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MULTIPLE POSITIVE SOLUTIONS FOR A SECOND-ORDER BOUNDARY VALUE PROBLEM ON THE HALF-LINE

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Abstract. In this paper we are concerned with the existence and localization of multiple positive solutions of a boundary value problem on the half-line associated to second-order differential equations. We use a variational approach based on critical point theory in conical shells and a Harnack type inequality. The specific compactness involved by the problems on infinite intervals is guaranteed by a growth condition on the nonlinear term which is tempered towards infinity.

Keywords. Critical point; Mountain-pass lemma; Compression; Positive solution; Harnack inequality.

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1. Introduction

We consider the existence and localization of positive solutions for the following second-order boundary value problem

$$\begin{cases} -u''(t)+ku(t) &= f(t,u(t)), \quad t\in(0,+\infty), \\ u(0)=u(+\infty) &= 0, \end{cases} \tag{1.1}$$

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where $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a continuous function with $f(\mathbb{R}^+\times\mathbb{R}^+)\subset\mathbb{R}^+$, and k is a positive number.

In addition we shall assume that there exists $\theta>0$, nonnegative functions a,b and a continuous function $p:\mathbb{R}^+\longrightarrow(0,+\infty)$ with

$$\lim_{t\to+\infty}\sqrt{t}\,p(t)=0, \tag{1.2}$$

such that $a,\ bp^{-\theta}\in L^2(\mathbb{R}^+)$ and

$$|f(t,s)|\le a(t)+b(t)\,|s|^{\theta}\quad \text{for all }\ t\in\mathbb{R}^+,\ s\in\mathbb{R}. \tag{1.3}$$

Then we can associate to the problem the functional

$$E(u)=\int_0^{+\infty}\left[\frac{1}{2}\left(u'^2+ku^2\right)-F(t,u(t))\right]dt,\ \text{ where }\ F(t,u)=\int_0^u f(t,s)ds, \tag{1.4}$$

defined on the space $H_0^1(\mathbb{R}^+)=\{u\in H^1(\mathbb{R}^+):\ u(0)=0\}$ whose critical points are exactly the solutions of the problem. Note that the asymptotic condition $u(+\infty)=0$ holds for every function $u\in H^1(\mathbb{R}^+)$ (see [1, Corollary 8.9]).

Thus we use a variational approach contrarily to most papers on boundary value problems on infinite intervals which are commonly based on fixed point arguments and reduction to a sequence of compact-interval problems, see, e.g., [2], [3], and particularly the papers [4] and [5], where the approach to problem (1.1) is based on Krasnoselskii's compression-expansion fixed point theorem.

Looking for positive solutions to (1.1) we are led to work in the positive cone of $H_0^1(\mathbb{R}^+)$, and accordingly to use a critical point theory in cones. In addition we should be interested to localize positive solutions u in an annular set, that is with

$$R_0\le|u|\le R_1, \tag{1.5}$$

where $0<R_0<R_1$ and $|u|$ is the norm of $H_0^1(\mathbb{R}^+)$. Then multiple solutions can be obtained if the hypotheses of the localization result are satisfied for different pairs of radii R_0,R_1. To guarantee the lower estimate $R_0\le|u|$, a Harnack type inequality is necessary. Unfortunately, for most problems including problem (1.1), such an inequality in terms of the energetic norm, here $|.|$, is not known and difficult to be done (see [6] and [7]). However, as shown in [7], this drawback can be over come by using together with the energetic norm of a second norm $\|.\|$

from a larger space, in our case the norm of $L^2(0,T)$ for some fixed finite T. Thus, instead of (1.5), we will be able to obtain a localization of solutions in the following way

$$R_0 \leq \|u\| \text{ and } |u| \leq R_1.$$

The aim of this paper is to apply the corresponding critical point theory to the semi-line problem (1.1). This requires first to give the variational structure of the problem, then to obtain a Harnack type inequality, and finally to guarantee the invariance, compactness and compression conditions from the general critical point theorems which are used. This way, our approach completely differs from those in [8] and [9].

2. Preliminaries

2.1. Spaces of functions

In this paper we consider the Hilbert spaces: $H_0^1(\mathbb{R}^+)$ endowed with the inner product and norm

$$(u,v) = \int_0^{+\infty} \left(u'v' + kuv\right) dt, \quad |u| = \left(\int_0^{+\infty} \left(u'^2 + ku^2\right) dt\right)^{\frac{1}{2}},$$

$L^2(\mathbb{R}^+)$ with the inner product and norm

$$(u,v)_{2,\infty} = \int_0^{+\infty} uvdt, \quad \|u\|_{2,\infty} = \left(\int_0^{+\infty} u^2 dt\right)^{\frac{1}{2}},$$

and $L^2(0,T)$, for $T>0$, endowed with the inner product and norm

$$(u,v)_{2,T} = \int_0^{T} uvdt, \quad \|u\|_{2,T} = \left(\int_0^{T} u^2 dt\right)^{\frac{1}{2}}.$$

Also, we consider the Banach space

$$C_{l,p}\left(\mathbb{R}^+\right) = \left\{u \in C\left(\mathbb{R}^+\right) : \lim_{t\to+\infty} p(t)u(t) = 0\right\},$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t\in\mathbb{R}^+} p(t)|u(t)|,$$

and the Banach space

$$C_l\left(\mathbb{R}^+\right) = \left\{u \in C\left(\mathbb{R}^+\right) : \lim_{t\to+\infty} u(t) \text{ exists}\right\},$$

with the norm

$$\|u\|_l = \sup_{t\in\mathbb{R}^+} |u(t)|.$$

2.2. Corduneanu compacteness criteria

In order to prove that $H_0^1(\mathbb{R}^+)$ embeds compactly in $C_{l,p}(\mathbb{R}^+)$, we need the following Corduneanu's compactness criterion in the space $C_l(\mathbb{R}^+)$.

