1^k = \frac{\alpha_k u}{1-h_k}, z_2^k = \frac{(1-\alpha_k)\lambda_k(I-F)T^k x^k}{1-h_k} \text{ and } z_3^k = \frac{(1-\alpha_k)(1-\lambda_k)(1-\beta_k)T_k x^k}{1-h_k}.
$$

Clearly,

$$\begin{aligned}
\|z_1^{k+1}-z_1^k\| &= \left|\frac{\alpha_{k+1}}{1-h_{k+1}}-\frac{\alpha_k}{1-h_k}\right|\|u\|,\\
\|z_2^{k+1}-z_2^k\| &= \left\|\frac{(1-\alpha_{k+1})\lambda_{k+1}(I-F)T^{k+1}x^{k+1}}{1-h_{k+1}}-\frac{(1-\alpha_k)\lambda_k(I-F)T^kx^k}{1-h_k}\right\|\\
&\le \frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}\big\|(I-F)T^{k+1}x^{k+1}-(I-F)T^{k+1}x^k\big\|\\
&\quad+\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}\big\|(I-F)T^{k+1}x^k-(I-F)T^kx^k\big\|\\
&\quad+\left|\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}-\frac{(1-\alpha_k)\lambda_k}{1-h_k}\right|\|(I-F)T^kx^k\|\\
&\le \frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}\big\|x^{k+1}-x^k\big\|+\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}4M_1\\
&\quad+\left|\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}-\frac{(1-\alpha_k)\lambda_k}{1-h_k}\right|2M_1,
\end{aligned}$$

and

$$\begin{aligned}
\|z_3^{k+1}-z_3^k\| &= \bigg\|\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})T_{k+1}x^{k+1}}{1-h_{k+1}}\\
&\qquad-\frac{(1-\alpha_k)(1-\lambda_k)(1-\beta_k)T_kx^k}{1-h_k}\bigg\|\\
&\le \frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}\big\|T_{k+1}x^{k+1}-T_kx^k\big\|\\
&\quad+\left|\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}-\frac{(1-\alpha_k)(1-\lambda_k)(1-\beta_k)}{1-h_k}\right|\|T_kx^k\|.
\end{aligned}$$

From Lemma 2.4 and the nonexpansive property of $J_{r_k}^A$, one finds that

$$\begin{aligned}
\big\|T_{k+1}x^{k+1}-T_kx^k\big\| &= \left\|J_{r_k}^A\left(\frac{r_k}{r_{k+1}}x^{k+1}+\left(1-\frac{r_k}{r_{k+1}}\right)J_{r_{k+1}}^Ax^{k+1}\right)-J_{r_k}^Ax^k\right\|\\
&\le \left\|\left(\frac{r_k}{r_{k+1}}x^{k+1}+\left(1-\frac{r_k}{r_{k+1}}\right)J_{r_{k+1}}^Ax^{k+1}\right)-x^k\right\|\\
&\le \|x^{k+1}-x^k\|+\left|1-\frac{r_k}{r_{k+1}}\right|2M_1.
\end{aligned}$$

Therefore, one has

$$
\begin{aligned}
\|z^{k+1}-z^k\| \le{}& \frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}\|x^{k+1}-x^k\| \\
&+\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}\|x^{k+1}-x^k\| \\
&+\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}4M_1+\left|\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}-\frac{(1-\alpha_k)\lambda_k}{1-h_k}\right|2M_1 \\
&+\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}\left|1-\frac{r_k}{r_{k+1}}\right|2M_1 \\
&+\left|\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}-\frac{(1-\alpha_k)(1-\lambda_k)(1-\beta_k)}{1-h_k}\right|M_1 \\
&+\left|\frac{\alpha_{k+1}}{1-h_{k+1}}-\frac{\alpha_k}{1-h_k}\right|M_1 \\
={}& \tilde{a}_k\|x^{k+1}-x^k\|+\tilde{c}_k,
\end{aligned}
\tag{3.8}
$$

where

$$
\tilde{a}_k=\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}+\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}
$$

and $\tilde{c}_k$ is the sum of the remain terms in the right hand-side of the before-last equality. Since $|1-(r_k/r_{k+1})|\le|r_{k+1}-r_k|/\varepsilon$, it is not difficult to verify that $\tilde{c}_k\to 0$ as $k\to\infty$ when r_k satidfies condition (iv),

$$
\lim_{k\to\infty}\frac{(1-\alpha_{k+1})\lambda_{k+1}}{1-h_{k+1}}=0 \text{ and } \lim_{k\to\infty}\frac{(1-\alpha_{k+1})(1-\lambda_{k+1})(1-\beta_{k+1})}{1-h_{k+1}}=1.
$$

