J. Nonlinear Funct. Anal. 2016 (2016), Article ID 48

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

ON A NON-LOCAL PROBLEM FOR A LOADED MIXED TYPE EQUATION WITH CAPUTO AND RIEMANN-LIOUVILLE OPERATORS

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Abstract. In this paper, we investigate the existence and uniqueness of solutions of non-local boundary value problems for the loaded parabolic-hyperbolic equation involving the Caputo fractional derivative and Riemann-Liouville integrals. The uniqueness of solutions was proved based on the method of integral energy by using an extremume principle for the mixed type equation. The existence was proved based on the method of integral equations.

Keywords. Loaded equation; Parabolic-hyperbolic type; Integral operation; Caputo fractional derivative; Integral equation.

1. Introduction

Development of the theory of the equations with fractional derivatives is stimulated with development of the theory of the integer order differential equations. About applications to physics, biology, mathematical modeling etc, one is referred to the works [1, 2, 3]. Precisely, many problems in viscoelasticity [4, 5, 6] dynamical processes in self-similar structures [1], biosciences [7], signal processing [8], system control theory [9], electrochemistry [10], diffusion processes [11], and linear time-invariant systems of any order with internal point delays [12] lead to differential equations of fractional orders. Notice that the works [13, 14, 15] are devoted to the study of BVPs for parabolic-hyperbolic equations involving fractional derivatives.

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Received July 3, 2016

BVPs for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators were investigated in [16, 17]. With intensive research on the problem of the optimal control of the agroeconomical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate a new class of equations called loaded equations. For the first time, it was given the most general definition of a loaded equations and various loaded equations are classified in detail by Nakhushev [18].

Definition 1.1. An equation $Au(x) = f(x)$ is called loaded equation in n dimensional Euclidean domain Ω if operator A depends of the restriction of the unknown function to a closed subset of $\overline{\Omega}$, of measure strictly less than n.

After this work, many interesting results on the theory of boundary value problems for the loaded equations parabolic, parabolic-hyperbolic and elliptic-hyperbolic types were published, for example, see [19, 20, 21] and the references therein. In this direction, some local and non-local problems for the loaded elliptic-hyperbolic type equations of the second and the third order in double-connected domains, one is referred to the works [22, 23, 24, 25] and the references therein.

2. Preliminaries

2.1. Euler's integrals

The elementary definition of the gamma function is the Euler's integral (see [26].p.24)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

For $z \in R^+$, this integral converges and satisfies the recurrence relation

$$\Gamma(z+1) = z\Gamma(z), \tag{1}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{2}$$

and for $z \in N$ is satisfies

$$\Gamma(z+1) = z! \tag{3}$$

There is another Euler's integral, which can be represented by gamma function (see [26].p.26)

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int\limits_0^1 t^{x-1}(1-t)^{y-1}dt,$$

which converges for $x > 0,\ y > 0$.

2.2. Riemann-Liouville integral-differential operator

Definition 2.2.1. Let $f(x)$ be an absolutely continuous function over(a,b). Then the left and right Riemann-Liouville fractional integrals order α $(\alpha \in R^+)$ (respectively) are (see [26].p.69)

$$\left(I_{a+}^{\alpha}f\right)x = \frac{1}{\Gamma(\alpha)}\int\limits_a^x f(t)(x-t)^{\alpha-1}dt,\ \ x > a, \tag{4}$$

$$\left(I_{-b}^{\alpha}f\right)x = \frac{1}{\Gamma(\alpha)}\int\limits_x^b f(t)(x-t)^{\alpha-1}dt,\ \ x < b. \tag{5}$$

The Riemann-Liouville fractional derivatives ${}_{RL}D_{ax}^{\alpha}f$ and ${}_{RL}D_{xb}^{\alpha}f$ of order $\alpha(\alpha \in R^+,)$ are defined by (see [26].p.26):

$$({}_{RL}D_{ax}^{\alpha}f)x = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int\limits_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\ \ n = [\alpha]+1,\ \ \ x > a, \tag{6}$$

$$({}_{RL}D_{xb}^{\alpha}f)x = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int\limits_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}dt,\ \ n = [\alpha]+1,\ \ \ x < b, \tag{7}$$

respectively, where $[\alpha]$ is the integer part of α. In particular, for $\alpha = N \cup \{0\}$ we have

$$\left({}_{RL}D_{ax}^{0}f\right)x = f(x),\ \ \left({}_{RL}D_{xb}^{0}f\right)x = f(x),\ \ ({}_{RL}D_{ax}^{n}f)x = f^{(n)}(x),$$

$$({}_{RL}D_{xb}^{n}f)x = (-1)^n f^{(n)}(x),\ \ n \in N,$$

where $f^{(n)}(x)$ is the usual derivative of $f(x)$of order n.

