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PHASE PLANE ANALYSIS OF THE SUSCEPTIBLE-INFECTED-REMOVED-SUSCEPTIBLE (SIRS) EPIDEMIC MODELS WITH NONLINEAR INCIDENCE RATES

THI DOAN DOAN[1], CHANGRONG ZHU[2,*], KUNQUAN LAN[1]

[1]Department of Mathematics, Ryerson University, Toronto, Ontario, Canada M5B 2K3

[2]School of Mathematics and Statistics, Chongqing University Chongqing, P. R. China 401331

Abstract. We study the phase portraits of a susceptible-infective-removed-susceptible epidemic model with a nonlinear incidence rate, where we consider the power of the population of the susceptible involved in the nonlinear incidence rate is 2. Previous results mainly considered the case when the power is 1. We reduce the model system of three first order equations into a system of two equations involving the fractions of infective and removed populations. We find out the ranges of the four parameters involved in the IR system under which the equilibria are positive. By carrying out the qualitative analysis, we show that the disease free equilibrium can be a saddle-node, saddle point or stable node, the endemic equilibrium with a smaller number of infected individuals may be a stable focus or node, and the endemic equilibrium with large infectives may be a saddle. This is different from other models with the power 1, where the endemic equilibrium with a smaller number of infected individuals may be a saddle while the endemic equilibrium with a larger number of infected individuals may be a stable, unstable node or focus.

Keywords. Epidemic model; Nonlinear incidence rate; Positive equilibria; Hopf bifurcation; Phase portraits.

1. Introduction

In recent decades, many mathematical models have been established to understand the mechanism of epidemic diseases, explain the phenomena of some realistic epidemic diseases and predict the transmission of some communicable diseases which can provide references to control

*Corresponding author.

E-mail addresses: thidoan.doan@hotmail.ca (T.D. Doan), crzmath@cqu.edu.cn (C. Zhu), klan@ryerson.ca (K.Q. Lan)

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the outbreaks. The prominences among these models are the following susceptible-infective-removed-susceptible (SIRS) models with incidence rates proposed by Liu, Levin and Iwasa [12]

$$\begin{cases} \dot{S} = b - dS - G(I,S) + \nu R, \\ \dot{I} = G(I,S) - (d+r)I, \\ \dot{R} = rI - (d+\nu)R, \end{cases} \tag{1.1}$$

where $S(t), I(t), R(t) \geq 0$ denote the fractions of the population of the susceptible, infective and removed, respectively at time t; the parameters b and d are the birth and death rates of the population, r is the recovery rate of the infective individuals, ν is the rate of removed individuals who lose immunity and return to susceptible class, and $G(I,S)$ is the incidence rate (see section 2 in [12]).

A special case of the incidence rate $G(I,S)$ is of the form $S^q H(I)$, where

$$H(I) = \frac{kI^{p_1}}{1 + aI^{p_2}}. \tag{1.2}$$

When $q = 1$, there have been widely studied on the dynamics of the model (1.1) with $G(I,S) = SH(I)$. For example, the authors in [12] consider the existence of equilibria and local stability when $p_2 = p_1 - 1 > 0$ (see section 5 in [12]). Xiao and Ruan [20] study the global dynamical behaviors of the model with $p_1 = 1$ and $p_2 = 2$. Ruan and Wang [16] study the case $p_1 = p_2 = 2$ and obtain some sufficient conditions under which the system has up to two limit cycles. Bifurcations of the system with $p_1 > 0$ and $p_2 > 0$ are discussed in [7]. We refer to [1, 3, 8, 11, 12] for the study on the model (1.1) with $G(I,S) = kS^q I^{p_1}$ and $q > 0$.

There are little study on the model (1.1) with $G(I,S) = S^q H(I)$, $q \neq 1$ and $a > 0$ although there are a lot of study on the case when $a = 0$ and $q \neq 1$ (see [2, 4, 12, 13, 17, 18, 19]. Many epidemic models with the incidence rates have rich dynamical behaviors such as Hopf bifurcation, saddle-node bifurcation, Bogdanov-Takens bifurcation, existence or coexistence of periodic and homoclinic sulotions (see [6, 11, 12, 16] and therein).

In this paper, we intend to study local stability of the system (1.1) with $G(I,S) = S^2 H(I)$, $a > 0$, and $p_1 = p_2 = 1$. Under the standard assumption that the total population is a constant at any time, and introducing suitable change of variables, we reduce the system which involves 6 parameters into an IR system of two differential equations with 4 parameters. We show that the IR system has up to three equilibria, one is the disease-free equilibrium and the other two are endemic equilibria. By using the qualitative theory of planar systems and normal form theory, we show that the disease-free equilibrium can be a saddle, a stable node or a saddle-node under suitable ranges of the 4 parameters, the endemic equilibrium with a smaller number of infected individuals may be a stable focus or node, and the endemic equilibrium with large infectives may be a saddle. This is different from other models with $q=1$, where the endemic equilibrium with a smaller number of infected individuals may be a saddle while the endemic equilibrium with a larger number of infected individuals may be a stable, unstable node or focus, see [1, 16] and references therein.

