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CRITICAL POINT THEOREMS CONCERNING STRONGLY INDEFINITE FUNCTIONALS AND APPLICATIONS TO HAMILTONIAN SYSTEMS

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Abstract. Let X be a Finsler manifold. We prove some abstract results on the existence of critical points for strongly indefinite functionals $f \in C^1(X,\mathbb{R})$ via a new deformation theorem. Different from the works in the literature, the new deformation theorem is constructed under a new version of Cerami-type condition instead of Palais-Smale condition. As applications, we prove the existence of multiple periodic solutions for a class of Hamiltonian systems.

Keywords. Finsler manifold; Deformation lemma; Critical point theory; Multiple periodic solution; Hamiltonian system.

1. Introduction

Let X be a Finsler manifold of class C^1 and let $f : X \longrightarrow \mathbb{R}$ be a continuously differentiable function. Recall that the differential of f at x, denoted $df(x)$, is an element of the cotangent space $T_x(X)^*$ of X at x. A point $x_0 \in X$ is said to be a critical point of f if $df(x_0) = 0$. The corresponding value $c = f(x_0)$ is called a critical value of f. In applications to differential equations, critical points correspond to weak solutions of the equations. Indeed, this fact makes critical point theory an important existence tool in studying differential equations. As an example, consider the Hamiltonian system

$$\dot{x} = J\nabla H(t,x), \tag{1.1}$$

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where J is the standard symplectic $(2N\times 2N)$-matrix: $H\in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is $T-$periodic in the first variable and $\nabla H(t,x)$ denotes the gradient of H with respect to x. The critical points of the functional

$$f(x)=\int_0^T[\frac{1}{2}J\dot{x}(t).x(t)+H(t,x)]dt$$

on an appropriate space are the $T-$periodic solutions of (1.1). Here $x.y$ denotes the inner product of $x,y\in\mathbb{R}^{2N}$.

The simplest sort of critical points of $f\in C^1(X,\mathbb{R})$ is its minimum

$$c=\inf_{u\in X}f(u)$$

when it is finite. In order to obtain other critical points, the minimax method was introduced by Lusternik and Schnirelman in 1934. The value c is then defined by

$$c=\inf_{A\in\mathscr{A}}\sup_{x\in A}f(x),$$

where $\mathscr{A}$ is a suitable class of subsets of X. When f is bounded from below, the set $\mathscr{A}$ was defined in 1934 by Lusternik and Schnirelman for compact manifolds by using a topological invariant: the Lusternik-Schnirelman category. The theory was extended to Riemannian manifolds by Schwartz in 1964 and to Finsler manifolds by Palais [1] in 1966. The main ingredients are a compactness condition, called Palais-Smale condition, and the notion of pseudo-gradient flow which generalizes for Finsler manifolds the notion of the gradient flow. In 1972, M. Reeken introduced in [2] a notion of relative category applicable to functionals which are unbounded from below. In 1992, the relative category was extended in [3] to strongly indefinite functionals by A. Szulkin using a device of Benci and Rabinowitz [4]. In the same year, Fournier, Lupo, Ramos and Willem have given in [5] another extension of the relative categoty applicable to strongly indefinite functionals. They introduced a new version of Palais-Smale condition $(PS)_c^*$ with respect to a sequence of closed, connected sub-manifolds of class C^2 of the C^1 Finsler manifold X. In the literature, deformation theorems have been proved under the assumption that $f\in C^1(X,R)$ satisfies Palais-Smale condition. This condition is not in general satisfied and a weakened version introduced by Cerami [6], called Cerami's condition and denoted by (C), seems to be essential in the study of variational problems in the strong resonance case. In fact in many situations the Palais-Smale condition is not in general satisfied. In [7], Bartolo, Benci and

Fortunato have obtained a deformation theorem under the Cerami's condition. The aim of this paper is to develop a method which permits to deal with nonlinear problems having a "strong resonance" at infinity. This paper is organized as follows. The second Section is devoted to some preliminaries concerning the notions of limit relative category, cuplength and some related results. In Section 3, we will show a new deformation theorem concerning strongly indefinite functionals under a new version of Cerami's condition $(C)_c^*$ with respect to a sequence of closed, connected sub-manifolds of class C^2 of the C^1 Finsler manifold X. In Section 4, we prove some new critical point theorems concerning strongly indefinite functionals with Cerami condition $(C)_c^*$ instead of Palais-Smale condition $(PS)_c^*$. More precisely, we prove the Generalized Saddle Point Theorem, the Generalized Linking Theorem and the Mountain Circle Theorem. As an application of the abstract theorems given in Section 4, we prove in Section 5 the existence of multiple periodic solutions for a class of non-autonomous subquadratic Hamiltonian systems.

2. Preliminaries

All the concepts of this Section are taken from paper [5].

By a map between topological spaces we mean a continuous function. Let (X,A) be a topological pair, a deformation $h_t : A \longrightarrow X$ is a map $h : [0,1] \times A \longrightarrow X$ such that $h_0(x) = x$ for every $x \in A$. The set A is contractible in X if there exists a deformation $h_t : A \longrightarrow X$ such that $h_1(x) = h_1(y)$ for every $x, y \in A$. Let A, B, Y be closed subsets of the topological space X. Then, by definition, $A <_Y B$ in X if $Y \subset A \cap B$ and if there exists a deformation $h_t : A \longrightarrow X$ such that $h_1(A) \subset B$ and $h_t(Y) \subset Y$ for every $t \in [0,1]$. We now recall the concept of relative category.

Definition 2.1. Let $Y \subseteq A$ be closed subsets of a topological space X. The relative category of A in X relative to Y is the least integer n such that there exist n+1 closed subsets A_0,A_1,...,A_n satisfying

a) $A = \bigcup_{i=0}^{n} A_i$;

b) A_1,...,A_n are contractible in X;

c) $A_0 <_Y Y$ in X.

When no such integer exists, the category of A in X, relative to Y is infinite. The relative category is denoted by $cat_{X,Y}(A)$. When $Y = \emptyset$, the relative category $cat_{X,\emptyset}(A)$ is by definition equal to the Lusternik-Schnirelman category $cat_X(A)$. Considering $A_0 = Y$ in the above definition we see that $cat_{X,Y}(A) \leq cat_X(A)$ whenever $Y \subset A$.

Theorem 2.2. *Let A, B, Y be closed subsets of X with $Y \subseteq A \cap B$. The relative category satisfies the following properties:*

a) Normalization: $cat_{X,Y}(Y) = 0$;

b) Sub-additivity: $cat_{X,Y}(A \cup B) \leq cat_{X,Y}(A) + cat_X(B)$;

c) Homotopy: if $A <_Y B$, then $cat_{X,Y}(A) \leq cat_{X,Y}(B)$;

d) Monotonicity: if $A \subseteq B$, then $cat_{X,Y}(A) \leq cat_{X,Y}(B)$.

Definition 2.3. A metric space X is an absolute neighborhood extensor, shortly an ANE if, for every metric space E, every closed subset F of E and every $f \in C^1(F,X)$ there exists a continuous extension of f defined on a neighborhood of F in E.

Proposition 2.4. *Let Y be a closed subset of X, and suppose both Y, X are ANE's. Then for any closed subset A of X there exists a closed neighborhood B of A such that*

$$cat_{X,Y}(A) = cat_{X,Y}(B).$$

Relative cuplength. We use singular homology and cohomology over the real field $\mathbb{R}$ and denote it by H_* and H^* respectively.

A subset A of a topological space X is a strong deformation retract of X if and only if there exists a deformation $h_t : X \longrightarrow X$ such that $h_1(X) \subset A$ and $h_t(x) = x$ for every $x \in A$, $t \in [0,1]$.