Definition 2.2.2. Caputo fractional derivatives ${}_{C}D_{ax}^{\alpha}f$ and ${}_{C}D_{xb}^{\alpha}f$of order $\alpha > 0$ $(\alpha \notin N \cup \{0\})$ are defined by (see [26].p.92):

$$({}_{C}D_{ax}^{\alpha}f)x = \frac{1}{\Gamma(n-\alpha)}\int\limits_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\ \ n = [\alpha]+1,\ \ \ x > a, \tag{8}$$

(9) $$\left({}_C D_{xb}^{\alpha} f\right) x = \frac{(-1)^n}{\Gamma(n-\alpha)} \int\limits_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \;\; n = [\alpha]+1, \;\; x < b,$$

respectively.

From 6-9, as a conclusion we have

$$\left({}_C D_{ax}^{\alpha} f\right) x = \mathrm{si}\,gn^k(x-a) \left(I_{ax}^{\alpha-k} f^{(k)}\right) x, \;\; k-1 < \alpha \leqslant k,$$

consequently, while for $\alpha \in N \cup \{0\}$ we have

$$\left({}_C D_{ax}^{0} f\right) x = f(x), \;\; \left({}_C D_{xb}^{0} f\right) x = f(x), \;\; \left({}_C D_{ax}^{n} f\right) x = f^{(n)}(x),$$

$$\left({}_C D_{xb}^{n} f\right) x = (-1)^n f^{(n)}(x), \;\; n \in N.$$

2.3. Wright type functions

The elementary definition of the Wright type function at $\alpha > \beta$, $\alpha > 0$ and for all $z \in C$, is represented as [27]

(10) $$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\mu)\Gamma(\delta-\beta n)}.$$

If $\alpha = \mu = 1$, then owing to 3 from 10 we have

(11) $$e_{1,\beta}^{1,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{n!\Gamma(\delta-\beta n)}.$$

If for the Wright type function

$$\pi \geqslant |\arg z| > \pi(\alpha+\beta)/2+\varepsilon, \; \varepsilon > 0, \;\; k = 0,1,2,...,$$

then at $z \to \infty$ takes place [27]

$$\lim_{|z|\to\infty} e_{\alpha,\beta}^{\mu,\delta}(z) = 0;$$

$$\lim_{|z|\to\infty} z e_{\alpha,\beta}^{\mu,\delta}(z) = -\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}.$$

3. Problem formulation and main functional relations

In the given paper, for the equation

$$u_{xx}-\frac{1-\operatorname{sgn}y}{2}\left[u_{yy}-\sum_{k=1}^{n}R_k(x,0)\right]-\frac{1+\operatorname{sgn}y}{2}{}_CD_{oy}^{\alpha}u=0 \tag{12}$$

with operators [28]:

$$R_k(x,y)=R_k\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)=\begin{cases} p_k(\xi)I_{\xi\,1}^{\beta_k}u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right), & \text{at } q\leqslant x\leqslant 1,\\ q_k(\eta)I_{-1\,\eta}^{\gamma_k}u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right), & \text{at } -1\leqslant x\leqslant -q,\end{cases} \tag{13}$$

where $0<\alpha,\ \beta_k,\gamma_k<1$

$$I_{xa}^{-\gamma_k}f(x)=\frac{1}{\Gamma(\gamma_k)}\int_x^a(t-x)^{\gamma_k-1}f(t)dt;\ I_{ax}^{-\beta_k}f(x)=\frac{1}{\Gamma(\beta_k)}\int_a^x(x-t)^{\beta_k-1}f(t)dt, \tag{14}$$

$${}_CD_{oy}^{\alpha}f=\frac{1}{\Gamma(1-\alpha)}\int_0^y(y-t)^{-\alpha}f'(t)dt. \tag{15}$$

We will investigate the uniqueness and the existence of solution of the non-local problem. Lets, Ω be special domain, bounded with segments:

$$A_1A_1^*=\{(x,y):\ x=1,\ 0<y<h\},\ \ B_1B_1^*=\{(x,y):\ x=q,\ 0<y<h\},$$

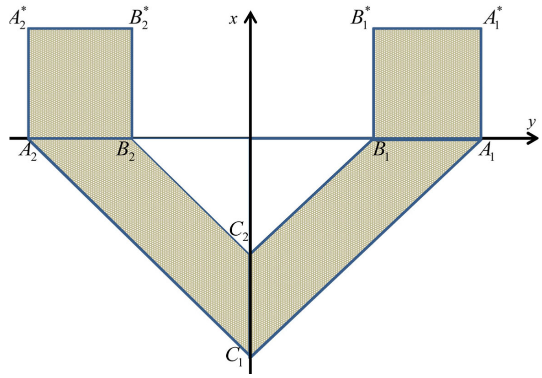
$$B_1^*A_1^*=\{(x,y):\ y=h,\ q<x<1\},\ \ A_2A_2^*=\{(x,y):\ x=-1,\ 0<y<h\},$$

$$B_2B_2^*=\{(x,y):\ x=-q,\ 0<y<h\},\ \ A_2^*B_2^*=\{(x,y):\ y=h,\ -1<x<-q\}$$

at $y>0$ and characteristics :

$$A_jC_1:x+(-1)^jy=(-1)^{j+1},\ \ B_jC_2:x+(-1)^jy=(-1)^{j+1}\cdot q;\ (0<q<1),\ (j=1,2)$$

of the equation (12) at $y<0$, where $x+y=\xi,\ x-y=\eta$, $A_1(1;0)$, $A_1^*(1;h)$, $A_2(-1;0)$, $A_2^*(-1;h)$, $B_1(q;0)$, $B_1^*(q;h)$, $B_2(-q;0)$, $B_2^*(-q;h)$, $C_1(0;-1)$, $C_2(0;-q)$.