2. Local stability of the SIRS model

In this section, we study local stability of the SIRS model (1.1) with $G(I,S)=S^2H(I)$, $a>0$, and $p_1=p_2=1$, namely,

$$\begin{cases} \dot{S} = b - dS - \dfrac{kS^2I}{1+aI} + \nu R, \\ \dot{I} = \dfrac{kS^2I}{1+aI} - (d+r)I, \\ \dot{R} = rI - (d+\nu)R, \end{cases} \tag{2.1}$$

where kS^2I describes the infection force of the disease, $1/(1+aI)$ measures the inhibition effect from the behavioral change of the susceptible individuals when the number of infectious individual increases. There are little study on (2.1) although there have been a lot of study on the case when $a=0$ (see [4, 12, 13, 17, 18, 19].

Summing up all the equations of (2.1), we obtain

$$N'(t) = b - dN(t),$$

where $N(t)=S(t)+I(t)+R(t)$ for $t\geq 0$. Solving the above first order linear equation, we obtain

$$S(t)+I(t)+R(t) = b/d + ce^{-dt},$$

where $c=N(0)\in\mathbb{R}$ is a constant. Taking limit as $t\to\infty$ implies

$$\lim_{t\to\infty} S(t)+I(t)+R(t) = \frac{b}{d} := N_\infty.$$

Following [12, 13], we always assume that the total population is in the equilibrium population N_∞ and investigate the behavior of the system (2.1) on the plane

$$S(t)+I(t)+R(t)=N_\infty \quad \text{for } t\geq 0. \tag{2.2}$$

Under the assumption (2.2), the last two equations of (2.1) can be changed into the following reduced IR system

$$\begin{cases} \dot{I}=\dfrac{kI}{1+aI}(N_\infty-I-R)^2-(d+r)I, \\ \dot{R}=rI-(d+\nu)R. \end{cases} \tag{2.3}$$

It is easy to verify that with $N_\infty=\frac{b}{d}$, (I,R) is a solution of (2.3) if and only if (I,R,S) is a solution of (2.1), where S satisfies (2.2). Let

$$t^*=(d+r)t, \quad u(t^*)=I(t)/N_\infty \quad \text{and } v(t^*)=R(t)/N_\infty,$$

Then the system (2.3) is changed into the following equivalent system:

$$\begin{cases} \dot{u}=\dfrac{\beta u}{1+\alpha u}(1-u-v)^2-u:=f(u,v), \\ \dot{v}=\gamma u-\delta v:=g(u,v), \end{cases} \tag{2.4}$$

where $\alpha=aN_\infty$, $\beta=\frac{kN_\infty^2}{d+r}$, $\gamma=\frac{r}{d+r}$ and $\delta=\frac{d+\nu}{d+r}$. All of these parameters are greater than 0. Indeed, we have

$$\begin{aligned} \frac{du(t^*)}{dt^*} &= \frac{1}{N_\infty(d+r)}\dot{I}(t) \\ &= \frac{1}{N_\infty(d+r)}\Big[\frac{kN_\infty u(t^*)}{1+aN_\infty u(t^*)}\big(N_\infty-N_\infty u(t^*)-N_\infty v(t^*)\big)^2 \\ &- (d+r)N_\infty u(t^*)\Big] \\ &= \frac{\beta u(t^*)}{1+\alpha u(t^*)}\big(1-u(t^*)-v(t^*)\big)^2-u(t^*) \end{aligned}$$

and

$$\begin{aligned} \frac{dv(t^*)}{dt^*} &= \frac{1}{N_\infty(d+r)}\dot{R}(t)=\frac{1}{N_\infty(d+r)}\Big[rN_\infty u(t^*)-(d+\nu)N_\infty v(t^*)\Big] \\ &= \gamma u(t^*)-\delta v(t^*). \end{aligned}$$

Recall that $(u,v)\in\mathbb{R}^2$ is an equilibrium of (2.4) if $f(u,v)=0$ and $g(u,v)=0$. An equilibrium (u,v) of (2.4) is said to be an interior (endemic) equilibrium if $u>0$ and $v>0$. It is clear that (u,v) is an equilibrium of (2.4) if and only if

$$\begin{cases} \dfrac{\beta u}{1+\alpha u}(1-u-v)^2-u=0, \\ \gamma u-\delta v=0. \end{cases} \tag{2.5}$$

Let

$$u_1=\frac{\delta[\alpha\delta+2\beta(\delta+\gamma)]-\delta\sqrt{\Delta}}{2\beta(\delta+\gamma)^2} \quad \text{and } u_2=\frac{\delta[\alpha\delta+2\beta(\delta+\gamma)]+\delta\sqrt{\Delta}}{2\beta(\delta+\gamma)^2}, \tag{2.6}$$

where

$$\Delta=\alpha^2\delta^2+4\alpha\beta\delta(\delta+\gamma)+4\beta(\delta+\gamma)^2>0. \tag{2.7}$$

Let

$$v_1=\frac{\gamma}{\delta}u_1 \quad \text{and } v_2=\frac{\gamma}{\delta}u_2. \tag{2.8}$$

Lemma 2.1. *Let $\alpha,\gamma,\delta>0$. Then the following assertions hold.*

(1) *If $\beta>0$, then $(0,0)$ is an equilibrium of (2.4) and (u_2,u_2) is an interior equilibrium of (2.4).*