Definition 2.5. Let Y be a closed subset of a space X. The cuplength of X relative to Y is the largest integer n such that there exist $\alpha_0 \in H^*(X,Y)$, $* \geq 0$ and $\alpha_1, ..., \alpha_n \in H^*(X)$, $* \geq 1$ with

$$\alpha_0 \cup \alpha_1 \cup ... \cup \alpha_n \neq 0.$$

We write then $cuplength(X,Y) = n$. We set $cuplength(X,Y) = -\infty$ if no such integer exists.

The relation between relative *cuplength* and relative *category* is given by the following.

Theorem 2.6. *Let Y be a closed subset of an ANE X. Then*

$$cat_{X,Y}(X) \geq 1 + cuplength(X,Y).$$

Limit relative category. *Consider a topological space X together with a sequence $(X_n)_{n\in\mathbb{N}}$ of closed subsets of X. We assume that there exists for every n, a retraction $r_n : X \longrightarrow X_n$. If A is any subspace of X we denote by A_n the set $A \cap X_n$.*

Definition 2.7. Let $Y \subseteq A$ be a closed subset of X. The limit relative category of A in X relative to Y, with respect to $(X_n)_{n\in\mathbb{N}}$ is defined by

$$cat^{\infty}_{X,Y}(A) = \limsup_{n\longrightarrow\infty} cat_{X_n,Y_n}(A_n).$$

If A,B,Y are closed subsets of X, then by definition, $A <^{\infty}_{Y} B$, with respect to $(X_n)_{n\in\mathbb{N}}$, if and only if $Y \subseteq A\cap B$ and $A_n <_{Y_n} B_n$ for every n large.

Proposition 2.8. *Let A,B,Y be closed subsets of X with $Y \subseteq A\cap B$. The limit relative category satisfies the following properties*

a) Normalization: $cat^{\infty}_{X,Y}(Y) = 0$;

b) Sub-additivity: $cat^{\infty}_{X,Y}(A\cup B) \leq cat^{\infty}_{X,Y}(A) + cat_X(B)$;

c) Homotopy: if $A <^{\infty}_{Y} B$, then $cat^{\infty}_{X,Y}(A) \leq cat^{\infty}_{X,Y}(B)$;

d) Monotonicity: if $A \subseteq B$, then $cat^{\infty}_{X,Y}(A) \leq cat^{\infty}_{X,Y}(B)$.

Example 2.9. *Let E be a normed space such that $E = W \oplus Z$ (topological direct sum) and for some $0 < r_1 < r_2$, define $A = \{w \in W : r_1 \leq \|w\| \leq r_2\}$, $\partial A = \{w \in W : \|w\| = r_1$ or $\|w\| = r_2\}$. Then, with respect to any sequence of vector subspaces $E_n = W_n \oplus Z_n$, $W_n \subseteq W$, $Z_n \subseteq Z$, $1 \leq dim(W_n) < \infty$ we have $cat^{\infty}_{X,\partial B\times V}A = 2$.*

Theorem 2.10. *Let E be a normed space such that $E = W \oplus Z$ (topological direct sum) and consider a sequence of vector subspaces $E_n = W_n \oplus Z_n$, $W_n \subseteq W$, $Z_n \subseteq Z$, $1 \leq dim(W_n) < \infty$. Let E be an ANE and $X = E\times V$. Define $B = \{w \in W : \|w\| \leq R\}$, $\partial B = \{w \in W : \|w\| = R\}$. Then, with respect to $X_n = E_n \times V$, $cat^{\infty}_{X,\partial B\times V}(B\times V) \geq 1 + cuplength(V)$.*

3. A deformation theorem

In this section, we prove a ”deformation theorem” which plays a fundamental role in applying the developed Lusternik-Schnirelmann type arguments to the search for critical points of functionals on Finsler manifolds. Let X be a connected Finsler manifold of class C^1. We denote the tangent bundle of X by $T(X)$ and the tangent space of X at a point $x \in X$ by $T_x(X)$. Recall that the cotangent bundle $T(X)^*$ has a dual Finsler structure given by

$$\|w\| = \sup\{< w,v >: v \in T_x(X),\ \|v\| = 1\},$$

where $w \in T_x(X)^*$ and $< .,. >$ is the duality pairing between $T_x(X)^*$ and $T_x(X)$. Let $f \in C^1(X,R)$. We shall repeatedly use the following notations:

$$K = \{x \in X : df(x) = 0\},\ K_c = K \cap f^{-1}(\{c\}),$$

$$f^c = \{x \in X : f(x) \leq c\},\ S_\delta = \{x \in X : d(x,S) \leq \delta\},$$

where $\delta > 0$ and $S \subseteq X$ is any subset (recall that X is a metric space).

Let X be a C^2 Finsler manifold. The following lemma will be needed.

Lemma 3.1. [13] *If $f \in C^1(X,R)$, there exists a locally Lipschitz continuous map $v : \tilde{X} = X \backslash K \longrightarrow TX$ satisfying the conditions:*

$$\begin{cases} a)\ \|v(x)\| \leq \frac{2}{\|df(x)\|},\ \forall x \in \tilde{X}, \\ b)\ < df(x), v(x) > \geq 1,\ \forall x \in \tilde{X}. \end{cases} \tag{3.1}$$

Proof. Consider the map

$$l : \tilde{X} \longrightarrow T(X)^*,\ x \longrightarrow l(x) = \frac{df(x)}{\|df(x)\|^2}.$$

Given $x_0 \in \tilde{X}$, we can find $w \in T_{x_0}(X)$ with $\|w\| = 1$ such us

$$< l(x_0), w > > \frac{2}{3}\|l(x_o)\|.$$

Set

$$z = \frac{3}{2}\|l(x_0)\|\, w, \tag{3.2}$$

we have

$$\begin{cases} \|z\| < 2\,\|l(x_0)\|\,, \\ < l(x_0), z > \; > \|l(x_0)\|^2 . \end{cases} \tag{3.3}$$

By the continuity of l, there exists an open neighborhood N_{x_0} of x_0 such that for all $x \in N_{x_0}$

$$\begin{cases} \|z\| < 2\,\|l(x)\|\,, \\ < l(x), z > \; > \|l(x)\|^2 . \end{cases} \tag{3.4}$$

The set of all such neighborhoods covers $\tilde{X}$. Therefore there exists a locally finite refinement (N_{x_i}). Let $\rho_i(x)$ denotes the distance from x to the complement of N_{x_i}. Then ρ_i is Lipschitz continuous and vanishes outside of N_{x_i}. Set

$$\beta_i(x) = \frac{\rho_i(x)}{\sum_r \rho_r(x)}.$$

Since (N_{x_i}) is a locally finite covering, for each $x \in \tilde{X}$, the denominator of $\beta_i(x)$ is only a finite sum. Finally let

$$\gamma(x) = \sum_i z_i \beta_i(x),$$

where z_i is given by (3.2) for $x_0 = x_i$. For each $x \in \tilde{X}$, γ is a convex combination of locally Lipschitz continuous maps satisfying the condition (3.4) and hence γ satisfies (3.4). Let

$$v(x) = \gamma(x)\,\|df(x)\|^2 .$$

Then v satisfies (3.1).

Consider now a given sequence $(X_n)_{n\geq 1}$ of closed connected submanifolds of class C^2 of the C^1 Finsler manifold X and let $f \in C^1(X,\mathbb{R})$. Denoting $f_n = f_{/X_n}$ we then have $f_n \in C^1(X_n,\mathbb{R})$, $n \geq 1$.