Introduce designations: $\Omega_{01}=\Omega\cap(y>0,\ x>0)$, $\Omega_{02}=\Omega\cap(y>0,\ x<0)$,

$$\Delta_1=\Omega\cap(x+y>q)\cap(y<0),,\ \ \Delta_2=\Omega\cap(y-x>q)\cap(y<0),$$

$$\Delta_3=\Omega\cap(-q<x+y<q)\cap(x>0),\ \ \Delta_4=\Omega\cap(-q<y-x<q)\cap(x<0),$$

$$\Delta_5=\Omega\cap(-1<x+y<-q)\cap(-1<y-x<-q),\ \ I_{2+j}=\left\{x:0<(-1)^{j-1}x<q\right\},$$

$I_j=\left\{x:q<(-1)^{j-1}x<1\right\}$ $(j=1,2)$.

We investigate the following problem in the domain Ω.

Problem I. To find a solution $u(x,y)$ of the equation (12) from the following class of functions:

$$W=\left\{u(x,y):\ u(x,y)\in C(\bar{\Omega})\cap C^2(\Delta_l)\ \ u_{xx}\in C\left(\Omega_{01}\cup\Omega_{02}\right),{}_CD_{oy}^{\alpha}u\in C\left(\Omega_{01}\cup\Omega_{02}\right)\right\},$$

$(l=\overline{1,5})$, satisfies boundary conditions:

(16) $$u(x,y)\Big|_{A_jA_j^*}=\varphi_j(y),\ \ 0\leqslant y\leqslant h,$$

(17) $$u(x,y)\Big|_{B_jB_j^*}=\psi_j(y),\ \ 0\leqslant y\leqslant h,$$

(18) $$\frac{d}{dx}u\left(\theta_1(x)\right)=a_1(x)u_y(x,0)+b_1(x)u_x(x,0)+c_1(x)u(x,0)+d_1(x),\ \ x\in I_1,$$

(19) $$\frac{d}{dx}u\left(\theta_2(x)\right)=a_2(x)u_y(x,0)+b_2(x)u_x(x,0)+c_2(x)u(x,0)+d_2(x),\ \ x\in I_2,$$

(20) $$u(x,y)\Big|_{B_jC_2}=g_j(x),\ \ x\in\overline{I_{2+j}},\ \ (j=1,2)$$

and gluing conditions:

$$\lim_{y\to+0} y^{1-\alpha}u_y(x,y)=\lambda_{11}(x)u_y(x,-0)+\lambda_{12}(x)\int_x^1 r_1(t)u(t,0)dt,\ \ q<x<1, \tag{21}$$

$$\lim_{y\to+0} y^{1-\alpha}u_y(x,y)=\lambda_{21}(x)u_y(x,-0)+\lambda_{22}(x)\int_{-1}^x r_2(t)u(t,0)dt,\ \ -1<x<-q, \tag{22}$$

where $\varphi_j(y)$, $\psi_j(y)$, $a_j(x)$, $b_j(x)$, $c_j(x)$, $d_j(x)$, $g_j(x)$, $\lambda_{ij}(x)$ $(i,j=1,2)$ are given functions,such that $\sum\limits_{j=1}^{2}\lambda_{ij}^2(x)\neq 0$, $i=1,2$, besides: $g_1(0)=g_2(0)$, $g_2(-q)=\varphi_2(0)$, $g_1(q)=\psi_1(0)$.

4. The uniqueness of solutions of Problem I.

Known that the equation (12) at $y\leqslant 0$ on the characteristics coordinate $\xi=x+y$ and $\eta=x-y$ has a form:

$$u_{\xi\eta}=\frac{1}{4}\sum_{k=1}^{n}R_k(\xi,0). \tag{23}$$

Introduce designations:

$$u(x,0)=\tau_1(x),\ \ q\leqslant x\leqslant 1;\ \ u(x,0)=\tau_2(x),\ \ -1\leqslant x\leqslant -q,$$

$$u_y(x,-0)=\nu_1^-(x),\ \ q<x<1;\ \ u_y(x,-0)=\nu_2^-(x),\ \ -1<x<-q,$$

$$\lim_{y\to 0} y^{1-\alpha}u_y(x,y)=\nu_1^+(x),\ \ q<x<1;\ \ \lim_{y\to 0} y^{1-\alpha}u_y(x,y)=\nu_2^+(x),\ \ -1<x<-q.$$