(2) *If $\beta>1$, then (u_1,v_1) is an interior equilibrium of (2.4).*

Proof. By (2.5), we see that $(0,0)$ is an equilibrium of (2.4), and (u,v) is an equilibrium point of (2.4) with $u\neq 0$ if and only if (u,v) with $u\neq 0$ satisfies the following system:

$$\begin{cases} \beta(1-u-v)^2=1+\alpha u, \\ v=(\gamma/\delta)u. \end{cases} \tag{2.9}$$

Substituting the second equation of (2.9) into the first one and simplifying the resulting equation, we get

$$u^2-\frac{\delta[\alpha\delta+2\beta(\delta+\gamma)]}{\beta(\delta+\gamma)^2}u-\frac{\delta^2(1-\beta)}{\beta(\delta+\gamma)^2}=0.$$

The above equation has two solutions u_1 and u_2 given in (2.6) and (u_1,v_1) and (u_2,v_2) are solutions of (2.9). It is clear that $u_2>0$ for $\alpha,\beta,\gamma,\delta>0$ and (u_2,v_2) is an interior equilibrium of (2.4) and the result (1) holds. It is easy to verify that $u_1>0$ if and only if $\alpha\delta+2\beta(\delta+\gamma)>\sqrt{\Delta}$ if and only if

$$[\alpha\delta+2\beta(\delta+\gamma)]^2>\alpha^2\delta^2+4\alpha\beta\delta(\delta+\gamma)+4\beta(\delta+\gamma)^2$$

if and only if $\beta > 1$. Hence, the result (2) holds.

By Lemma 2.1, we obtain the following result which gives the number of positive equilibria of (2.4).

Theorem 2.1. *Assume that $\alpha, \gamma, \delta > 0$. Then the following assertions hold.*

(i) If $0 < \beta \leq 1$, then (2.4) *has the disease free equilibrium* $(0,0)$ *and the endemic equilibrium* (u_2, u_2).

(ii) If $\beta > 1$, then (2.4) *has the disease free equilibrium* $(0,0)$ *and two the endemic equilibria* (u_1, u_1) *and* (u_2, u_2).

To study the local stability of (2.4) near the equilibria given in Theorem 2.1, we need some results from the qualitative theory of the planar systems of the form

$$\begin{cases} \dfrac{du}{dt} = f_1(u,v), \\ \dfrac{dv}{dt} = g_1(u,v), \end{cases} \tag{2.10}$$

where $f_1, g_1 : (U,V) \to \mathbb{R}$ are functions having continuous first partial derivatives and (U,V) is an open subset in $\mathbb{R}^2$, see [9, 10, 14, 21]. We denote by $A(u,v)$ the Jacobian matrix of f_1 and g_1 at (u,v), that is,

$$A(u,v) = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \end{pmatrix} \tag{2.11}$$

and by $|A(u,v)|$ and $T(A(u,v))$ the determinant and trace of $A(u,v)$, respectively.

The following well-known results can be found in [15] and have been used in [5, 9, 10, 21].

Lemma 2.2. *If $(\bar{u}, \bar{v})$ is an equilibrium of* (2.4)*, then the following assertions hold.*
(i) If $|A(\bar{u}, \bar{v})| < 0$, then $(\bar{u}, \bar{v})$ is a saddle of (2.4).
(ii) If $|A(\bar{u}, \bar{v})| > 0$ and $(T(A(\bar{u}, \bar{v})))^2 - 4|A(\bar{u}, \bar{v})| \geq 0$, then $(\bar{u}, \bar{v})$ is a node of (2.4)*; it is stable if $T(A(\bar{u}, \bar{v})) < 0$ and unstable if $T(A(\bar{u}, \bar{v})) > 0$.*
(iii) If $|A(\bar{u}, \bar{v})| > 0$, $(T(A(\bar{u}, \bar{v})))^2 - 4|A(\bar{u}, \bar{v})| < 0$ and $T(A(\bar{u}, \bar{v})) \neq 0$, then $(\bar{u}, \bar{v})$ is a focus of (2.4)*; it is stable if $T(A(\bar{u}, \bar{v})) < 0$ and unstable if $T(A(\bar{u}, \bar{v})) > 0$.*

By Lemma 2.2 (ii) and (ii), we see that if $|A(\bar{u}, \bar{v})| > 0$, then $(\bar{u}, \bar{v})$ is stable if $T(A(\bar{u}, \bar{v})) < 0$ while it is unstable if $T(A(\bar{u}, \bar{v})) > 0$. In applications, the difficulty often encountered is to determine the signs of the term $(T(A(\bar{u}, \bar{v})))^2 - 4|A(\bar{u}, \bar{v})|$ in order to justify the equilibrium $(\bar{u}, \bar{v})$ is a node or it is a focus.