Definition 3.2. Given $c \in \mathbb{R}$, we say that $f \in C^1(X,\mathbb{R})$ satisfies the Cerami's condition with respect to $(X_n)_{n\in\mathbb{N}}$ at level c if every sequence $(x_{n_p})_{p\in\mathbb{N}}$ satisfying

$$x_{n_p} \in X_{n_p},\ \forall p \in \mathbb{N};\ f(x_{n_p}) \longrightarrow c,\ \left\|df_{n_p}(x_{n_p})\right\|(d(x_{n_p},x_0)+1) \to 0 \ as\ p \to \infty,$$

(where $x_0 \in X$ is fixed), possesses a convergent subsequence which converges in X to a critical point of f. The above property will be referred as the $(C)^*_c$ condition with respect to $(X_n)_{n\geq 1}$.

This definition can be expressed as follows.

Definition 3.3. Given $c \in \mathbb{R}$, we say that $f \in C^1(X,\mathbb{R})$ satisfies the Cerami's condition with respect to (X_n) at level c if
(i) every bounded sequence $(x_{n_p})_{p\in\mathbb{N}} \subset X$, for which

$$x_{n_p} \in X_{n_p},\ \forall p \in \mathbb{N};\ f(x_{n_p}) \longrightarrow c,\ \left\| df_{n_p}(x_{n_p}) \right\| \longrightarrow 0 \ as\ p \longrightarrow \infty$$

possesses a convergent subsequence,
(ii) $\exists \sigma, R, \alpha > 0 / \forall x \in f^{-1}([c-\sigma, c+\sigma]),\ d(x_0,x) \geq R:\ \|df(x)\|\, d(x_0,x) \geq \alpha,$
where $x_0 \in X$ is fixed.

Condition (i) is a "compactness" condition which is satisfied by a large class of functionals. Condition (ii) gives a priori bound on the critical points of functionals with strong resonance at infinity.

Lemma 3.4. *Let g be a locally Lipschitz continuous vector field on a C^2 Finsler manifold X. Then, for $x \in X$, the Cauchy problem*

$$\begin{cases} \dot{\sigma}(t) = g(\sigma(t)), \\ \sigma(0) = x, \end{cases} \tag{3.5}$$

has a unique maximal solution $\sigma(.,x)$ defined on some open interval $]w_-(x), w_+(x)[$ containing 0. The set $D = \{(t,x):\ w_-(x) < t < w_+(x)\}$ is open in $\mathbb{R}\times X$ and the flow $\sigma : D \longrightarrow X$, $(t,x) \longmapsto \sigma(t,x)$ is continuous. Moreover, if $\{\sigma(t,x):\ w_-(x) < t < w_+(x))\}$ is contained in a complete subset of X on which the function g satisfies

$$\|g(x)\| \leq ad(x_0,x) + b,$$

where a, b are two constants and $x_0 \in X$ is fixed; then $w_-(x) = -\infty$, $w_+(x) = +\infty$.

For the proof see [8].

Lemma 3.5. *Let f be a C^1 function defined on a C^1 Finsler manifold X and let $c \in \mathbb{R}$ and $\bar{\varepsilon} > 0$ be such that $f^{-1}([c-2\bar{\varepsilon}, c+2\bar{\varepsilon}])$ is complete. Assume that f satisfies the Cerami's condition $(C)_c^*$.*
Then, given any neighborhood N of K_c and any constant $0 < \varepsilon < \bar{\varepsilon}$, there exist a deformation $\eta_t : X \longrightarrow X$ satisfying the following properties:
(a) $\eta_1(f^{c+\varepsilon} \backslash N) \subset f^{c-\varepsilon}$;

(b) $\eta_1(f^{c+\varepsilon}) \subset f^{c-\varepsilon}$, *if* $K_c = \emptyset$;

(c) $\eta_t(x) = x$ *if* $x \notin f^{-1}([c-2\bar{\varepsilon}, c+2\bar{\varepsilon}])$;

(d) $fo\eta(.,x)$ *is non increasing,* $\forall x \in X$.

Proof. Let $c \in \mathbb{R}$. We assume $K_c \neq \emptyset$. First of all we observe, by condition $(C)^*_c$, that K_c is compact. Let N be a neighborhood of K_c and let M_λ be the open $\lambda-$ neighborhood of K_c, i.e.

$$M_\lambda = \{x \in X : \ d(x, K_c) < \lambda\},$$

where $\lambda > 0$ and $d(x, K_c)$ denotes the distance from x to the set K_c. Hence, in order to prove (a), we can assume that there exits $\delta > 0$ such that $N = M_\delta$. The hypothesis (i), namely the Palais-Smale condition on bounded sets, implies that there exist $\bar{\varepsilon}, b, \bar{b}_1 > 0$ such that

$$\text{(3.6)} \qquad \begin{cases} a) \ \|df(x)\| > b \ for \ x \in \left(f^{c+\bar{\varepsilon}} \backslash f^{c-\bar{\varepsilon}}\right) \bigcap \left(M_\delta \backslash M_{\frac{\delta}{8}}\right), \\ b) \ \|df(x)\| > b_1 \ for \ x \in \left(f^{c+\bar{\varepsilon}} \backslash f^{c-\bar{\varepsilon}}\right) \bigcap \left(B_R \backslash M_{\frac{\delta}{8}}\right), \end{cases}$$

where $B_R = B(x_0, R)$. In fact, the condition (3.5)a) can be written as

$$\exists b > 0, \ \exists \bar{\varepsilon} > 0 : \ \forall n \in \mathbb{N}, \ \forall x \in X_n \bigcap \left(f^{c+\bar{\varepsilon}} \backslash f^{c-\bar{\varepsilon}}\right) \bigcap \left(M_\delta \backslash M_{\frac{\delta}{8}}\right), \ \|df(x)\| > b.$$

For otherwise,

$$\forall b > 0, \ \forall \bar{\varepsilon} > 0 : \ \exists n \in \mathbb{N}, \ \exists x \in X_n \bigcap \left(f^{c+\bar{\varepsilon}} \backslash f^{c-\bar{\varepsilon}}\right) \bigcap \left(M_\delta \backslash M_{\frac{\delta}{8}}\right), \ \|df(x)\| \leq b.$$

Let us take $\bar{\varepsilon} = b = \frac{1}{p}$, $p \in \mathbb{N}$. Then

$$\exists n_p \in \mathbb{N}, \ \exists x_{n_p} \in X_{n_p} \bigcap \left(f^{c+\frac{1}{p}} \backslash f^{c-\frac{1}{p}}\right) \bigcap \left(M_\delta \backslash M_{\frac{\delta}{8}}\right), \ \left\|df(x_{n_p})\right\| \leq \frac{1}{p}.$$

So, we have

$$x_{n_p} \in X_{n_p}, \ f(x_{n_p}) \longrightarrow c, \ df(x_{n_p}) \longrightarrow 0, \ (x_{n_p}) \ is \ bounded \ and \ x_{n_p} \notin M_{\frac{\delta}{8}}.$$

By (i) of $(C)^*_c$, (x_{n_p}) possesses a subsequence which converges in X to $x \in X$ satisfying

$$f(x) = c, \ df(x) = 0, \ x \notin M_{\frac{\delta}{8}}.$$

But $x \in K_c \subset M_{\frac{\delta}{8}}$, a contradiction. So (3.6)a) is satisfied. Similarly, we prove (3.6)b).
Since (3.6) still holds if $\bar{\varepsilon}$ is decreased, we can assume

$$\text{(3.7)} \qquad \bar{\varepsilon} < \min\left\{\frac{b\delta}{8}, \sigma\right\},$$

where σ is the positive constant which appears in (ii) of condition $(C)_c^*$. Then by (ii) and (3.6) (b) we also have that

$$\|df(x)\| > 0 \; for \; x \in \left(f^{c+\bar{\varepsilon}} \backslash f^{c-\bar{\varepsilon}}\right) \backslash M_{\frac{\delta}{8}}. \tag{3.8}$$

Now, let $0 < \varepsilon < \bar{\varepsilon}$. If we set

$$A = \{x \in X : \; f(x) \geq c + \bar{\varepsilon} \; or \; f(x) \leq c - \bar{\varepsilon}\},$$

$$B = \{x \in X : \; c - \varepsilon \leq f(x) \leq c + \varepsilon\},$$

the function

$$\chi_1(x) = \frac{d(x,A)}{d(x,A) + d(x,B)},$$

is Lipschitz continuous with $\chi_1 = 0$ on A, $\chi_1 = 1$ on B, and $0 \leq \chi_1(x) \leq 1$, $\forall x \in X$.