Known that solution of the Cauchy problem for the equation (12) in the domain of Δ_1 can be represented as follows:

$$u(x,y)=\frac{\tau_1(x+y)+\tau_1(x-y)}{2}-\frac{1}{2}\int_{x+y}^{x-y}\nu_1^-(t)dt+\frac{1}{4}\int_{x+y}^{x-y}d\xi\int_{\xi}^{x-y}\sum_{k=1}^{n}p_k(\xi)I_{\xi\,1}^{-\beta_k}\tau_1(\xi)d\eta. \tag{24}$$

Using condition (18), we find from (24) that

$$(2a_1(x)-1)\,\nu_1^-(x)=\frac{1}{2}\sum_{k=1}^{n}p_k(x)(x-1)I_{x1}^{-\beta_k}\tau_1(x)$$

$$+(1-2b_1(x))\,\tau_1'(x)-2c_1(x)\tau_1(x)-2d_1(x), \tag{25}$$

Precisely also, from the solution

$$u(x,y)=\frac{\tau_1(x+y)+\tau_1(x-y)}{2}-\frac{1}{2}\int\limits_{x+y}^{x-y}\nu_1^-(t)dt+\frac{1}{4}\int\limits_{x+y}^{x-y}d\eta\int\limits_{x+y}^{\eta}\sum_{k=1}^{n}q_k(\eta)I_{-1\eta}^{\gamma_k}\tau_2(\eta)d\xi \tag{26}$$

of the Cauchy problem for the Eq.(12) in the domain of Δ_2 with conditions

$$u(x,0)=\tau_2(x),\ \ x\in\overline{A_2B_2};\ \ u_y(x,-0)=\nu_2^-(x),\ \ x\in A_2B_2$$

and on the base of (19), we obtain

$$(2a_2(x)+1)\,\nu_2^-(x)=-\frac{1}{2}\sum_{k=1}^{n}q_k(x)(x+1)I_{x\,1}^{-\gamma_k}\tau_2(x)+$$

$$+(1-2b_2(x))\,\tau_2'(x)-2c_2(x)\tau_2(x)-2d_2(x). \tag{27}$$

Considering designations and gluing conditions (21) and (22), we have

$$\nu_1^+(x)=\lambda_{11}(x)\nu_1^-(x)+\lambda_{12}(x)\int\limits_x^1 r_1(t)\tau(t)dt, \tag{28}$$

$$\nu_2^+(x)=\lambda_{21}(x)\nu_2^-(x)+\lambda_{22}(x)\int\limits_{-1}^x r_2(t)\tau(t)dt. \tag{29}$$

Further from the Eq. (12) at $y\to+0$ owing to account (15), (28), (29) and

$$\lim_{y\to0}D_{0y}^{\alpha-1}f(y)=\Gamma(\alpha)\lim_{y\to0}y^{1-\alpha}f(y)$$

we get

$$\tau_1''(x)-\Gamma(\alpha)\lambda_{11}(x)\nu_1^-(x)-\Gamma(\alpha)\lambda_{12}(x)\int\limits_x^1 r_1(t)\tau_1(t)dt=0 \tag{30}$$

and

$$\tau_2''(x)-\Gamma(\alpha)\lambda_{21}(x)\nu_2^-(x)-\Gamma(\alpha)\lambda_{22}(x)\int\limits_{-1}^x r_2(t)\tau_2(t)dt=0. \tag{31}$$

Theorem 4.1. *If the following conditions*

$$p_k(q),q_k(-q)\geq0,\ \ (p_k(x)A_1(x))',(q_k(x)A_2(x))'\geq0,\ \ (k=1,\ 2,...,n) \tag{32}$$

$$A_j((-1)^{j-1}q),\ \ B_j'(x),C_j(x)\leq0,\ \frac{\lambda_{j2}(x)}{r_j(x)},\ \frac{\lambda_{j2}((-1)^{j-1}q)}{r_j((-1)^{j-1}q)}\geq0(j=1,\ 2), \tag{33}$$

hold, then the solution $u(x,y)$ of the Problem I is unique.

Proof. If the homogeneous problem has only trivial solutions, then we can state that the original problem has a unique solution. To this end, we assume that the Problem I has two solutions. Denote the difference of these as $u(x,y)$ we get the appropriate homogenous problem. Multiplying equations (30), (31) respectively to functions $\tau_1(x)$, $\tau_2(x)$ and integrating from q to 1 and from -1 to Cq accordingly, we obtain

$$\int_q^1 \tau_1''(x)\tau_1(x)dx - \Gamma(\alpha)\int_q^1 \lambda_{11}(x)\nu_1^-(x)\tau_1(x)dx -$$

$$\Gamma(\alpha)\int_q^1 \lambda_{12}(x)\tau_1(x)dx\int_t^1 r_1(z)\tau_1(z)dz = 0 \tag{34}$$

and

$$\int_{-1}^{-q} \tau_2''(x)\tau_2(x)dx - \Gamma(\alpha)\int_{-1}^{-q} \lambda_{21}(x)\nu_2^-(x)\tau_2(x)dx -$$

$$\Gamma(\alpha)\int_{-1}^{-q} \lambda_{22}(x)\tau_2(x)dx\int_{-1}^{t} r_2(z)\tau_2(z)dz = 0. \tag{35}$$