Lemma 2.3. [9] *Let $(\bar{u},\bar{v})$ be an equilibrium of* (2.4). *Assume that $|A(\bar{u},\bar{v})| = 0$, $T(A(\bar{u},\bar{v})) \neq 0$ and* (2.4) *is equivalent to the following system*

$$\begin{cases} \dot{u} = p(u,v), \\ \dot{v} = \rho v + q(u,v) \end{cases} \tag{2.12}$$

with an isolated equilibrium point $(0,0)$, where $\rho \neq 0$,

$$p(u,v) = \sum_{i+j=2,i,j\geq 0}^{\infty} a_{ij}u^i v^j \quad \text{and } q(u,v) = \textstyle\sum_{i+j=2,i,j\geq 0}^{\infty} b_{ij}u^i v^j$$

are convergent power series. If $a_{20} \neq 0$, then $(\bar{u},\bar{v})$ is a saddle-node of (2.4).

Let $(\bar{u},\bar{v})$ be an equilibrium of (2.4) and let $h(\bar{u},\bar{v}) = 1-\bar{u}-\bar{v}$. By (2.11), the Jacobian matrix of f and g at $(\bar{u},\bar{v})$ is

$$A(\bar{u},\bar{v}) = \begin{pmatrix} -1 + \frac{\beta h(\bar{u},\bar{v})^2}{1+\alpha u}\big(1 - \frac{\alpha\bar{u}}{1+\alpha\bar{u}}\big) - \frac{2\beta u h(\bar{u},\bar{v})}{1+\alpha\bar{u}} & -\frac{2\beta\bar{u}h(\bar{u},\bar{v})}{1+\alpha\bar{u}} \\ \gamma & -\delta \end{pmatrix}. \tag{2.13}$$

We first study the phase portraits of the disease-free equilibrium $(0,0)$.

Theorem 2.2. *Assume that $\alpha,\gamma,\delta > 0$. Then the following assertions hold.*

(1) *If $\beta > 1$, then $(0,0)$ is a saddle of* (2.4).

(2) *If $0 < \beta < 1$, then $(0,0)$ is a stable node of* (2.4).

(3) *If $\beta = 1$, then $(0,0)$ is a saddle-node of* (2.4).

Proof. By (2.13) with $(\bar{u},\bar{v}) = (0,0)$, we have

$$A(0,0) = \begin{pmatrix} \beta - 1 & 0 \\ \gamma & -\delta \end{pmatrix}.$$

Then,

$$|A(0,0)| = -\delta(\beta-1) \quad \text{ and } T(A(0,0)) = (\beta-1)-\delta.$$

(1) If $\beta > 1$, then $|A(0,0)| < 0$. The result (1) follows from Lemma 2.2 (i).

(2) If $0 < \beta < 1$, then $|A(0,0)| > 0$ and $T(A(0,0)) < 0$, and

$$T(A(0,0))^2 - 4|A(0,0)| = [(\beta-1)-\delta]^2 + 4\delta(\beta-1) = [\delta + (\beta-1)]^2 \geq 0.$$

The result (2) follows from Lemma 2.2 (ii).

(3) If $\beta = 1$, then $|A(0,0)| = 0$ and $T(A(0,0)) < 0$. Using Taylor's expansion, we have

$$\frac{1}{1+\alpha u} = 1 - \alpha u + R_2(u),$$

where $R_2(u) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}(\alpha u)^n$. Then

$$\begin{aligned}
f(u,v) &= u(1-u-v)^2[1-\alpha u + R_2(u)] - u \\
&= u[(1-2u-2v)+(u+v)^2][1-\alpha u + R_2(u)] - u \\
&= [u(1-2u-2v)+u(u+v)^2][1-\alpha u + R_2(u)] - u \\
&= u(1-2u-2v)[1-\alpha u + R_2(u)] + u(u+v)^2[1-\alpha u + R_2(u)] - u \\
&= -(2+\alpha)u^2 - 2uv + O_3(u,v),
\end{aligned}$$

where $O_3(u,v) = uR_2(u)(1-2u-2v) + 2\alpha u^2(u+v) + u(u+v)^2[1-\alpha u + R_2(u)]$. Hence, (2.4) becomes

$$\begin{cases} \dot{u} = -(2+\alpha)u^2 - 2uv + O_3(u,v), \\ \dot{v} = \gamma u - \delta v. \end{cases} \tag{2.14}$$

Let $u_1 = u$ and $v_1 = \gamma u - \delta v$. Then $u = u_1,\ v = \frac{1}{\delta}(\gamma u_1 - v_1)$ and

$$\begin{aligned}
\dot{u}_1 = \dot{u} &= -(2+\alpha)u^2 - 2uv + O_3(u,v) \\
&= -(2+\alpha)u_1^2 - \frac{2u_1}{\delta}(\gamma u_1 - v_1) + O_3(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)) \\
&= -(2+\alpha)u_1^2 - \frac{2}{\delta}(\gamma u_1^2 - u_1 v_1) + O_3(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)) \\
&= -(2+\alpha+\frac{2\gamma}{\delta})u_1^2 + \frac{2}{\delta}u_1 v_1 + O_3(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\dot{v}_1 = \gamma\dot{u} - \delta\dot{v} &= \gamma[-(2+\alpha)u^2 - 2uv + O_3(u,v)] - \delta[\gamma u - \delta v] \\
&= -\delta v_1 + \gamma\big[-(2+\alpha)u_1^2 - \frac{2u_1}{\delta}\big(\gamma u_1 - v_1\big) + O_3\big(u_1, \frac{1}{\delta}\big(\gamma u_1 - v_1\big)\big)\big] \\
&= -\delta v_1 + \gamma\big[-(2+\alpha)u_1^2 - \frac{2}{\delta}\big(\gamma u_1^2 - u_1 v_1\big) + O_3\big(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)\big)\big] \\
&= -\delta v_1 - \gamma\big(2+\alpha+\frac{2\gamma}{\delta}\big)u_1^2 + \frac{2\gamma}{\delta}u_1 v_1 + \gamma O_3\big(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)\big).
\end{aligned}$$