Also there is a Lipschitz continuous function χ_2 with $\chi_2 = 1$ on $X \backslash M_{\frac{\delta}{4}}$, $\chi_2 = 0$ on $M_{\frac{\delta}{8}}$ and $0 \leq \chi_2(x) \leq 1$, $\forall x \in X$. Then the Lipschitz continuous function $\chi : X \longrightarrow [0,1]$ defined by $\chi(x) = \chi_1(x)\chi_2(x)$, $\forall x \in X$ is such that

$$\chi(x) = \begin{cases} 0 \; if \; x \notin f^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]) \; or \; x \in M_{\frac{\delta}{8}}, \\ 1 \; if \; x \in f^{-1}([c - \varepsilon, c + \varepsilon]) \; and \; x \notin M_{\frac{\delta}{4}}. \end{cases} \tag{3.9}$$

Furthermore consider the map $V : X \longrightarrow X$ defined by

$$V(x) = \begin{cases} -\chi(x)v(x), & x \in \tilde{X}, \\ 0, & x \notin \tilde{X}, \end{cases} \tag{3.10}$$

where v is the map defined by Lemma 3.1. Obviously V is locally Lipschitz continuous in $\tilde{X}$ and by (3.1) a)

$$\|V(x)\| \leq \frac{2}{\|df(x)\|}, \; \forall x \in \tilde{X}. \tag{3.11}$$

Now we shall show that there exist two constants $a, b > 0$ independent of $x \in X$ such that

$$\|V(x)\| \leq ad(x_0, x) + b, \; \forall x \in X. \tag{3.12}$$

We can suppose that $x \in f^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]) \backslash M_{\frac{\delta}{8}}$ and we distinguish two cases.

1) $d(x_0, x) \geq R$: then, because $\bar{\varepsilon} < \sigma$, (ii) of condition $(C)_c^*$ implies

$$\|df(x)\| \geq \frac{\alpha}{d(x_0, x)}.$$

So by (3.11) we get

$$\|V(x)\| \leq \frac{2}{\alpha} d(x_0,x).$$

2) $d(x_0,x) < R$: then by (3.6) b) and (3.11), $\|V(x)\|$ is bounded. So, we conclude that (3.12) holds everywhere.

Consider now the following initial value problem

$$\begin{cases} \frac{d\eta}{dt} = V(\eta), \\ \eta(0) = x. \end{cases} \tag{3.13}$$

Since V is locally Lipschitz continuous, for each initial value $x \in X$, (3.13) possesses a unique solution $\eta(.,x)$ which by virtue of Lemma 3.2 and (3.12), is defined in $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. If we fix $t \in \mathbb{R}^+$, by well known theorems and (3.12), the map $\eta(t,.) : X \longrightarrow X$ is a bounded homeomorphism of X onto X. Observe that (cf (3.9) and (3.10))

$$x \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \; or \; x \in M_{\frac{\delta}{8}} \implies \eta(t,x) = x, \; \forall t \in \mathbb{R}^+. \tag{3.14}$$

So, for each $t \in \mathbb{R}^+$, $\eta(t,.)$ satisfies (c). For each $x \in X$ the real valued map defined by $s_x(t) = f(\eta(t,x))$, $t \in \mathbb{R}^+$ is not increasing in $\mathbb{R}^+$. In fact, for $t \geq 0$, we have

$$\frac{ds_x}{dt}(t) = < df(\eta(t,x)), \frac{d\eta}{dt}(t,x) > = < df(\eta(t,x)), V(\eta(t,x)) >$$

Then, if $\eta(t,x) \notin \tilde{X}$, we have $\frac{ds_x}{dt}(t) = 0$ and if $\eta(t,x) \in \tilde{X}$, we have

$$\frac{ds_x}{dt}(t) = -\chi(\eta(t,x)) < df(\eta(t,x)), v(\eta((t,x)) >, \tag{3.15}$$

then by (3.1) b), it follows that $\frac{ds_x}{dt}(t) \leq 0$. So (d) is proved. It remains to prove that there exists $\bar{t}$ such that $\eta(\bar{t},.)$ satisfies (a). For this, it is enough to consider the set

$$Y = \left(f^{c+\varepsilon} \backslash f^{c-\varepsilon}\right) \backslash M_\delta \tag{3.16}$$

and to show that

$$\exists \bar{t} > 0 \; s.t. \; \forall x \in Y, \; \eta(\bar{t},x) \in f^{c-\varepsilon}. \tag{3.17}$$

In fact $f^{c+\varepsilon} \backslash M_\delta = \left[\left(f^{c+\varepsilon} \backslash f^{c-\varepsilon}\right) \backslash M_\delta\right] \bigcup \left(f^{c-\varepsilon} \backslash M_\delta\right)$. If $x \in f^{c-\varepsilon} \backslash M_\delta$, $s_x(t) \leq s_x(0) = f(\eta(0,x)) = f(x) \leq c - \varepsilon$ and then $\eta(t,x) \in f^{c-\varepsilon}$. To this end, we set

$$Z = \left(f^{c+\varepsilon} \backslash f^{c-\varepsilon}\right) \backslash M_{\frac{\delta}{2}} \tag{3.18}$$

and we first show that

α) For each $x \in Y$, there is $t_x \leq 2\varepsilon$ such that $\eta(t_x, x) \in Z$. In fact, let $x \in Y$ and $t > 0$ such that $\eta(\tau, x) \in Z$, $\forall \tau \in [0,t]$. We shall prove that $t < 2\varepsilon$. By (3.8), $Z \subset \tilde{X}$; hence by (3.1) b), (3.9) and (3.15) we obtain

$$\forall \tau \in [0,t],\ \frac{ds_x}{dt}(\tau) = -< df(\eta(\tau,x)), v(\eta(\tau,x)) > \leq -1.$$

Then

$$2\varepsilon > s_x(0) - s_x(t) = -\int_0^t \frac{ds_x}{d\tau}(\tau) d\tau \geq t.$$

Then α) is proved.