First, we investigate the integral

$$J_1 = \Gamma(\alpha)\int_q^1 \lambda_{11}(x)\tau_1(x)\nu_1^-(x)dx + \Gamma(\alpha)\int_q^1 \lambda_{12}(x)\tau_1(x)dx\int_x^1 r_1(t)\tau_1(t)dt.$$

Taking (25) into account $d_1(x)\equiv 0$, $2a_1(x)\neq 1$, we get

$$J_1 = \frac{\Gamma(\alpha)}{2}\int_q^1 \frac{(x-1)}{2a_1(x)-1}\lambda_{11}(x)\tau_1(x)\sum_{k=1}^{n} p_k(x) I_{x\,1}^{-\beta_k}\tau_1(x)dx +$$

$$\Gamma(\alpha)\int_q^1 \frac{(1-2b_1(x))\,\lambda_{11}(x)}{2a_1(x)-1}\tau_1(x)\tau_1'(x)dx - 2\Gamma(\alpha)\int_q^1 \frac{\lambda_{11}(x)c_1(x)}{2a_1(x)-1}\tau^2(x)dx +$$

$$\Gamma(\alpha)\int_q^1 \lambda_{22}(x)\tau_1(x)dx\int_x^1 r_1(t)\tau_1(t)dt =$$

$$= \sum_{k=1}^{n}\frac{\Gamma(\alpha)}{2\Gamma(\beta_k)}\int_0^1 \frac{(x-1)\lambda_{11}(x)p_k(x)}{2a_1(x)-1}\tau_1(x)dx\int_x^1 (t-x)^{\beta_k-1}\tau_1(t)dt +$$

$$\frac{\Gamma(\alpha)}{2}\int\limits_0^1\frac{1-2b_1(x)}{2a_1(x)-1}\lambda_{11}(x)d\left(\tau_1^2(x)\right)-$$

$$2\Gamma(\alpha)\int\limits_q^1\frac{\lambda_{11}(x)c_1(x)}{2a_1(x)-1}\tau_1^2(x)dx-\frac{\Gamma(\alpha)}{2}\int\limits_q^1\frac{\lambda_2(x)}{r(x)}d\left(\int\limits_x^1 r_1(t)\tau_1(t)dt\right)^2. \tag{36}$$

Considering $\tau_1(1)=0,\ \tau_1(q)=0$ (which deduced from the conditions (16), (17) at $j=1$ in homogeneous case) and due to the formulate [28]

$$|x-t|^{-\gamma}=\frac{1}{\Gamma(\gamma)\,cos\frac{\pi\gamma}{2}}\int\limits_0^\infty z^{\gamma-1}cos\left[z(x-t)\right]dz,\ \ 0<\gamma<1. \tag{37}$$

From (36), we get

$$J_1=\sum_{k=1}^n\frac{\Gamma(\alpha)A_1(q)\cos\frac{\pi\beta_k}{2}}{2\pi}p_k(q)\int\limits_0^\infty z^{-\beta_k}\left[M_1^2(q,z)+N_1^2(q,z)\right]dz+$$

$$\sum_{k=1}^n\frac{\Gamma(\alpha)\cos\frac{\pi\beta_k}{2}}{2\pi}\int\limits_0^\infty z^{-\beta_k}dz\int\limits_q^1\left[M_1^2(x,z)+N_1^2(x,z)\right](A_1(x)p_k(x))'dx-$$

$$\frac{\Gamma(\alpha)}{2}\int\limits_q^1\tau_1^2(x)B_1'(x)dx-2\Gamma(\alpha)\int\limits_q^1 C_1(x)\tau_1^2(x)dx+$$

$$\frac{\Gamma(\alpha)}{2}\frac{\lambda_{12}(q)}{r_1(q)}\left(\int\limits_q^1 r_1(t)\tau_1(t)dt\right)^2+\frac{\Gamma(\alpha)}{2}\int\limits_q^1\left(\frac{\lambda_{12}(x)}{r_1(x)}\right)'\left(\int\limits_x^1 r_1(t)\tau_1(t)dt\right)^2dx, \tag{38}$$

where $A_1(x)=\frac{(x-1)\lambda_{11}(x)}{2a_1(x)-1}$, $B_1(x)=\frac{(1-2b_1(x))\lambda_{11}(x)}{2a_1(x)-1}$, $C_1(x)=\frac{\lambda_{11}(x)c_1(x)}{2a_1(x)-1}$, $M_1(x,z)=\int\limits_x^1\tau_1(t)\cos zt dt$ and $N_1(x,z)=\int\limits_x^1\tau_1(t)\sin zt dt$. Similarly, taking (27) into account $d_2(x)\equiv 0$, $2a_2(x)\neq -1$, we investigate the integral:

$$J_2=\Gamma(\alpha)\int\limits_{-1}^{-q}\lambda_{21}(x)\tau_2(x)\nu_2^-(x)dx+\Gamma(\alpha)\int\limits_{-1}^{-q}\lambda_{22}(x)\tau_2(x)dx\int\limits_{-1}^{x}r_2(t)\tau_2(t)dt=$$