Thus, (2.14) can be transformed into the following system

$$\begin{cases} \dot{u}_1 = -(2+\alpha+\frac{2\gamma}{\delta})u_1^2 + \frac{2}{\delta}u_1v_1 + O_3\big(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)\big), \\ \dot{v}_1 = -\delta v_1 - \gamma(2+\alpha+\frac{2\gamma}{\delta})u_1^2 + \frac{2\gamma}{\delta}u_1v_1 + \gamma O_3\big(u_1, \frac{1}{\delta}(\gamma u_1 - v_1)\big). \end{cases} \tag{2.15}$$

Note that $a_{20} = -(2+\alpha+\frac{2\gamma}{\delta}) \neq 0$. The result (3) follows from Lemma 2.3.

Remark 2.1. When $0 < \beta < 1$, the disease-free equilibrium is stable. If we can decrease β such that $\beta < 1$, then we can control a disease spread and eventually eradicate it. Note that $\beta = kN_\infty^2/(d+r)$. To decease the value β, we can choose the parameter $k < (d+r)N_\infty^2$ or reduce the total population N_∞ who contact the infected individual and have a risk of disease. Hence, when one disease outbreaks, we can take suitable measurements such as isolation to shrink N_∞ such that $\beta < 1$ and eventually eradicate the disease in a short period.

Now, we consider the local stability of the interior equilibria of (2.4). By (2.13) and the first equation of (2.9), we have

$$A(\bar{u},\bar{v}) = \begin{pmatrix} -\frac{\alpha\bar{u}}{1+\alpha\bar{u}} - \frac{2\beta\bar{u}(1-\bar{u}-\bar{v})}{1+\alpha\bar{u}} & -\frac{2\beta\bar{u}(1-\bar{u}-\bar{v})}{1+\alpha\bar{u}} \\ \gamma & -\delta \end{pmatrix} \tag{2.16}$$

and

$$\begin{aligned} |A(\bar{u},\bar{v})| &= \frac{\alpha\delta\bar{u}}{1+\alpha\bar{u}} + \frac{2\beta\delta\bar{u}(1-\bar{u}-\bar{v})}{1+\alpha\bar{u}} + \frac{2\beta\gamma\bar{u}(1-\bar{u}-\bar{v})}{1+\alpha\bar{u}} \\ &= \frac{\bar{u}}{1+\alpha\bar{u}}[\alpha\delta + 2\beta\delta(1-\bar{u}-\bar{v}) + 2\beta\gamma(1-\bar{u}-\bar{v})] \\ &= \frac{\bar{u}}{1+\alpha\bar{u}}[\alpha\delta + 2\beta(\delta+\gamma)(1-\bar{u}-\bar{v})] \\ &= \frac{\bar{u}}{1+\alpha\bar{u}}\Big[\alpha\delta + 2\beta(\delta+\gamma)(1-\frac{\delta+\gamma}{\delta}\bar{u})\Big]. \end{aligned} \tag{2.17}$$

Theorem 2.3. *If $\alpha,\beta,\gamma,\delta > 0$, then (u_2,v_2) is a saddle of* (2.4).

Proof. By (2.6), we have

$$\begin{aligned} 1 - \frac{\delta+\gamma}{\delta}u_2 = 1 - \frac{\delta+\gamma}{\delta}\,\frac{\delta[\alpha\delta+2\beta(\delta+\gamma)]+\delta\sqrt{\Delta}}{2\beta(\delta+\gamma)^2} &= 1 - \frac{\alpha\delta+2\beta(\delta+\gamma)+\sqrt{\Delta}}{2\beta(\delta+\gamma)} \\ = -\frac{\alpha\delta+\sqrt{\Delta}}{2\beta(\delta+\gamma)}. \end{aligned}$$

This together with (2.17) with $(\bar{u},\bar{v})=(u_2,v_2)$, implies that

$$|A(u_2,v_2)|=\frac{u_2}{1+\alpha u_2}\Big[\alpha\delta-2\beta(\delta+\gamma)\frac{\alpha\delta+\sqrt{\Delta}}{2\beta(\delta+\gamma)}\Big]=-\frac{u_2\sqrt{\Delta}}{1+\alpha u_2}<0.$$

The result follows from Lemma 2.2 (i).

Now, we study the dynamics of the endemic equilibrium (u_1,v_1) of (2.4).

Theorem 2.4. *If $\alpha,\beta,\delta>0$ and $\beta>1$, then (u_1,v_1) is a stable focus or a stable node of* (2.4).