Let us now consider the case in which an orbit $\eta(.,x)$, $x \in Y$ enters into $M_{\frac{\delta}{2}}$. Let t_2 be the first instant in which $\eta(.,x)$ touches the boundary of $M_{\frac{\delta}{2}}$ ($\forall t \in [0, t_2[$, $\eta(t,x) \notin M_{\frac{\delta}{2}}$). We can show that

β) there is an instant $t_0 \leq t_2$ such that $\eta(t_0, x) \in f^{c-\varepsilon}$. In order to prove β) we argue indirectly. In fact, if β) is not true, clearly $\eta(t,x) \in Z$, $\forall t \in [0,t_2]$. Then by the proof of α) we have

$$t_2 < 2\varepsilon. \tag{3.19}$$

On the other hand, let t_1 be the last instant before t_2 in which $\eta(.,x)$ touches the boundary of M_δ. Obviously

$$\eta(t,x) \in \left(f^{c+\varepsilon} \backslash f^{c-\varepsilon}\right) \bigcap \left(M_\delta \backslash M_{\frac{\delta}{2}}\right),\ \forall t \in]t_1, t_2[$$

and by (3.6) a) we have

$$\|df(\eta(t,x))\| > b,\ \forall t \in]t_1, t_2].$$

Then using (3.1) a), we can write

$$\begin{aligned}
\frac{\delta}{2} = \delta - \frac{\delta}{2} &\leq d(\eta(t_1,x), K_c) - d(\eta(t_2,x), K_c) \leq d(\eta(t_1,x), \eta(t_2,x)) \\
&\leq \int_{t_1}^{t_2} \left\| \frac{d\eta}{dt}(t,x) \right\| dt = \int_{t_1}^{t_2} \|V(\eta(t,x))\| \, dt \\
&\leq 2\int_{t_1}^{t_2} \frac{dt}{\|df(\eta(t,x))\|} < \frac{2}{b}(t_2 - t_1) \leq \frac{2}{b} t_2.
\end{aligned}$$

The above formula implies that $t_2 > \frac{b\delta}{4}$ and using (3.7), we obtain $t_2 > 2\bar{\varepsilon} > 2\varepsilon$ which contrasts with (3.19). Hence β) is true. Then by α) and β) we deduce that for each $x \in Y$ there exists t_x s.t. $\eta(t_x, x) \in f^{c-\varepsilon}$, it is obvious that in any case we can choose $t_x \leq 2\varepsilon$. After that, setting

$\bar{t} = 2\varepsilon$, we get $s_x(\bar{t}) \leq s_x(t_x) \leq c - \varepsilon$. The property (3.17) is verified and hence Lemma 3.5 is proved.

Proposition 3.6. *Let $c \in \mathbb{R}$, $f \in C^1(X, \mathbb{R})$ and $Y \subseteq X$ be closed. Assume that*

a) $\sup_Y f < c$*;*

b) f satisfies the $(C)_c^*$ condition with respect to $(X_n)_{n\in\mathbb{N}}$;

c) there exists $\bar{\varepsilon}_1 > 0$ such that $f^{-1}([c - \bar{\varepsilon}_1, c + \bar{\varepsilon}_1])$ is complete.

Then, for every open neighborhood N of K_c such that $N \cap Y = \emptyset$ and for every $0 < \varepsilon < \bar{\varepsilon} <$ $\inf(\bar{\varepsilon}_1, c - \sup_Y)$*, we have*

$$f^{c+\varepsilon} \backslash N <_Y^\infty f^{c-\varepsilon}.$$

Proof. Let N be an open neighborhood of K_c such that $N \cap Y = \emptyset$ and let $0 < \varepsilon < \bar{\varepsilon} < \inf(\bar{\varepsilon}_1, c - \sup_Y)$. Since f satisfies the $(C)_c^*$ condition, then for all integer n, f_n satisfies the $(C)_c^*$ condition. So, according to Lemma 3.5, there exists a deformation $\eta_t : X_n \longrightarrow X_n$ satisfying

$$\eta_1(f_n^{c+\varepsilon} \backslash N) \subseteq f_n^{c-\varepsilon}, \tag{3.20}$$

$$\eta_t(x) = x, \ \forall x \notin f_n^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]). \tag{3.21}$$

Since $\bar{\varepsilon} < c - \sup_{Y_n} f$, then $\sup_{Y_n} f_n < c - \bar{\varepsilon}$ and then $Y_n \cap f_n^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]) = \emptyset$. Therefore by (3.21), $\eta_t(y) = y$, $\forall y \in Y_n$ and $\eta_t(Y_n) \subset Y_n$. Finally, it is clear that $Y_n \subseteq (f_n^{c+\varepsilon} \backslash N) \cap f_n^{c-\varepsilon}$. So $f_n^{c+\varepsilon} \backslash N <_{Y_n} f_n^{c-\varepsilon}$, $\forall n \in \mathbb{N}$ and then $f^{c+\varepsilon} \backslash N <_Y^\infty f^{c-\varepsilon}$.

4. Critical point theorems

In this section, we consider a connected Finsler manifold X of class C^1 together with a sequence $(X_n)_{n\in\mathbb{N}}$ of closed connected sub-manifolds of class C^2 of X. We assume that there exists, for every $n \geq 0$, a retraction $r_n : X \longrightarrow X_n$. The limit relative category is computed with respect to $(X_n)_{n\in\mathbb{N}}$. Let $f \in C^1(X, \mathbb{R})$ and Y be a closed subset of X. Define, for each $j \geq 1$,

$$\mathscr{A}_j = \left\{ A \subseteq X : \ A \textit{ is closed}, \ A \supseteq Y, \ cat_{X,Y}^\infty A \geq j \right\},$$

$$c_j = \inf_{A \in \mathscr{A}_j} \sup_{x \in A} f(x).$$

Theorem 4.1. *Assume that*

a) $\sup_Y f < c_k = c_{k+1} = ... = c_{k+m} = c < \infty$*;*

b) f *satisfies the* $(C)_c^*$ *condition with respect to* $(X_n)_{n\in\mathbb{N}}$*;*

c) there exists $\eta > 0$ *such that* $f^{-1}([c-\eta, c+\eta])$ *is complete;*

then c *is a critical value of* f *and* $cat_X(K_c) \geq m+1$.

Proof. Proposition 2.4 implies the existence of an open neighborhood N of K_c such that $cat_X(\bar{N}) = cat_X(K_c)$. Using assumption a), we can assume that $N \cap Y = \emptyset$. From Proposition 3.1, assumptions a), b), c) imply that there exists $\varepsilon > 0$ such that

$$f^{c+\varepsilon} \backslash N <_Y^\infty f^{c-\varepsilon}.$$

We have $f^{c+\varepsilon} \subseteq X$ is closed and $Y \subseteq f^{c+\varepsilon}$. By definition of c_{k+m}, we have

$$c = c_{k+m} = \inf_{A \in \mathscr{A}_{k+m}} \sup_{x\in A} f(x).$$

Then there exists $A_0 \in \mathscr{A}_{k+m}$ such that $\sup_{A_0} f < c + \varepsilon$. Hence, $A_0 \subseteq f^{c+\varepsilon}$. By Proposition 3.6, we obtain

$$k+m \leq cat_{X,Y}^\infty(A_0) \leq cat_{X,Y}^\infty(f^{c+\varepsilon})$$

$$\leq cat_{X,Y}^\infty(f^{c+\varepsilon} \backslash N) + cat_X(\bar{N}) \leq cat_{X,Y}^\infty(f^{c-\varepsilon}) + cat_X(K_c).$$

By taking $\varepsilon > 0$ small enough, we can assume $\sup_Y f < c - \varepsilon$. Then we have $Y \subseteq f^{c-\varepsilon}$, $f^{c-\varepsilon}$ is closed in X and $\sup_{f^{c-\varepsilon}} f < c = c_k$ and so $f^{c-\varepsilon} \notin \mathscr{A}_k$. Therefore $cat_{X,Y}^\infty(f^{c-\varepsilon}) < k$. We deduce that $k+m \leq k-1+cat_X(K_c)$ and then $cat_X(K_c) \geq m+1$.

Now,we consider the Generalized Saddle Point Theorem. We assume that $X = E \times V$ where E is a Banach space and V is a complete connected Finsler manifold of class C^2. Let $E = W \oplus Z$ (topological direct sum) and $E_n = W_n \oplus Z_n$ be a sequence of closed subspaces with $Z_n \subseteq Z$, $W_n \subseteq W$, $1 \leq dim W_n < \infty$. Define $X_n = E_n \times V$.