$$=\sum_{k=1}^n\frac{\Gamma(\alpha)A_2(-q)\cos\frac{\pi\gamma_k}{2}}{2\pi}q_k(-q)\int\limits_0^\infty z^{-\gamma_k}\left[M_2^2(-q,z)+N_2^2(-q,z)\right]dz+$$

$$\sum_{k=1}^n\frac{\Gamma(\alpha)\cos\frac{\pi\gamma_k}{2}}{2\pi}\int\limits_0^\infty z^{-\gamma_k}dz\int\limits_{-1}^{-q}\left[M_2^2(x,z)+N_2^2(x,z)\right](A_2(x)q_k(x))'dx-$$

$$\frac{\Gamma(\alpha)}{2}\int\limits_{-1}^{-q}\tau_2^2(x)B_2'(x)dx-2\Gamma(\alpha)\int\limits_{-1}^{-q}C_2(x)\tau_2^2(x)dx+$$

$$\frac{\Gamma(\alpha)}{2}\frac{\lambda_{22}(-q)}{r_1(-q)}\left(\int\limits_{-1}^{-q}r_2(t)\tau_2(t)dt\right)^2+\frac{\Gamma(\alpha)}{2}\int\limits_{-1}^{-q}\left(\frac{\lambda_{22}(x)}{r_2(x)}\right)'\left(\int\limits_{-1}^{x}r_2(t)\tau_2(t)dt\right)^2dx, \tag{39}$$

where $A_2(x)=\frac{(x+1)\lambda_{21}(x)}{2a_2(x)+1}$, $B_2(x)=\frac{(1-2b_2(x))\lambda_{21}(x)}{2a_2(x)+1}$, $C_2(x)=\frac{\lambda_{21}(x)c_2(x)}{2a_2(x)+1}$, $M_2(x,z)=\int\limits_{-1}^{x}\tau_2(t)\cos zt\,dt$ and $N_2(x,z)=\int\limits_{-1}^{x}\tau_2(t)\sin zt\,dt$.

Thus, owing to (32) and (33) from (38) and (39) it is accordingly concluded, that $\tau_1(x)\equiv 0$ and $\tau_2(x)\equiv 0$. Hence, based on the solution of the first boundary problem for the Eq.(1) [29], owing to account (16) and (17) we will get $u(x,y)\equiv 0$ in $\overline{\Omega}_{0j}$, $(j=1,2)$.

Further, from functional relations (25) and (27), taking $\tau_1(x)\equiv\tau_2(x)\equiv 0$ into account, we deduce that $\nu_j^-(x)\equiv 0$. Consequently, based on the solution (24) and (26) we obtain $u(x,y)\equiv 0$ in closed domain $\overline{\Delta}_j$ $(j=1,2)$. Owing to uniqueness of solution of the Gaurset problem, we get that $u(x,y)\equiv 0$ in the domains of $\bar{\Delta}_{j+2}$ $(j=1,2,3)$. Thus, we received that $u(x,y)\equiv 0$ in the domain $\overline{\Omega}$ (see [22], [23]). This completes the proof.

5. The existence of solutions of Problem I.

Theorem 5.1. *If the conditions* (32)*,* (33) *and*

$$\varphi_j(y),\ \psi_j(y)\in C[0,h]\cap C^1(0,h), \tag{40}$$

$$g_j(x)\in C\left(\overline{I_{2+j}}\right)\cap C^2\left(I_{2+j}\right),\ a_j(x),b_j(x),c_j(x),d_j(x)\in C^1\left(\overline{I_j}\right)\cap C^2\left(I_j\right), \tag{41}$$

$$p_k(x)\in C\left(\overline{I_1}\right)\cap C^1(I_1),\ q_k(x)\in C\left(\overline{I_2}\right)\cap C^1(I_2)\quad (k=\overline{1,\ n}), \tag{42}$$

$$\lambda_{1j}(x)\in C\left(\overline{I_1}\right)\cap C^1(I_1),\ \lambda_{2j}(x)\in C\left(\overline{I_2}\right)\cap C^1(I_2)\quad (j=\overline{1,2}), \tag{43}$$

hold, then there exist a unique solution.

Proof. Substituting (25) and (27) into (30) and (31) respectively, we obtain

$$\tau_j''(x)=f_j(x),\quad -1<(-1)^jx<-q, \tag{44}$$

$$f_1(x)=\frac{\Gamma(\alpha)}{2}A_1(x)\sum_{k=1}^{n}p_k(x)I_{x1}^{-\beta_k}\tau_1(x)+\Gamma(\alpha)B_1(x)\tau_1'(x)-$$

$$\Gamma(\alpha)C_1(x)\tau_1(x)+\Gamma(\alpha)\lambda_{12}(x)\int\limits_x^1 r_1(x)\tau_1(x)dx-D_1(x) \tag{45}$$

$$f_2(x)=\frac{\Gamma(\alpha)}{2}A_2(x)\sum_{k=1}^{n}q_k(x)I_{-1x}^{-\gamma_k}\tau_2(x)+\Gamma(\alpha)B_2(x)\tau_2'(x)-$$

