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ON A BOUNDARY BLOW-UP PROBLEM FOR THE MONGE-AMPÈRE EQUATION

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Abstract. We consider boundary blow-up solutions to the Monge-Ampère equation $detD^2u = p(x)e^{q(x)u}$ in bounded, smooth and strictly convex domain in $\mathbb{R}^N$. Our main concern is the effect of the non-constant weight function $q(x)$ on solvability of the problem. Existence and non-existence results are obtained through sub-super solution method and comparison principle.

Keywords. Monge-Ampère equation; Singular boundary value problem; Solvability; Existence.

1. Introduction

Let us consider the Monge-Ampère equation

$$\begin{cases} det\ D^2u = p(x)e^{q(x)u}, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, and $p(x)$ is a positive smooth function on $\overline{\Omega}$. Problem (1.1) has a singular boundary condition, and it is usually called boundary blow-up problem in literature. Such problem was first studied by Bieberbach [1] for the equation $\Delta u = e^u$ in a smooth bounded domain in $\mathbb{R}^2$. In recent years, there are extensive works studying boundary blow-up problems. We refer the reader to [2, 3, 4, 5, 6, 7], which focus on such problems governed by the Monge-Ampère operator.

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Recall that when $q(x) \equiv 1$, problem (1.1) was studied in [4] by Lazer and Mckenna as a generalized Bieberbach problem, and they established the following result.

Theorem A (see Theorem 2.3 in [4]) *Let Ω be a smooth, bounded strictly convex domain in $\mathbb{R}^N$, $N \geq 2$. If $p \in C^\infty(\overline{\Omega})$ and $p(x) > 0$ for all $x \in \overline{\Omega}$, then there exists a unique $u \in C^\infty(\Omega)$ such that*

$$\begin{cases} det\ D^2u = p(x)e^{u(x)}, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega. \end{cases} \tag{1.2}$$

In this paper, we introduce a weight function in the nonlinearity, as described by (1.1), and study its effect on existence and nonexistence of boundary blow-up solutions. This is motivated by the recent work [8], where the authors introduced new weight functions when dealing with boundary blow-up problems governed by the Laplace operator. As the authors of [8] found, introducing certain weight functions in the nonlinearity may make it more difficult to study problems such as existence, nonexistence, uniqueness and blow-up rates of solutions. For related works we refer the reader to [9, 10] and the pioneering paper [11]. None of these nor any other articles, however, consider equations governed by the Monge-Ampère operator as we do here.

Note that a natural question for problem (1.2) is whether it can admit classical solutions with e replaced by other positive constants, written

$$\begin{cases} det\ D^2u = p(x)a^u, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega. \end{cases} \tag{1.3}$$

More generally, assume a is replaced by a positive function, say $a(x)$ in (1.3), since one can write $a(x) = e^{q(x)}$ for some $q(x)$, the question arises: How does the behavior of $q(x)$ affect the solvability of (1.1).

Our results concerning these questions can be summarized as follows. Firstly, (1.3) has a solution(which is unique) for $a > 1$ and it admits no solution for $0 < a < 1$, as indicated by Lemma 3.1 and Theorem 3.3 in Section 3. Secondly, our Theorem 3.2, which is obtained by sub-sup solution method, asserts that (1.1) has a classical solution if $q(x) \in C^\infty(\overline{\Omega})$ is positive in Ω. We also obtain a nonexistence result in Theorem 3.3, as a counterpart of Theorem 3.2. That is, once $q(x)$ is non-positive in $B(x_0,\delta)\cap\Omega$ for some $x_0\in\partial\Omega$ and $\delta>0$, no solution for (1.1) would exist.

2. Preliminaries

In this section, we list out some known facts for the reader's convenience, and give some remarks on the part of them, which is prepared for proving our results in Section 3.

First, we recall the following two comparison lemmas. The first is the Comparison Principle for the Monge-Ampère equation, see Theorem 1.4.6 in [12]. We write it out in a form concerning smooth functions.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$. If u, $v\in C^2(\Omega)\cap C(\overline{\Omega})$ are two convex functions satisfying*

$$\begin{cases} \det D^2u \geq \det D^2v, & x\in\Omega; \\ u(x)\leq v(x), & x\in\partial\Omega, \end{cases}$$

then $u(x)\leq v(x)$ for any $x\in\Omega$.

The second one is taken from Lemma 2.1 in [4].