Theorem 4.2. (Generalized Saddle Point Theorem) *Assume there exist* $r > 0$ *and* $\alpha < \beta \leq \gamma$ *such that*

a) f *satisfies the* $(C)_c^*$ *condition with respect to* $(X_n)_{n\in\mathbb{N}}$ *for every* $c \in [\beta, \gamma]$*;*

b) $f(w,v) \leq \alpha$ *for every* $(w,v) \in W \times V$ *such that* $\|w\| = r$*;*

c) $f(z,v) \geq \beta$ *for every* $(z,v) \in Z \times V$*;*

d) $f(w,v) \leq \gamma$ *for every* $(w,v) \in W \times V$ *such that* $\|w\| \leq r$.

Then $f^{-1}([\beta,\gamma])$ *contains at least* $cuplength(V)+1$ *critical points of* f.

Proof. We apply Theorem 4.1 with

$$Y = \{w \in W : \|w\| = r\} \times V.$$

Let us define $m = cuplength(V)$ and

$$A = \{w \in W : \|w\| \leq r\} \times V.$$

It follows from Theorem 2.10 that

$$cat^{\infty}_{X,Y}(A) \geq m+1.$$

Thus $A \in \mathscr{A}_j, j = 1,...,m+1$. Assumption d) implies

$$c_1 \leq c_2 \leq ... \leq c_{m+1} \leq \gamma.$$

Assume now that $\sup_B f < \beta$ for some $B \in \mathscr{A}_1$. Assumption c) implies that $B \cap (Z \times V) = \emptyset$. Thus the deformations $h_n : [0,1] \times B_n \longrightarrow X_n$ given by

$$h_n(t,w,z,v) = \big(1-t+\frac{tr}{\|w\|})w,(1-t)z,v\big)$$

are well defined and show that $cat^{\infty}_{X,Y}(B) = 0$, contracting the definition of $\mathscr{A}_1$. We conclude from assumption b) that

$$\sup_Y f \leq \alpha < \beta \leq c_1$$

and the Theorem follows.

Remarks 4.3. 1) In c) we may replace Z by $\varphi_1 + Z$, $\varphi_1 \in W$. Likewise in b) we may have $\varphi_2 + W$, $\varphi_2 \in Z$, in place of W.

2) Theorem 4.2 extends the Saddle Point Theorem of Rabinowitz (see Theorem 4.6 in [9]), Theorem 3.2 of Liu [11], Theorem 3.8 of Szulkin [12] and Theorem 6.3 of Fournier, Lupo, Ramos and Willem [5].

Now, consider the Generalized Linking Theorem. Let X, $(X_n)_{n\in\mathbb{N}}$ be as above. Let $R>0$, $r>0$, $\rho\in]0,r[$ and suppose $e\in\bigcap_{n=0}^{\infty}Z_n$, $\|e\|=1$. Define

$$Q=\{w\in W:\ \|w\|\leq R\}\oplus\{\lambda e:\ 0\leq\lambda\leq r\},$$

$$\partial Q=\big(\{w\in W:\ \|w\|=R\}\oplus\{\lambda e:\ 0\leq\lambda\leq r\}\big)\bigcup\big(\{w\in W:\ \|w\|\leq R\}\oplus\{0,re\}\big).$$

Theorem 4.4. (Generalized Linking Theorem) *Assume there exist* $\alpha<\beta\leq\gamma$ *such that*

a) f *satisfies the* $(C)_c^*$ *condition with respect to* $(X_n)_{n\in\mathbb{N}}$ *for every* $c\in[\beta,\gamma]$*;*

b) $f(x,v)\leq\alpha$ *for every* $(x,v)\in\partial Q\times V$*;*

c) $f(z,v)\geq\beta$ *for every* $(z,v)\in Z\times V$ *such that* $\|z\|=\rho$*;*

d) $f(x,v)\leq\gamma$ *for every* $(x,v)\in Q\times V$.

Then $f^{-1}([\beta,\gamma])$ *contains at least* $cuplength(V)+1$ *critical points of* f.

Proof. We apply Theorem 4.1 with $Y=\partial Q\times V$. It is easy to see that $cat_{X,Y}^{\infty}(Q\times V)\geq cuplength(V)+1$. Assume that $\sup_B f<\beta$ for some $B\in\mathscr{A}_1$. Assumption c) implies that $B\cap\big(\{z\in Z:\|z\|=\rho\}\times V\big)=\emptyset$. Let $\theta_n:W_n\oplus\{\lambda e:\lambda\in\mathbb{R}\}\setminus\{\rho e\}\longrightarrow\partial Q\cap E_n$ be a retraction. Then the deformation $h_n:[0,1]\times B_n\longrightarrow X_n$ given by

$$h_n(t,w,z,v)=\big((1-t)(w+z)+t\theta_n(w+\|z\|e),v\big)$$

are well defined and show that $cat_{X,Y}^{\infty}(B)=0$, contracting the definition of $\mathscr{A}_1$. The argument then follows the one of the preceding proof.

Remark 4.5. Theorem 4.4 extends the Linking Theorem (Theorem 5.3) in [9], The Montain-Pass Theorem of Ambrosetti-Rabinowitz [9] and Theorem 6.5 in [5].

Our last result in this section concerns the Mountain Circle Theorem proved in [5] with the $(PS)_c^*$ condition. Let E,E_n as above.

Theorem 4.6. *(Mountain Circle Theorem). Assume there exist constants* $0<r_1<r_2<r_3$, $\alpha<\beta\leq\gamma$ *such that*

a) f *satisfies the* $(C)_c^*$ *condition with respect to* (E_n) *for every* $c\in[\beta,\gamma]$*;*

b) $f(w,z)\leq\alpha$ *for every* $(w,z)\in W\times Z$*,* $\|w\|\leq r_1$*;*

c) $f(w,z)\geq\beta$ *for every* $(w,z)\in W\times Z$*,* $\|w\|=r_2$*;*

d) $f(w,0) \leq \alpha$ for every $w \in W$, $\|w\| = r_3$;

b) $f(w,0) \leq \gamma$ for every $w \in W$, $\|w\| \leq r_3$.

Then $f^{-1}([\beta,\gamma])$ contains at least two critical points of f.

Proof. Define $X = \left\{w \in W : \|w\| > \frac{r_1}{2}\right\} \times Z$, $X_n = X \cap E_n$ and $\tilde{f} = f_{|X}$. We apply Theorem 4.1 with $Y = \{w \in W : \|w\| = r_1 \text{ } or \text{ } \|w\| = r_3\}$. Define $A = \{w \in W : r_1 \leq \|w\| \leq r_3\}$. It follows that

$$cat^{\infty}_{X,Y}(A) = cat^{\infty}_{A\times Z,Y}(A) = 2.$$

Assumption b) implies that $\tilde{f}^{-1}([\frac{\alpha+\beta}{2},+\infty[)$ is complete. It follows from assumption c) that $c_1 \leq c_2 \leq \gamma$. Assume that $\sup_B \tilde{f} < \beta$ for some $B \in \mathscr{A}_1$. Assumption c) implies that $|w| \neq r_2$ for every $(w,z) \in B$. The deformations $h_n : [0,1] \times B_n \longrightarrow X_n$ given by

$$h_n(t,w,z) = \left((1-t) + \frac{tr_3}{\|w\|}\right)w + (1-t)z \text{ } if \text{ } \|w\| > r_2$$

$$h_n(t,w,z) = \frac{(1-t)+tr_1}{\|w\|}w + (1-t)z \text{ } if \text{ } \|w\| < r_2$$

are well-defined and show that $cat^{\infty}_{X,Y}(B) = 0$, contradicting the definition of $\mathscr{A}_1$. We conclude from assumptions b) and d) that

$$\sup_Y \tilde{f} \leq \alpha < \beta \leq c_1.$$

Remark 4.7. Theorem 4.6 extends Theorem 6.7 in [5].