$$\Gamma(\alpha)C_2(x)\tau_2(x)+\Gamma(\alpha)\lambda_{22}(x)\int\limits_{-1}^x r_2(x)\tau_2(x)dx-D_2(x), \tag{46}$$

where $D_j(x)=\Gamma(\alpha)\frac{2\lambda_{j1}(x)d_j(x)}{2a_j(x)+(-1)^j}$. Obviously, a solution of equations (44) together with conditions

$$\tau_j\left((-1)^{j-1}q\right)=\psi_j(0),\ \ \tau_1\left((-1)^{j-1}\right)=\varphi_j(0) \tag{47}$$

has a form

$$\tau_1(x)=\int\limits_x^1(z-x)f_1(z)dz+\frac{x-1}{1-q}\int\limits_q^1(z-q)f_1(z)dz+$$

$$\frac{x-q}{1-q}\varphi_1(0)-\frac{x-1}{1-q}\psi_1(0), \tag{48}$$

$$\tau_2(x)=\int\limits_{-1}^x(x-z)f_2(z)dz-\frac{x+1}{q-1}\int\limits_{-1}^{-q}(z+q)f_2(z)dz+$$

$$\frac{x+q}{q-1}\varphi_2(0)-\frac{x+1}{q-1}\psi_2(0). \tag{49}$$

Taking (45) and (46) into account from (48) and (49) and using (14), we deduce

$$\tau_1(x)+\int\limits_q^1 K_1(x,s)\tau_1(s)ds=F_1(x), \tag{50}$$

$$\tau_2(x)+\int\limits_{-1}^{-q} K_2(x,s)\tau_2(s)ds=F_2(x), \tag{51}$$

where

(52)

$$K_1(x,s)=\begin{cases}\frac{\Gamma(\alpha)}{2}\int\limits_x^s(z-x)A_1(z)\sum\limits_{k=1}^n\frac{p_k(z)}{\Gamma(\beta_k)}(s-z)^{\beta_k-1}dz+\Gamma(\alpha)r_1(s)\int\limits_x^s(z-x)\lambda_{12}(z)dz+\\ \frac{\Gamma(\alpha)(x-1)}{2(1-q)}\int\limits_q^s(z-q)A_1(z)\sum\limits_{k=1}^n\frac{p_k(z)}{\Gamma(\beta_k)}(s-z)^{\beta_k-1}dz+\Gamma(\alpha)r_1(s)\int\limits_q^s(z-q)\lambda_{12}(z)dz\\ -\Gamma(\alpha)\left(B_1'(s)(s-x)+B_1(s)\right)-\Gamma(\alpha)(s-x)C_1(s)-\frac{\Gamma(\alpha)(x-1)}{1-q}\left(B_1'(s)(s-q)+B_1(s)\right)-\\ \frac{\Gamma(\alpha)(x-1)}{1-q}(s-q)C_1(s),\quad x\leqslant s\leqslant 1,\\ \frac{\Gamma(\alpha)(x-1)}{2(1-q)}\int\limits_q^s(z-q)A_1(z)\sum\limits_{k=1}^n\frac{p_k(z)}{\Gamma(\beta_k)}(s-z)^{\beta_k-1}dz+\Gamma(\alpha)r_1(s)\int\limits_q^s(z-q)\lambda_{12}(z)dz-\\ \frac{\Gamma(\alpha)(x-1)}{1-q}\left(B_1'(s)(s-q)+B_1(s)\right)-\frac{\Gamma(\alpha)(x-1)}{1-q}(s-q)C_1(s),\quad q\leqslant s\leqslant x,\end{cases}$$

(53)

$$K_2(x,s)=\begin{cases}\frac{\Gamma(\alpha)}{2}\int\limits_s^x(x-z)A_2(z)\sum\limits_{k=1}^n\frac{q_k(z)}{\Gamma(\gamma_k)}(z-s)^{\gamma_k-1}dz+\Gamma(\alpha)r_2(s)\int\limits_s^x(x-z)\lambda_{22}(z)dz-\\ \frac{\Gamma(\alpha)(x+1)}{2(q-1)}\int\limits_s^{-q}(z+q)A_2(z)\sum\limits_{k=1}^n\frac{q_k(z)}{\Gamma(\gamma_k)}(z-s)^{\gamma_k-1}dz+\Gamma(\alpha)r_2(s)\int\limits_s^{-q}(z+q)\lambda_{22}(z)dz-\\ \Gamma(\alpha)\left(B_2'(s)(x-s)-B_2(s)\right)-\Gamma(\alpha)(x-s)C_2(s)+\frac{\Gamma(\alpha)(x+1)}{q-1}\left(B_2'(s)(s+q)-B_2(s)\right)-\\ \frac{\Gamma(\alpha)(x+1)}{1-q}(s+q)C_2(s),\quad -1\leqslant s\leqslant x,\\ -\frac{\Gamma(\alpha)(x+1)}{2(q-1)}\int\limits_s^{-q}(z+q)A_2(z)\sum\limits_{k=1}^n\frac{q_k(z)}{\Gamma(\gamma_k)}(z-s)^{\beta_k-1}dz+\Gamma(\alpha)r_2(s)\int\limits_s^{-q}(z+q)\lambda_{22}(z)dz\\ -\frac{\Gamma(\alpha)(x+1)}{1-q}\left(B_2'(s)(s+q)-B_2(s)\right)-\frac{\Gamma(\alpha)(x+1)}{1-q}(s+q)C_2(s),\quad x\leqslant s\leqslant -q,\end{cases}$$