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N\geq 2$, and let $u_k\in C^2(\Omega)\cap C(\overline{\Omega})$ for $k=1,2$. Let $f(x,\xi)$ be defined for $x\in\Omega$ and ξ in some interval containing the ranges of u_1 and u_2 and assume that $f(x,\xi)$ is strictly increasing in ξ for all $x\in\Omega$. If*

- **(i):** *the Hessian matrix D^2u_1 is positive definite in Ω,*
- **(ii):** *$\det D^2u_1\geq f(x,u_1(x))$, $\forall x\in\Omega$,*
- **(iii):** *$\det D^2u_2\leq f(x,u_2(x))$, $\forall x\in\Omega$,*
- **(iv):** *$u_1(x)\leq u_2(x)$, $\forall x\in\partial\Omega$,*

then we have

$$u_1(x)\leq u_2(x),\ \forall x\in\overline{\Omega}.$$

Remark 2.3. In the statement of lemma 2.2, if $u_2\in C^2(\Omega)\cap C(\overline{\Omega})$ is replaced by $\widetilde{u_2}\in C^2(\Omega)$ with $\widetilde{u_2}(x)\to+\infty$ as $x\to\partial\Omega$(such that condition (iv) is automatically valid in more general sense), then it holds $u_1(x)\leq\widetilde{u_2}(x),\ \forall x\in\Omega$. Indeed, if $u_1(x_0)>\widetilde{u_2}(x_0)$ for some $x_0\in\Omega$, there will be a compact sub-domain Ω' of Ω such that $x_0 \in \Omega'$, $u_1(x) > \widetilde{u_2}(x)$ in Ω', and $u_1(x) = \widetilde{u_2}(x)$ for $x \in \partial\Omega'$. However, one can use Lemma 2.2 in Ω' to deduce $u_1(x) \leq \widetilde{u_2}(x)$ in Ω', a contradiction. Similarly, one deduce that, the function v in Lemma 2.1 can be replaced by a function $\widetilde{v} \in C^2(\Omega)$ which may take infinite values on part or whole of $\partial\Omega$.

We end this section with a known regularity result concerning Monge-Ampère equations. A slight adjustment of Lemma 2.2 in [4] will yield the following lemma, whose proof can be given in a similar way, using Proposition 2.4 (ii) of [13].

Lemma 2.4. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$. Let $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ with $f(x,\xi) > 0$ for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}$. Let $u \in C^\infty(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\begin{cases} det\ D^2u = f(x,u), & x \in \Omega, \\ u(x) = c = constant, & x \in \partial\Omega, \end{cases}$$

with $u(x) < c$ in Ω. Let Ω' be a compact sub-domain of Ω and assume that $a \leq u(x) \leq b$ for $x \in \overline{\Omega'}$ and let $k \geq 1$ be an integer. There exists a constant C^* which depends only on k, a, b, bounds for the derivatives of $f(x,\xi)$ for $(x,\xi) \in \overline{\Omega'} \times [a,b]$, and $dist(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^k(\overline{\Omega'})} \leq C^*.$$

3. Existence and nonexistence results

First of all, we show that (1.3) has a unique solution for $a > 1$. Indeed, denote $a = e^q$, we have the following.

Lemma 3.1. *Let Ω be a bounded smooth and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, and let $p(x) \in C^\infty(\overline{\Omega})$ be a positive function on $\overline{\Omega}$. If $q > 0$ is a constant then there exists a unique $u \in C^\infty(\Omega)$ such that*

$$\begin{cases} det\ D^2u = p(x)e^{qu}, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega. \end{cases} \tag{3.1}$$

Proof. Direct calculation shows (3.1) can be transformed to

$$\begin{cases} det\ D^2 v = q^N p(x) e^v, & x \in \Omega, \\ v(x) \to +\infty, & x \to \partial\Omega \end{cases} \tag{3.2}$$

by letting $v(x) = qu(x)$. Thus the unique solvability of (3.2) (see Theorem 2.3 in [4]) implies the unique solvability of (3.1).

The main result of this paper is contained in the following two theorems. Since we cannot find explicit sub-sup solutions as in [4], our proof for the existence result is not straight forward. The idea comes from [8]. We first establish the following theorem.

Theorem 3.2. *Let Ω be a bounded smooth and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$. Let $p(x) \in C^\infty(\overline{\Omega})$, $p(x) > 0$ on $\overline{\Omega}$ and $q(x) \in C^\infty(\overline{\Omega})$ with $q(x) > 0$ in Ω. Then there exists $u \in C^\infty(\Omega)$ that solves problem (1.1).*

Proof. Consider the following problem

$$\begin{cases} det\ D^2 u = p(x) e^{q(x)u}, & x \in \Omega, \\ u(x) = n, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

where n is a positive integer. By Theorem 1.1 in [14], there exists a unique strictly convex $\underline{u_n} \in C^\infty(\overline{\Omega})$ that solves

$$\begin{cases} det\ D^2 u = p(x) e^{nq(x)}, & x \in \Omega, \\ u(x) = n, & x \in \partial\Omega, \end{cases}$$

and it is a sub-solution of (3.3) in the sense that

$$\begin{cases} det\ D^2 \underline{u_n} \geq p(x) e^{q(x)\underline{u_n}}, & x \in \Omega, \\ \underline{u_n}(x) = n, & x \in \partial\Omega. \end{cases}$$