5. Multiple periodic solutions of Hamiltonian systems

Consider a decomposition $\mathbb{R}^{2N} = A \oplus B$ of $\mathbb{R}^{2N}$ with

$$A = space\left\{e_{i_1},...,e_{i_p}\right\},\ B = space\left\{e_{i_{p+1}},...,e_{i_{2N}}\right\},$$

where $0 \leq p \leq 2N-1$ and $(e_i)_{1\leq i\leq 2N}$ is the standard basis of $\mathbb{R}^{2N}$, and let us denote P_A (resp. P_B) the projection of $\mathbb{R}^{2N}$ into A (resp. B). Let $H : \mathbb{R}\times\mathbb{R}^{2N} \longrightarrow \mathbb{R}$, $(t,x) \longmapsto H(t,x)$ be a continuous function $T-$periodic in the first variable, and differentiable with respect to the second variable with continuous derivative $\nabla H(t,x) = \frac{\partial H}{\partial x}(t,x)$. Consider the following assumptions:

(H_0) H is periodic in the variables $x_{i_1},...,x_{i_p}$;

(H_1) $\lim_{|P_B(x)|\longrightarrow\infty} \frac{|\nabla H(t,x)|}{|P_B(x)|} = 0$, uniformly in $(t,P_A(x)) \in [0,T]\times A$;

(H_2) There exist a nonempty open subset C of $[0,T]$ and two functions $f,g\in L^1(0,T;\mathbb{R})$ such that

$$(i)\qquad \lim_{|P_B(x)|\longrightarrow\infty}[\nabla H(t,x).x-2H(t,x)]=-\infty,\ a.e.\ t\in C,$$

$$(ii)\qquad \nabla H(t,x).x-2H(t,x)\leq f(t),\ \forall x\in\mathbb{R}^{2N},\ a.e.\ t\in[0,T],$$

$$(iii)\qquad H(t,x)\geq g(t),\ \forall x\in\mathbb{R}^{2N},\ a.e.\ t\in[0,T].$$

Our main result in this section is:

Theorem 5.1. *Assume $(H_0)-(H_2)$ hold. Then the Hamiltonian system*

$$(\mathscr{H})\qquad \dot{x}=J\nabla H(t,x)$$

possesses at least $(p+1)$ $T-$periodic solutions $x^1,...,x^{p+1}$ geometrically distinct.

Remark 5.2. Let us remark that if $x(t)$ is a periodic solution of $(\mathscr{H})$, then by replacing t by $-t$ in $(\mathscr{H})$, we obtain $\dot{x}(-t)=J\nabla H(-t,x(-t))$. So it is clear that the function $y(t)=x(-t)$ is a periodic solution of the system $\dot{y}(t)=-J\nabla H(-t,y(t))$. Moreover, $-H(-t,x)$ satisfies (H_2) whenever $H(t,x)$ satisfies the following assumption
(H_2') There exist a nonempty open subset C of $[0,T]$ and two functions $f,g\in L^1(0,T;\mathbb{R})$ such that

$$(i)\qquad \lim_{|P_B(x)|\longrightarrow\infty}[\nabla H(t,x).x-2H(t,x)]=+\infty,\ a.e.\ t\in C,$$

$$(ii)\qquad H'(t,x).x-2H(t,x)\geq f(t),\ \forall x\in\mathbb{R}^{2N},\ a.e.\ t\in[0,T],$$

$$(iii)\qquad H(t,x)\leq g(t),\ \forall x\in\mathbb{R}^{2N},\ a.e.\ t\in[0,T].$$

Therefore Theorem 5.1 remains true if we replace (H_2) by (H_2').

Proof of Theorems 5.1. Consider the Hilbert space $E=H^{\frac{1}{2}}(S^1,\mathbb{R}^{2N})$ where $S^1=\mathbb{R}/(T\mathbb{Z})$ and associate to the system $(\mathscr{H})$ the functional φ defined on the space E by

$$\varphi(u)=\frac{1}{2}\int_0^T J\dot{u}.udt+\int_0^T H(t,u)dt.$$

It is well known that the functional φ is continuously differentiable in E and critical points of φ on E correspond to $T-$periodic solutions of the system $(\mathscr{H})$, moreover one has

$$\varphi'(u)v = \int_0^T [J\dot{u} + \nabla H(t,u)].vdt$$

for all $u, v \in E$. Consider the subspaces $W = E^-$, $Z = E^+ \oplus B$ of E where E^+ and E^- are respectively the subspaces of E on which the quadratic form $Q(u) = \frac{1}{2}\int_0^T J\dot{u}.udt$ is positive definite and negative definite. Let us recall that the space E is compactly embedded in $L^2(0,T;\mathbb{R}^{2N})$ [9] and the expression

$$\|u\| = \left[Q(u^+) - Q(u^-) + \left|u^0\right|^2\right]^{\frac{1}{2}}$$

for $u = u^+ + u^- + u^0$, is an equivalent norm. In particular, there exists a constant $c_0 > 0$ such that

$$\|u\|_{L^2} \leq c_0 \|u\|,\ \forall u \in E. \tag{5.1}$$

Consider the quotient space

$$V = A/\left\{x \approx x + e_i,\ i = i_1, ..., i_p\right\},$$

which is nothing but the torus T^p. We regard the functional φ as defined on the space $(W \oplus Z) \times V$ as follows

$$\varphi(u+v) = \frac{1}{2}\int_0^T J\dot{u}.udt + \int_0^T H(t,u+v)dt.$$

To find critical points of φ, we will apply Theorem 4.2 to this functional with respect to the sequence of subspaces $X_n = E_n \times V$, where

$$E_n = \left\{u \in E:\ u(t) = \sum_{|m|\leq n} exp(\frac{2\pi}{T}mtJ)\hat{u}_m\ a.e.\right\},\ n \geq 0.$$

The following lemma will be needed.

Lemma 5.3. [10] *Let $G : \mathbb{R} \times \mathbb{R}^{2N} \longrightarrow \mathbb{R}$ be a continuous function such that*

$$\lim_{|P_B(x)| \longrightarrow \infty} G(t,x) = -\infty\ uniformly\ in\ P_A(x) \in A,\ a.e\ t \in C,$$

where C is a nonempty open subset of $[0,T]$. Then for every $\delta > 0$ there exists a subset C_δ of C with meas$(C - C_\delta) < \delta$ such that

$$\lim_{|P_B(x)| \longrightarrow \infty} G(t,x) = -\infty, \text{ uniformly in } (t, P_A(x)) \in C_\delta \times A.$$

Now, let us check the $(C)_c^*$ condition.