Consequently, $\lim_{k\to\infty}\tilde{a}_k=1$, and hence, from (3.8) and the property of $\tilde{c}_k$, it follows that

$$
\limsup_{k\to\infty}\big(\|z^{k+1}-z^k\|-\|x^{k+1}-x^k\|\big)\le 0.
$$

By virtue of Lemma 2.7, one has

$$
\lim_{k\to\infty}\|x^k-z^k\|=0. \tag{3.9}
$$

Thus, from (3.8) and (3.9), we obtain that

$$
\lim_{k\to\infty}\|x^{k+1}-x^k\|=\lim_{k\to\infty}(1-h_k)\|z^k-x^k\|=0. \tag{3.10}
$$

Next, from (1.2) it follows that

$$
\|x^{k+1}-T^kx^k\|\le\alpha_k\big(\|u\|+\|(I-\lambda_kF)T^kx^k\|\big)+\lambda_k\|FT^kx^k\|\le 3\alpha_kM_1+\lambda_kM_1\to 0,
$$

as $k \to \infty$. This together with (3.10) implies

$$\lim_{k\to\infty} \|x^k - T^k x^k\| = 0.$$

Consequently, by (1.2) and the property of α_k,

$$\lim_{k\to\infty} \|x^k - T_k x^k\| = \lim_{k\to\infty} \|x^k - J^A_{r_k} x^k\| = 0. \tag{3.11}$$

Now, we prove that, for each $r \geq \varepsilon$,

$$\lim_{k\to\infty} \|x^k - J^A_r x^k\| = 0. \tag{3.12}$$

From Proposition 2.8 with $t = r_k$ and $x = x^k$, we have

$$\frac{1}{r}\|J^A_{r_k} x^k - J^A_r J^A_{r_k} x^k\| \leq \frac{1}{r_k}\|x^k - J^A_{r_k} x^k\| \leq \frac{1}{\varepsilon}\|x^k - J^A_{r_k} x^k\|,$$

which together with (3.11) implies (3.12). It means that $\{x^k\}$, defined by (1.2), statisfies (3.6). Now, we estimate the value $\|x^{k+1} - p_*\|^2$ as follows

$$\begin{aligned}
\|x^{k+1} - p_*\|^2 &= \|\alpha_k(u - p_*) + (1-\alpha_k)((I - \lambda_k F)T^k x^k - p_*)\|^2 \\
&= \|(1-\alpha_k)(T^k x^k - p_*) + \alpha_k(u - p_*) - (1-\alpha_k)\lambda_k F T^k x^k\|^2 \\
&\leq (1-\alpha_k)^2\|T^k x^k - p_*\|^2 + 2\alpha_k\langle u - p_*, j(x^{k+1} - p_*)\rangle \\
&\quad - 2(1-\alpha_k)\lambda_k\langle F T^k x^k, j(x^{k+1} - p_*)\rangle \\
&\leq (1-\alpha_k)^2\|x^k - p_*\|^2 + 2\alpha_k\langle u - p_*, j(x^{k+1} - p_*)\rangle \\
&\quad + 2\lambda_k\|F T^k x^k\|\|x^{k+1} - p_*\| \\
&\leq (1-\alpha_k)\|x^k - p_*\|^2 + \alpha_k 2\big[\langle u - p_*, j(x^{k+1} - p_*)\rangle + (\lambda_k/\alpha_k)\tilde{M}_1\big] \\
&= (1-b_k)\|x^k - p_*\|^2 + b_k c_k,
\end{aligned} \tag{3.13}$$

where $\tilde{M}_1 = M_1(M_1 + \|p_*\|)$, $b_k = \alpha_k$ and

$$c_k = 2\big[\langle u - p_*, j(x^{k+1} - p^*)\rangle + (\lambda_k/\alpha_k)\tilde{M}_1\big].$$

From Lemmas 3.2 and the conditions on $\alpha_k, \lambda_k/\alpha_k$, it follows that $\limsup_{k\to\infty} c_k \leq 0$. Finally, by Lemma 2.3, $\lim_{k\to\infty} \|x^{k+1} - p_*\|^2 = 0$. This completes the proof.