By Theorem 7.1 in [14], (3.3) admits a unique strictly convex solution $u_n \in C^\infty(\overline{\Omega})$ with $u_n \geq \underline{u_n}$ on $\overline{\Omega}$. From Lemma 2.2, we infer $u_{n_2} \geq u_{n_1}$ if $n_2 \geq n_1$. So $u_n(x)$ is increasing in n for each $x \in \Omega$.

Claim. $\{u_n(x)\}_{n=1}^\infty$ is uniformly bounded from above in any compact sub-domain of Ω.

Let D be an arbitrary compact sub-domain of Ω. Since $u_1(x) = 1$ for $x \in \partial\Omega$, we can choose a small $\delta_0 > 0$ such that $u_1(x) \geq 0$ in $\Omega_{\delta_0} := \{x \in \Omega : dist(x, \partial\Omega) < \delta_0\}$, and thus $u_n(x) \geq 0$

in Ω_{δ_0} for each $n \in \mathbb{N}$. Then there exists a δ_1 with $0 < \delta_1 < \delta_0$ such that $D \subset \Omega' := \{x \in \Omega : dist(x, \partial\Omega) > \delta_1\}$. Let $x_0 \in \partial\Omega'$, because $q(x_0) > 0$ and $q(x)$ is continuous, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset \Omega_{\delta_0}$ and $q(x) \geq q_0 := \frac{1}{2}q(x_0) > 0$ in $B(x_0, \varepsilon)$. Obviously, $u_n(x) \geq 0$ in $B(x_0, \varepsilon)$ for each $n \in \mathbb{N}$. Thus we have

$$det\ D^2 u_n = p(x)e^{q(x)u_n} \geq p(x)e^{q_0 u_n},\ \forall x \in B(x_0, \varepsilon).$$

By Lemma 3.1, there exists a unique $v \in C^\infty(B(x_0, \varepsilon))$ such that

$$\begin{cases} det\ D^2 v = p(x)e^{q_0 v}, & x \in B(x_0, \varepsilon), \\ v(x) \to +\infty, & x \to \partial B(x_0, \varepsilon). \end{cases}$$

Now Lemma 2.2 and Remark 2.3 give the comparison that $u_n \leq v$ in $B(x_0, \varepsilon)$ for all $n \in \mathbb{N}$. So $\{u_n(x)\}_{n=1}^\infty$ is uniformly bounded from above in $B(x_0, \frac{1}{2}\varepsilon)$. Since $x_0 \in \partial\Omega'$ is arbitrarily chosen, there exists a small ball centered at each point of $\partial\Omega'$ in which $\{u_n(x)\}_{n=1}^\infty$ has a uniformly upper bound, which yield an uniformly upper bound of $\{u_n(x)\}_{n=1}^\infty$ on $\partial\Omega'$. Notice that $u_n \in C^\infty(\overline{\Omega})$ is strictly convex, so the Hessian matrix of u_n is positively definite on $\overline{\Omega'}$ and its eigenvalues are all positive on $\overline{\Omega'}$. We have $\Delta u > 0$ in Ω', which implies $u_n(x)|_{\overline{\Omega'}}$ attains its maximum on $\partial\Omega'$ by the maximum principle ([15]). Hence $\{u_n(x)\}_{n=1}^\infty$ is uniformly bounded from above on $\overline{\Omega'}$, and our **claim** holds.

Now we can define a real valued function $u(x) := \lim_{n\to\infty} u_n(x)$ for each $x \in \Omega$. What's more, let $f(x, \xi) := p(x)e^{q(x)\xi}$, Ω' be a compact sub-domain of Ω, and $k \geq 1$ a positive integer, then Lemma 2.4 shows there exists a constant C^* which depends only on k, $a := \min_{x\in\overline{\Omega'}} u_1(x)$, $b :=$ the uniformly upper bound of $\{u_n(x)\}_{n=1}^\infty$ on $\overline{\Omega'}$, bounds for the derivatives of $f(x, \xi)$ for $(x, \xi) \in \overline{\Omega'} \times [a, b]$, and $dist(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^k(\overline{\Omega'})} \leq C^*.$$