Proof. By (2.6), we have

$$1-\frac{\delta+\gamma}{\delta}u_1=1-\frac{\alpha\delta+2\beta(\delta+\gamma)-\sqrt{\Delta}}{2\beta(\delta+\gamma)}=-\frac{\alpha\delta-\sqrt{\Delta}}{2\beta(\delta+\gamma)}.$$

By (2.17) with $(\bar{u},\bar{v})=(u_2,v_2)$, we have

$$|A(u_1,v_1)|=\frac{u_1}{1+\alpha u_1}[\alpha\delta-2\beta(\delta+\gamma)\frac{\alpha\delta-\sqrt{\Delta}}{2\beta(\delta+\gamma)}]=\frac{u_1\sqrt{\Delta}}{1+\alpha u_1}>0.$$

Finally, (2.16) implies that

$$\begin{aligned}
T(A(u_1,v_1))&=-\delta-\frac{\alpha u_1}{1+\alpha u_1}-\frac{2\beta u_1(1-u_1-v_1)}{1+\alpha u_1}\\
&=-\delta-\frac{u_1}{1+\alpha u_1}[\alpha+2\beta(1-u_1-v_1)]\\
&=-\delta-\frac{u_1}{1+\alpha u_1}[\alpha+2\beta(1-\frac{\delta+\gamma}{\delta}u_1)]\\
&=-\delta-\frac{u_1}{1+\alpha u_1}[\alpha-2\beta\frac{\alpha\delta-\sqrt{\Delta}}{2\beta(\delta+\gamma)}]\\
&=-\delta-\frac{u_1}{1+\alpha u_1}[\alpha-\frac{\alpha\delta-\sqrt{\Delta}}{\delta+\gamma}]=-\delta-\frac{u_1(\alpha\gamma+\sqrt{\Delta})}{(\delta+\gamma)(1+\alpha u_1)}<0.
\end{aligned}$$

The result follows from Lemma 2.2 (ii) and (iii).

Remark 2.2. When one disease invades, we can predict its transmission and keeps the disease on a low level if we take suitable measurements. The condition $\beta>1$ corresponds to $kN_\infty^2>d+r$. Some new drugs can be found to cure the infected individuals and in return lessen the death rate, d, and shorten the recovery rate r. If these measurements are adopted such that $d+r<kN_\infty^2$, the disease is controlled. Although we can not remove the epidemic disease, the disease can be kept on a low level since the endemic equilibrium with low infective individuals is stable.

In Theorem 2.4, we see that (u_1,v_1) can be a stable focus or node of (2.4). In the following, we provide sufficient conditions for (u_1,v_1) to be a focus and for (u_1,v_1) to be a node.

Let

$$\gamma_1=\frac{-\delta[\delta-6(\sqrt{\beta}-1)]-\sqrt{\tilde{\Delta}}}{\delta-8(\sqrt{\beta}-1)},\ \gamma_2=\frac{-\delta[\delta-6(\sqrt{\beta}-1)]+\sqrt{\tilde{\Delta}}}{\delta-8(\sqrt{\beta}-1)}, \tag{2.18}$$

and

$$\tilde{\Delta}=\delta^2[\delta-6(\sqrt{\beta}-1)]^2-\delta[\delta-8(\sqrt{\beta}-1)][\delta-2(\sqrt{\beta}-1)]^2.$$

Theorem 2.5. (1) *If* $\beta>1,\ \delta>8(\sqrt{\beta}-1),\ \gamma>0$ *or* $\beta>1,\ 0<\delta<8(\sqrt{\beta}-1)$ *and* $0<\gamma<\gamma_2$, *then there exists* $\tilde{\alpha}>0$ *such that* (u_1,v_1) *is a stable node of* (2.4) *for* $\alpha\in(0,\tilde{\alpha})$.

(2) *If* $\beta>1,\ 0<\delta<8(\sqrt{\beta}-1)$ *and* $\gamma>\gamma_2$, *then there exists* $\tilde{\alpha}>0$ *such that* (u_1,v_1) *is a stable focus of* (2.4) *for* $\alpha\in(0,\tilde{\alpha})$.

Proof. By (2.7) and (2.6), we have

$$\lim_{\alpha\to 0}\Delta=4\beta^2(\delta+\gamma)^2+4\beta(1-\beta)(\delta+\gamma)^2=4\beta(\delta+\gamma)^2$$

and

$$\lim_{\alpha\to 0}u_1=\frac{2\beta\delta(\delta+\gamma)-\delta\lim_{\alpha\to 0}\sqrt{\Delta}}{2\beta(\delta+\gamma)^2}=\frac{2\beta\delta(\delta+\gamma)-2\delta\sqrt{\beta}(\delta+\gamma)}{2\beta(\delta+\gamma)^2}=\frac{\delta(\beta-\sqrt{\beta})}{\beta(\delta+\gamma)}.$$

Hence, we have

$$\lim_{\alpha\to 0}T(A(u_1,v_1))=-\delta-\frac{\delta(\beta-\sqrt{\beta})}{\beta(\delta+\gamma)}\frac{2\sqrt{\beta}(\delta+\gamma)}{\delta+\gamma}=-\delta-\frac{2\delta(\sqrt{\beta}-1)}{\delta+\gamma}$$

and

$$\lim_{\alpha\to 0}|A(u_1,v_1)|=\frac{\delta(\beta-\sqrt{\beta})}{\beta(\delta+\gamma)}2\sqrt{\beta}(\delta+\gamma)=2\delta(\sqrt{\beta}-1).$$