Remark 3.4. 1. The above results have value, if E is a reflexive and strcitly convex Banach space with a uniformly Gâteaux differentiable norm. Indeed, we show that C^* contains a fixed point of J^A_ε. Since E is a strictly convex and reflexive Banach space, any closed and convex subset in E is a Chebyshev set (see [13]). Then, for a point $y \in Fix(J^A_\varepsilon)$, there exists a unique $\tilde{y} \in C^*$ such that

$$\|y-\tilde{y}\| = \inf_{x\in C^*} \|y-x\|.$$

From $y = J^A_\varepsilon y$ and $J^A_\varepsilon \tilde{y} \in C^*$, we obtain that

$$\|y - J^A_\varepsilon \tilde{y}\| = \|J^A_\varepsilon y - J^A_\varepsilon \tilde{y}\| \le \|y-\tilde{y}\|.$$

Consequently, $J^A_\varepsilon \tilde{y} = \tilde{y}$.

2. Taking $\lambda_k = 0$ for all $k \ge 0$, we obtain the following result.

Theorem 3.5. *Let F, A and E be as in Proposition 3.1. Assume that, for the parameters α_k, β_k and r_k, there hold the following:*
(i) $\alpha_k \in (0,1), \lim_{k\to\infty} \alpha_k = 0$, and $\sum_{k=1}^\infty \alpha_k = \infty$;
(ii) $\beta_k \in (a,b) \subset (0,1)$ for all k; and
(iii) $r_k \ge \varepsilon > 0$ for all $k \ge 1$ with $\lim_{k\to\infty}(r_k/r_{k+1}) = 1$ or $\lim_{k\to\infty} |r_{k+1} - r_k| = 0$.
Then, the sequence $\{x^k\}$, defined by (1.3), converges strongly to the element p_* in (1.1).

So, the requirement of weak continuity of J for E in algorithm (1.3) is removed.

3. Let $q > 1$ and $M > 0$ be two fixed real numbers. It is well-known that for a uniformly convex Banach space E, there exists a continuous strictly increasing convex function $g : [0,+\infty) \to [0,+\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda \|x\|^q + (1-\lambda)\|y\|^q - \omega_q(\lambda) g(\|x-y\|), \tag{3.15}$$

for all $x, y \in B_M(0) = \{x \in E : \|x\| \le M\}$ and $\lambda \in [0,1]$, where $\omega(\lambda) = \lambda^q(1-\lambda) + \lambda^(1-\lambda)^q$. For more details, see [14]. By using the proof technique of Zhang and Song in [15] with $\lambda = 1/2, q = 2$ and Lemma 2.4, we obtain

$$J^A_{r_k} x^k = J^A_{\frac{r_k}{2}}\left(\frac{1}{2}x^k + \frac{1}{2}J^A_{r_k}x^k\right),$$

and hence,

$$\begin{aligned}\|J_{r_k}^A x^k - p\|^2 &\le \left\|\frac{1}{2}(x^k - p) + \frac{1}{2}(\|J_{r_k}^A x^k - p)\right\|^2 \\ &\le \frac{1}{2}\|x^k - p\|^2 + \frac{1}{2}\|J_{r_k}^A x^k - p\|^2 - \frac{1}{4}g(\|x^k - J_{r_k}^A x^k\|) \\ &\le \|x^k - p\|^2 - \frac{1}{4}g(\|x^k - J_{r_k}^A x^k\|).\end{aligned}$$