(54) $$F_1(x)=\frac{x-q}{1-q}\varphi_1(0)-\frac{x-1}{1-q}\psi_1(0)-\int\limits_x^1(z-x)D_1(z)dz-\frac{x-1}{1-q}\int\limits_q^1(z-q)D_1(z)dz,$$

(55) $$F_2(x)=\frac{x+q}{q-1}\varphi_2(0)-\frac{x+1}{q-1}\psi_2(0)-\int\limits_{-1}^x(x-z)D_2(z)dz-\frac{x+1}{q-1}\int\limits_{-1}^{-q}(z+q)D_2(z)dz.$$

Since $\left|A_j(x)\right|\leqslant const$, $\left|B_j(x)\right|\leqslant const$, $\left|C_j(x)\right|\leqslant const$ and taking into account (41), (42), (43) from (52), (53), (54), (55), we deduce

$$\left|K_j(x,\xi)\right|\leqslant const,\quad \left|F_j(x)\right|\leqslant const,\ \ (j=1,2).$$

Consequently, solution of the integral equations (50) and (51) we can write via resolvent-kernel:

(56) $$\tau_1(x)=F_1(x)-\int\limits_q^1\Re_1(x,s)F_1(s)ds,$$

$$\tau_2(x)=F_2(x)-\int\limits_{-1}^{-q}\Re_2(x,s)F_2(s)ds, \tag{57}$$

where $\Re_j(x,s)$ is the resolvent-kernel of $K_j(x,s)$, $(j=1,2)$. Unknown functions $\nu_1^-(x)$ and $\nu_2^-(x)$ will be found from (25) and (27). Hence, after founding $\tau_1(x)$ and $\tau_2(x)$ a solution of the Problem I in domains Ω_{01} and Ω_{02} accordingly we write as follows

$$u(x,y)=\int\limits_0^y G_\xi(x,y,q,\eta)\psi_1(\eta)d\eta-\int\limits_0^y G_\xi(x,y,1,\eta)\varphi_1(\eta)d\eta+\int\limits_q^1\tilde{G}(x,\xi,y)\tau_1(\xi)d\xi$$

$$u(x,y)=\int\limits_0^y G_\xi(x,y,-1,\eta)\varphi_2(\eta)d\eta-\int\limits_0^y G_\xi(x,y,-q,\eta)\psi_2(\eta)d\eta+\int\limits_{-1}^{-q}\tilde{G}^*(x,\xi,y)\tau_2(\xi)d\xi,$$

where

$$\tilde{G}(x,\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^y\eta^{-\alpha}G(x,y,\xi,\eta)d\eta,$$

$$\tilde{G}^*(x,\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^y\eta^{-\alpha}G^*(x,y,\xi,\eta)d\eta,$$

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{+\infty}\left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{\left|x-\frac{\xi-q}{1-q}+2n\right|}{(y-\eta)^{\alpha/2}}\right)-e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{\left|x+\frac{\xi-q}{1-q}+2n\right|}{(y-\eta)^{\alpha/2}}\right)\right];$$

$$G^*(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{+\infty}\left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{\left|x-\frac{\xi+q}{q-1}+2n\right|}{(y-\eta)^{\alpha/2}}\right)-e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{\left|x+\frac{\xi+q}{q-1}+2n\right|}{(y-\eta)^{\alpha/2}}\right)\right],$$

are the Greens function of the first boundary problem for the Eq.(12), with the Riemann-Liouville fractional differential operator instead of the Caputo ones [30]

$$e^{1,\delta}_{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)}$$

is the Wright type functions [29, 30]. Solution of the Problem I can be restored in the domain Δ_j (j=1,2) as the solution of the Cauchy problem (see (24) and (26)). The solution of the Problem I in domains of Δ_{j+2} (j=1,2), we can restore as a solution of the Goursat problem with conditions (20) and $u(x,y)\Big|_{B_jE_j}=h_j(x)$, where $h_j(x)$ (j=1,2) are traces of the solution of the Cauchy problems in domains Δ_j (j=1,2), on the line $y-(-1)^jx=q$, and consequently in domain Δ_5 as solution of the Goursat problem with conditions $u(x,y)\Big|_{C_2F_j}=\tilde{h}_j(x)$ (j=1,2) where $\tilde{h}_j(t)$ (j=1,2) are traces of solution of the Goursat problems in domains Δ_{j+2} (j=1,2). This completes the proof.

Acknowledgment

The author is grateful to the reviewers for useful suggestions which improved the contents of this paper.

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