Let

$$\tilde{A}=(T(A(u_1,v_1)))^2-4|A(u_1,v_1)|. \tag{2.19}$$

By (2.18), we have

$$\begin{aligned}
\lim_{\alpha\to 0}\tilde{A} &= (\lim_{\alpha\to 0} T(A(u_1,v_1)))^2 - 4\lim_{\alpha\to 0}|A(u_1,v_1)| \\
&= \delta^2[1+\frac{2(\sqrt{\beta}-1)}{\delta+\gamma}]^2 - 8\delta(\sqrt{\beta}-1) \\
&= \delta^2\frac{[\delta+\gamma+2(\sqrt{\beta}-1)]^2}{(\delta+\gamma)^2} - 8\delta(\sqrt{\beta}-1) \\
&= \frac{\delta}{(\delta+\gamma)^2}\{\delta[\delta+\gamma+2(\sqrt{\beta}-1)]^2 - 8(\sqrt{\beta}-1)(\delta+\gamma)^2\} \\
&= \frac{\delta}{(\delta+\gamma)^2}\{\delta\gamma^2+2\gamma\delta[\delta+2(\sqrt{\beta}-1)]+\delta[\delta+2(\sqrt{\beta}-1)]^2 \\
&\quad -8(\sqrt{\beta}-1)(\delta^2+2\delta\gamma+\gamma^2)\} \\
&= \frac{\delta}{(\delta+\gamma)^2}\{[\delta-8(\sqrt{\beta}-1)]\gamma^2+2\gamma\delta[\delta+2(\sqrt{\beta}-1)-8(\sqrt{\beta}-1)] \\
&\quad +\delta[\delta+2(\sqrt{\beta}-1)]^2-8\delta^2(\sqrt{\beta}-1)\} \\
&= \frac{\delta}{(\delta+\gamma)^2}\{[\delta-8(\sqrt{\beta}-1)]\gamma^2+2\delta[\delta-6(\sqrt{\beta}-1)]\gamma+\delta[\delta-2(\sqrt{\beta}-1)]^2\} \\
&= \frac{\delta[\delta-8(\sqrt{\beta}-1)]}{(\delta+\gamma)^2}(\gamma-\gamma_1)(\gamma-\gamma_2).
\end{aligned}$$

Note that $\gamma_1 < 0$ and $\gamma_2 < 0$ if $\delta > 8(\sqrt{\beta}-1)$, and $\gamma_1 < 0$ and $\gamma_2 > 0$ if $\delta < 8(\sqrt{\beta}-1)$. Hence, we have

$$\tilde{A} > 0 \text{ if either } \delta > 8(\sqrt{\beta}-1) \text{ or } \delta < 8(\sqrt{\beta}-1) \text{ and } 0 < \gamma < \gamma_2$$

and $\tilde{A} < 0$ if either$\delta < 8(\sqrt{\beta}-1)$ and $\gamma > \gamma_2$. By the continuity of $\tilde{A}$ in α, the results follow from Lemma 2.2 (ii) and (iii) and Theorem 2.4.

Theorem 2.5 holds for small α. In the following, we deal with the case when α is large.

Notations: Let $0 < \delta < 1$ and

$$\alpha_0 := \alpha_0(\delta) = \frac{\delta^2-7\delta+8}{(1-\delta)^2},\ \beta_0 = \frac{\alpha^2(1-\delta^2)}{4+4\alpha\delta}. \tag{2.20}$$

Lemma 2.4. (1) *If* $0 < \delta < 1/4$, *then* $8 < \alpha_0 < 101/9$.

(2) *If* $1/4 < \delta < 1/3$, *then* $101/9 < \alpha_0 < 13$.

(3) *If* $0 < \delta < 1/4$ *and* $4 < \alpha < 8$, *then* $\beta_0 > 1$.

(4) *If* $1/4 < \delta < 1/3$ *and* $13 < \alpha < \infty$, *then* $\beta_0 > 1$.

Proof. Taking the derivative of α_0 given in (2.20) relative to δ, we have

$$\frac{d\alpha_0}{d\delta} = \frac{(2\delta-7)(1-\delta)+2(\delta^2-7\delta+8)}{(1-\delta)^3} = \frac{9-5\delta}{(1-\delta)^3} > 0 \text{ for } 0 < \delta < \frac{1}{3}.$$

Thus, α_0 is increasing and (1) and (2) follow. By (2.20), we have

$$\begin{aligned}\beta_0 - 1 &= \frac{\alpha^2(1-\delta^2)}{4+4\alpha\delta} - 1 = \frac{\alpha^2-\alpha^2\delta^2-4\alpha\delta-4}{4+4\alpha\delta} \\ &= \frac{\alpha^2-(\alpha\delta+2)^2}{4+4\alpha\delta} = \frac{(\alpha-\alpha\delta-2)(\alpha+\alpha\delta+2)}{4+4\alpha\delta}.\end{aligned}$$

If $0 < \delta < 1/4$ and $4 < \alpha < 8$, then $3 < \alpha(1-\delta) < 6$. Hence, $\alpha - \alpha\delta - 2 > 0$ and $\beta_0 > 1$. Hence, the result (3) holds. If $1/4 < \delta < 1/3$ and $13 < \alpha < \infty$, then $26/3 < \alpha(1-\delta) < \infty$. Hence, $\alpha - \alpha\delta - 2 > 0$, $\beta_0 > 1$ and the result (4) follows.