Lemma 5.4. *For all $c \in \mathbb{R}$ the functional φ satisfies the $(C)_c^*$ condition with respect to the sequence $(X_n)_{n\in\mathbb{N}}$.*

Proof. Let $(u_{n_j}, v_{n_j})_{j\in\mathbb{N}} \subset X_{n_j}$ be a sequence such that

$$\varphi(u_{n_j} + v_{n_j}) \longrightarrow c, \ \left\|\varphi'_{n_j}(u_{n_j} + v_{n_j})\right\| (\left\|u_{n_j} + v_{n_j}\right\| + 1) \longrightarrow 0 \text{ as } j \longrightarrow \infty, \tag{5.2}$$

where φ_{n_j} is the restriction of φ to X_{n_j}. Set $u_{n_j} = u_{n_j}^+ + u_{n_j}^- + u_{n_j}^0$, where $u_{n_j}^+ \in E^+$, $u_{n_j}^- \in E^-$, $u_{n_j}^0 \in B$. By assumption (H_1), there exist two constants $a > 0$, $b \in \mathbb{R}$ such that

$$|\nabla H(t,x)| \leq a\,|P_B(x)| + b, \ \forall t \in [0,T], \ \forall x \in \mathbb{R}^{2N}. \tag{5.3}$$

Since $\varphi'_{n_j}(u_{n_j} + v_{n_j}) \longrightarrow 0$ as $j \longrightarrow \infty$, there exists a constant $c_1 > 0$ such that

$$\left|\varphi'_{n_j}(u_{n_j} + v_{n_j})u_{n_j}^+\right| \leq c_1 \left\|u_{n_j}^+\right\|. \tag{5.4}$$

On the other hand, we have

$$\varphi'_{n_j}(u_{n_j} + v_{n_j})u_{n_j}^+ = 2\left\|u_{n_j}^+\right\|^2 + \int_0^T \nabla H(t, u_{n_j} + v_{n_j}).u_{n_j}^+ dt. \tag{5.5}$$

By (5.1), (5.3)-(5.5) and Hölder's inequality, there exists a constant $c_2 > 0$ such that

$$\left\|u_{n_j}^+\right\|^2 \leq c_1 \left\|u_{n_j}^+\right\| + c_2(\left\|P_B(u_{n_j})\right\| + 1)\left\|u_{n_j}^+\right\|. \tag{5.6}$$

So there exists a positive constant c_3 satisfying

$$\left\|u_{n_j}^+\right\| \leq c_3(\left\|P_B(u_{n_j})\right\| + 1). \tag{5.7}$$

Analogously, we have

$$\left\|u_{n_j}^-\right\| \leq c_3(\left\|P_B(u_{n_j})\right\| + 1). \tag{5.8}$$

Consequently, we deduce from (5.7) and (5.8) that the sequence (u_{n_j}) is bounded if and only if $(P_B(u_{n_j}))$ is bounded. Thus, we assume that $\left\|P_B(u_{n_j})\right\| \longrightarrow \infty$ as $j \longrightarrow \infty$. Taking $y_j = \frac{P_B(u_{n_j})}{\left\|P_B(u_{n_j})\right\|}$

we claim that there exists a constant $y \in B$ such that $\|y\| = 1$ and $y_j \longrightarrow y$ as $j \longrightarrow \infty$ in E. In fact, by (H_1), for all $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$(5.9) \qquad |\nabla H(t,x)| \leq \frac{2\varepsilon}{c_0^2}\,|P_B(x)| + c(\varepsilon).$$

Arguing as in (5.7), we obtain by (5.9)

$$(5.10) \qquad \left\|P_B(u_{n_j}^+)\right\| \leq \varepsilon\left\|P_B(u_{n_j})\right\| + c_4,$$

where c_4 is a constant. As ε was arbitrary given, we obtain

$$(5.11) \qquad \frac{P_B(u_{n_j}^+)}{\left\|P_B(u_{n_j})\right\|} \longrightarrow 0 \; as \; j \longrightarrow \infty.$$

Analogously

$$(5.12) \qquad \frac{P_B(u_{n_j}^-)}{\left\|P_B(u_{n_j})\right\|} \longrightarrow 0 \; as \; j \longrightarrow \infty.$$

From (5.11) and (5.12), it follows that $y_j \longrightarrow y \in B$, $\|y\| = 1$ as claimed. Using the Sobolev's embedding $E \hookrightarrow L^2(0,T;\mathbb{R}^{2N})$, we may suppose without loss of generality, that

$$(5.13) \qquad y_j(t) \longrightarrow y \; as \; j \longrightarrow \infty, \; a.e. \; t \in [0,T]$$

and consequently

$$(5.14) \qquad \left|P_B(u_{n_j})(t)\right| \longrightarrow +\infty \; as \; j \longrightarrow \infty, \; a.e. \; t \in [0,T].$$

Let $\delta = \frac{1}{2} meas(C)$. By $(H_2)(i)$ and Lemma 5.1, there exists a measurable subset C_δ of C with $meas(C - C_\delta) < \delta$ such that

$$(5.15) \quad 2H(t,x) - \nabla H(t,x).x \longrightarrow +\infty \; as \; |P_B(x)| \longrightarrow +\infty, \; uniformly \; in \; (t, P_A(x)) \in C_\delta \times A.$$

Combining (5.14), (5.15) and Fatou's lemma, yields

$$(5.16) \qquad \int_{C_\delta} \left[2H(t,u_{n_j}+v_{n_j}) - \nabla H(t,u_{n_j}+v_{n_j}).(u_{n_j}+v_{n_j})\right] dt \longrightarrow +\infty \; as \; j \longrightarrow \infty.$$

Furthermore, we have by $(H_2)(ii)$

$$\left|\varphi_{n_j}'(u_{n_j}+v_{n_j})(u_{n_j}+v_{n_j})\right| \geq -2\varphi(u_{n_j}+v_{n_j})$$
$$+ \int_0^T \left[2H(t,u_{n_j}+v_{n_j}) - \nabla H(t,u_{n_j}+v_{n_j}).(u_{n_j}+v_{n_j})\right] dt$$

(5.17)

$$\geq -2\varphi(u_{n_j}+v_{n_j})+\int_{C_\delta}\left[2H(t,u_{n_j}+v_{n_j})-\nabla H(t,u_{n_j}+v_{n_j})\right].(u_{n_j}+v_{n_j})dt-\int_0^T|f(t)|\,dt.$$

Since $(\varphi(u_{n_j}+v_{n_j}))$ is bounded, we deduce from (5.16), (5.17) that

$$\left\|\varphi'(u_{n_j}+v_{n_j})\right\|\left\|u_{n_j}+v_{n_j})\right\| \longrightarrow \infty \text{ as } j \longrightarrow \infty, \tag{5.18}$$

which contradicts (5.2). So $(u_{n_j}+v_{n_j})$ is bounded. By a standard argument, $(u_{n_j}+v_{n_j})$ possesses a convergent subsequence. The proof of Lemma 5.4 is complete.

Now, for $(u,v)=(u^++u^0,v)\in Z\times V$, we have

$$\varphi(u+v)=\left\|u^+\right\|^2+\int_0^T H(t,u+v)dt \geq \int_0^T g(t)dt=\beta. \tag{5.19}$$

Let $\alpha<\beta$ be a fixed real. We have for all $(u,v)\in W\times V$

$$\varphi(u+v)=-\|u\|^2+\int_0^T H(t,u+v)dt.$$

By the Mean Value Theorem, (5.1) and (5.9), we have

$$\varphi(u+v)\leq -\|u\|^2+\varepsilon c_5\|u\|^2+c_6\|u\|+\int_0^T H(t,v)dt,$$

where c_5, c_6 are two positive constants. Taking $\varepsilon c_5<1$, we deduce that

$$\varphi(u+v)\longrightarrow -\infty \text{ as } \|u\|\longrightarrow\infty,\ u\in W,\ uniformly\ in\ v\in V. \tag{5.20}$$

So there exists $r>0$ such that

$$\varphi(u+v)\leq\alpha,\ \forall(u,v)\in W\times V,\ \|u\|=r. \tag{5.21}$$

The functional φ is also bounded from above on $B_r\times V$ by a constant $\gamma\geq\beta$, where B_r is the closed disc in W with center zero and radius r. So the link conditions are satisfied. The functional φ satisfies all the assumptions of Theorem 4.2, so it has at least $cuplength(V)+1$ critical points on X, and since V is the torus T^p, then $cuplength(V)=p$ and Theorem 5.1 is proved.

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