Next, we estimate the value $\|x^{k+1} - p\|^2$ as follows

$$\begin{aligned}\|x^{k+1} - p\|^2 &= \|\alpha_k u + (1-\alpha_k)(I - \lambda_k F)T^k x^k - p\|^2 \\ &= \|(I - \lambda_k F)T^k x^k - p + \alpha_k u - \alpha_k (I - \lambda_k F)T^k x^k\|^2 \\ &= \|T^k x^k - T^k p - \lambda_k F T^k x^k + \alpha_k u - \alpha_k (I - \lambda_k F)T^k x^k\|^2 \\ &\le \|T^k x^k - T^k p\|^2 - 2\lambda_k \langle F T^k x^k, j(x^{k+1} - p)\rangle \\ &\quad + 2\alpha_k \langle u - (I - \lambda_k F)T^k x^k, j(x^{k+1} - p)\rangle \\ &\le \beta_k \|x^k - p\|^2 + (1-\beta_k)\|J_{r_k}^A x^k - p\|^2 + 2\lambda_k M_1 (M_1 + \|p\|) \\ &\quad + 2\alpha_k (\|u\| + 2M_1)(M_1 + \|p\|) \\ &\le \|x^k - p\|^2 - \frac{1-\beta_k}{4} g(\|x^k - J_{r_k}^A x^k\|) + \tilde{M}(\lambda_k + \alpha_k),\end{aligned}$$

where $\tilde{M} = 2(\|u\| + 2M_1)(M_1 + \|p\|)$. Therefore, one has

$$\frac{1-b}{4} g(\|x^k - J_{r_k}^A x^k\|) - \tilde{M}(\lambda_k + \alpha_k) \le \|x^k - p\|^2 - \|x^{k+1} - p\|^2.$$

We need only to discuss two cases. When $\frac{1-b}{4} g(\|x^k - J_{r_k}^A x^k\|) \le \tilde{M}(\lambda_k + \alpha_k)$, by the condition on α_k and λ_k, we have

$$\lim_{k\to\infty} g(\|x^k - J_{r_k}^A x^k\|) = 0.$$

When $\frac{1-b}{4} g(\|x^k - J_{r_k}^A x^k\|) > \tilde{M}(\lambda_k + \alpha_k)$, we obtain

$$\sum_{k=0}^{N}\left[\frac{1-b}{4} g(\|x^k - J_{r_k}^A x^k\|) - \tilde{M}(\lambda_k + \alpha_k)\right] \le \|x^0 - p\|^2 - \|x^{N+1} - p\|^2 \le \|x^0 - p\|^2.$$

Thus, one has

$$\sum_{k=0}^{\infty}\left[\frac{1-b}{4} g(\|x^k - J_{r_k}^A x^k\|) - \tilde{M}(\lambda_k + \alpha_k)\right] < +\infty.$$

Consequently, one has

$$\lim_{k\to\infty} \frac{1-b}{4} g(\|x^k - J^A_{r_k} x^k\|) - \tilde{M}(\lambda_k + \alpha_k) = 0,$$

that together with condition α_k and λ_k implies

$$\lim_{k\to\infty} g(\|x^k - J^A_{r_k} x^k\|) = 0.$$

From the property of the function g, it follows (3.11). So, we obtain the following result.

Theorem 3.6. *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma \geq 1$ and let A be an m-accretive mapping in a uniformly convex Banach space E, having a uniformly Gâteaux differentiable norm. Assume that $\alpha_k, \lambda_k, \beta_k$ and r_k satisfy conditions (i), (ii) and (iii). Then, the sequence $\{x^k\}$, defined by* (1.2), *converges strongly to the element p_*, solving* (1.1).

4. Algorithm (1.3) can be re-written in the form

$$x^{k+1} = \alpha_k u + \beta'_k x^k + (1 - \alpha_k - \beta'_k) J^A_{r_k} x^k, \tag{3.14}$$

with $\beta'_k = (1-\alpha_k)\beta_k$, and hence, $0 < \liminf_{k\to\infty} \beta'_k \leq \limsup_{k\to\infty} \beta'_k < 1$. The strong convergence of (3.14) has been considered in [16]. It means that the result in [16] is a particular case of our main result, Theorem 3.3 with $\lambda_k = 0$ for all $k \geq 1$.

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