Theorem 2.6. (1) *If $4 < \alpha < 8$, $0 < \delta < 1/4$, $\gamma = 1-\delta$ and $\beta = \beta_0$, then (u_1, v_1) is a stable node.*

(2) *If $13 < \alpha < \infty$, $1/4 < \delta < 1/3$, $\gamma = 1-\delta$ and $\beta = \beta_0$, then (u_1, v_1) is a stable focus.*

Proof. If $\delta + \gamma = 1$, by (2.7), we get $\Delta = \alpha^2\delta^2 + 4\alpha\beta\delta + 4\beta$. By (2.20), we have

$$\beta - \beta_0 = \frac{1}{4+4\alpha\delta}[\beta(4+4\alpha\delta) - \alpha^2(1-\delta^2)] = \frac{1}{4+4\alpha\delta}(\Delta - \alpha^2) = 0$$

and $\alpha = \sqrt{\Delta}$. By (2.6), we have

$$\begin{aligned}u_1 &= \frac{\delta(\alpha\delta+2\beta_0)-\alpha\delta}{2\beta_0} = \delta + \frac{\alpha\delta(\delta-1)}{2\beta_0} = \delta + \alpha\delta(\delta-1)\frac{2+2\alpha\delta}{\alpha^2(1-\delta^2)} \\ &= \delta - \frac{2\delta(1+\alpha\delta)}{\alpha(1+\delta)}.\end{aligned}$$

By (2.19), we show that

$$\tilde{A} = \frac{U}{4(\delta-1)^2}(\alpha_0 - \alpha), \tag{2.21}$$

where $U = \delta/[(1+\delta)(1+\alpha u_1)^2] > 0$. In fact,

$$\begin{aligned}
\tilde{A} &= \Big[\delta + \frac{u_1(\alpha\gamma+\alpha)}{1+\alpha u_1}\Big]^2 - \frac{4\alpha u_1}{1+\alpha u_1} = \Big[\frac{\delta+\alpha u_1(\delta+\gamma)+\alpha u_1}{1+\alpha u_1}\Big]^2 - \frac{4\alpha u_1}{1+\alpha u_1} \\
&= \Big[\frac{\delta+2\alpha u_1}{1+\alpha u_1}\Big]^2 - \frac{4\alpha u_1}{1+\alpha u_1} = \frac{1}{(1+\alpha u_1)^2}\big[(\delta+2\alpha u_1)^2 - 4\alpha u_1(1+\alpha u_1)\big] \\
&= \frac{1}{(1+\alpha u_1)^2}(\delta^2 + 4\alpha(\delta-1)u_1) \\
&= \frac{1}{(1+\alpha u_1)^2}\Big\{\delta^2 + 4\alpha(\delta-1)\big[\delta - \frac{2\delta(1+\alpha\delta)}{\alpha(1+\delta)}\big]\Big\} \\
&= \frac{1}{(1+\alpha u_1)^2}\Big[\delta^2 + 4\alpha\delta(\delta-1) - \frac{8\delta(\delta-1)(1+\alpha\delta)}{(1+\delta)}\Big] \\
&= U[\delta(1+\delta) + 4\alpha(\delta^2-1) - 8(\delta-1)(1+\alpha\delta)] \\
&= U[\delta(1+\delta) - 8(\delta-1) + 4\alpha(\delta-1)((\delta+1)-2\delta)] \\
&= U[\delta(1+\delta) - 8(\delta-1) - 4\alpha(\delta-1)^2] = \frac{U}{4(\delta-1)^2}(\alpha_0-\alpha).
\end{aligned}$$

By Lemma 2.4 (1), $\alpha_0 > 8$ if $0 < \delta < 1/4$. Since $4 < \alpha < 8$, we have $\alpha_0 - \alpha > 0$. The result (1) follows from (2.21). By Lemma 2.4 (2), $\alpha_0 < 13$ if $1/4 < \delta < 1/3$. Since $\alpha > 13$, $\alpha_0 - \alpha < 0$ and the result (2) follows from (2.21).

3. Conclusion

In this paper, we use the planar qualitative analysis to study the dynamical behaviors of an SIRS epidemic model with a saturated incidence rate $kS^2I/(1+\alpha I)$. Our results show that the ratio of the infection force and the summation of the death and recovery rates are important in determining the eradication or persistence of disease. Firstly, when the infection force k is small or the recovery rate r is large, that is $\beta < 1$, the disease-free equilibrium is stable and hence the disease will disappear as time evolves. So the outbreak of the epidemic disease does not happen. when the epidemic infection force is smaller than the recovery rate (or removal rate). Secondly, when the infection force rate k is larger than the summation of the death and recovery rates, then the disease will persist.

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