A compactness argument similar to the one used in the proof of Theorem 2.1 in [4] shows $u(x)$ belongs to $C^\infty(\Omega')$ thus $C^\infty(\Omega)$, and it satisfies

$$det\ D^2 u = p(x)e^{q(x)u},\ x \in \Omega.$$

Finally, let us prove the function $u(x)$ is such that $u(x) \to +\infty$ as $x \to \partial\Omega$. For arbitrarily fixed $M > 0$, consider the function $u_{[M]+1}$ which satisfies

$$\begin{cases} det\ D^2 u_{[M]+1} = p(x)e^{q(x)u_{[M]+1}}, & x \in \Omega, \\ u_{[M]+1}(x) = [M]+1, & x \in \partial\Omega. \end{cases}$$

Set

$$\Omega_{M,\varepsilon} = \{x \in \overline{\Omega} | u_{[M]+1}(x) \leq [M]+1-\varepsilon\},$$

where ε is a small positive number such that $[M]+1-\varepsilon > M$ and

$$\min_{x\in\overline{\Omega}} u_{[M]+1}(x) < [M]+1-\varepsilon.$$

It is easy to see that $\Omega_{M,\varepsilon}$ is a nonempty compact subset of $\mathbb{R}^N$. Thus there exist $x_1 \in \partial\Omega$ and $x_2 \in \Omega_{M,\varepsilon}$ such that

$$dist(x_1,x_2) = dist(\partial\Omega, \Omega_{M,\varepsilon}) =: \delta_{M,\varepsilon}.$$

Moreover, $\delta_{M,\varepsilon} > 0$ because $\partial\Omega \cap \Omega_{M,\varepsilon} = \emptyset$. Now, for each $x \in \Omega$ such that $dist(x,\partial\Omega) < \frac{\delta_{M,\varepsilon}}{2}$ we have

$$u_{[M]+1}(x) > [M]+1-\varepsilon > M,$$

which completes the proof because $u(x) = \lim_{n\to\infty} u_n(x) \geq u_{[M]+1}(x) > M$ for each $x \in \Omega$ with $dist(x,\partial\Omega) < \frac{\delta_{M,\varepsilon}}{2}$.

Theorem 3.3. *Let Ω be a bounded smooth and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$. Let $p(x) \in C^\infty(\overline{\Omega})$, $p(x) > 0$ on $\overline{\Omega}$ and $q(x) \in C^\infty(\overline{\Omega})$. If $q(x) \leq 0$ in a ball relative to Ω, say $B(x_0,\delta)\cap\Omega$ for some $x_0 \in \partial\Omega$ and $\delta > 0$, then problem (1.1) admits no classical solution.*

Proof. We argue by contradiction. Suppose u is a classical solution of (1.1). Without loss of generality we can assume $\delta < diam(\Omega)$ is small, such that $u(x) > 0$ in $B(x_0,\delta)\cap\Omega$ due to the singular boundary condition. Let D be a smooth strictly convex sub-domain of Ω such that $D \subset B(x_0,\delta)\cap\Omega$, and $\partial\Omega\cap\partial D$ contains $B(x_0,\frac{2}{3}\delta)\cap\partial\Omega$. Let φ be a smooth function supported on ∂D satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B(x_0,\frac{1}{3}\delta)\cap\partial\Omega$ and $\varphi = 0$ on $\partial D \setminus (B(x_0,\frac{2}{3}\delta)\cap\partial\Omega)$. By Theorem 1.1 in [14], the problem

$$\begin{cases} det\ D^2 z = n^N, & x \in D, \\ z = n\varphi, & x \in \partial D \end{cases}$$

with $n \in \mathbb{N}$ has a unique smooth strictly convex solution z_n. We can choose n large such that

$$det\ D^2 z_n = n^N \geq \max_{x\in\overline{\Omega}} p(x) \geq det\ D^2 u,\ x \in D.$$

By Lemma 2.1 and Remark 2.3, for large n, $z_n \le u$ in D because $z_n \le u$ on ∂D. However, $w_n := \frac{1}{n} z_n$ satisfies

$$\begin{cases} det\ D^2 w_n = 1, & x \in D, \\ w_n = \varphi, & x \in \partial D. \end{cases}$$

Hence $z_n = n w_0$, where w_0 is the unique smooth strictly convex solution of

$$\begin{cases} det\ D^2 w = 1, & x \in D, \\ w = \varphi, & x \in \partial D. \end{cases}$$

Notice $w_0(x_0) = 1$, so there exists a $\varepsilon > 0$ small such that $(B(x_0, \varepsilon) \cap \Omega) \subset D$ and $w_0(x) \ge \frac{1}{2}$ in $B(x_0, \varepsilon) \cap \Omega$. From above we know for large n, $\frac{1}{2} n \le u$ in $B(x_0, \varepsilon) \cap \Omega$, which is impossible. This completes the proof.

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