Lemma 2.1. [10] *A bounded set $D \subset C_l(\mathbb{R}^+)$ is relatively compact if the following conditions hold:*

(i) *D is equicontinuous on any compact subinterval of $\mathbb{R}^+$, i.e.,*

$$\text{for each compact } \mathscr{I} \subset \mathbb{R}^+ \text{ and } \varepsilon > 0, \text{ there exists } \delta_\varepsilon > 0 \text{ such that}$$
$$|u(t_1) - u(t_2)| \leq \varepsilon \text{ for all } t_1, t_2 \in \mathscr{I} \text{ with } |t_1 - t_2| < \delta_\varepsilon \text{ and all } u \in D.$$

(ii) *D is equiconvergent at $+\infty$, i.e.,*

$$\text{for each } \varepsilon > 0, \text{ there exists } T_\varepsilon > 0 \text{ such that}$$
$$|u(t) - u(+\infty)| \leq \varepsilon \text{ for every } t \geq T_\varepsilon \text{ and all } u \in D.$$

Using Lemma 2.1, we have the following result.

Lemma 2.2. *A bounded set $D \subset C_{l,p}(\mathbb{R}^+)$ is relatively compact if the following conditions hold:*

(a) *D is p-equicontinuous on any compact subinterval of $\mathbb{R}^+$, i.e.,*

$$\text{for each compact } \mathscr{I} \subset \mathbb{R}^+ \text{ and } \varepsilon > 0, \text{ there exists } \delta_\varepsilon > 0 \text{ such that}$$
$$|(pu)(t_1) - (pu)(t_2)| \leq \varepsilon \text{ for all } t_1, t_2 \in \mathscr{I} \text{ with } |t_1 - t_2| < \delta_\varepsilon \text{ and all } u \in D.$$

(b) *D is p-equiconvergent at $+\infty$, i.e.,*

$$\text{for each } \varepsilon > 0, \text{ there exists } T_\varepsilon > 0 \text{ such that}$$
$$|(pu)(t) - (pu)(+\infty)| \leq \varepsilon \text{ for every } t \geq T_\varepsilon \text{ and all } u \in D.$$

Proof. Assume that D is a bounded set of $C_{l,p}(\mathbb{R}^+)$ for which the conditions (a) and (b) hold. Then the set pD is bounded in $C_l(\mathbb{R}^+)$ and satisfies (i) and (ii). Hence, in virtue of Lemma 2.1, the set pD is relatively compact in $C_l(\mathbb{R}^+)$, which immediately implies that D is relatively compact in $C_{l,p}(\mathbb{R}^+)$ as wished.

2.3. Some embedding results

Lemma 2.3. *The embedding $H_0^1(\mathbb{R}^+) \subset C_{l,p}(\mathbb{R}^+)$ is compact.*

Proof. First we show that $H_0^1(\mathbb{R}^+)$ is a subspace of $C_{l,p}(\mathbb{R}^+)$. Indeed, if $u \in H_0^1(\mathbb{R}^+)$, then

$$p(t)\,|u(t)| = p(t)\left|\int_0^t u'(s)ds\right| \le p(t)\int_0^t |u'(s)|\,ds,$$

and using the Cauchy-Schwarz inequality we derive

$$\begin{aligned} p(t)|u(t)| &\le p(t)\left(\int_0^t u'^2(s)ds\right)^{\frac{1}{2}}\left(\int_0^t ds\right)^{\frac{1}{2}} \\ &\le \sqrt{t}\,p(t)\left(\int_0^{+\infty} u'^2(s)ds\right)^{\frac{1}{2}} \le \sqrt{t}p(t)|u|. \end{aligned} \tag{2.1}$$

Since $\sqrt{t}p(t) \to 0$ as $t \to \infty$, this yields $p(t)u(t) \to 0$ as $t \to \infty$. Hence $u \in C_{l,p}(\mathbb{R}^+)$. From (2.1), we also deduce

$$p(t)|u(t)| \le |u| \sup_{t\in\mathbb{R}^+} \sqrt{t}p(t).$$

This yields

$$\|u\|_{\infty,p} \le c_{\infty,p}|u|, \tag{2.2}$$

where $c_{\infty,p} = \sup_{t\in[0,+\infty)} \sqrt{t}p(t)$. The inequality (2.2) which holds for all u in $H_0^1(\mathbb{R}^+)$ shows that the embedding of $H_0^1(\mathbb{R}^+)$ into $C_{l,p}(\mathbb{R}^+)$ is continuous.

Next, we prove that the closed unit ball B of $H_0^1(\mathbb{R}^+)$ is relatively compact in $C_{l,p}(\mathbb{R}^+)$. From (2.2), B is bounded in $C_{l,p}(\mathbb{R}^+)$. The conclusion will follow from Lemma 2.2 once we have proved for B the p-equicontinuity on compacts of $\mathbb{R}^+$ and the p-equiconvergence at $+\infty$.

(a) B is p-equicontinuous on every compact subinterval $\mathscr{I}$ of $\mathbb{R}^+$. Indeed, let $u \in B$ and $t_1, t_2 \in \mathscr{I}$, $t_1 < t_2$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(pu)(t_1) - (pu)(t_2)| &= \left|p(t_1)\int_0^{t_1} u'(s)ds - p(t_2)\int_0^{t_2} u'(s)ds\right| \\ &= \left|p(t_1)\int_0^{t_1} u'(s)ds - p(t_2)\left(\int_0^{t_1} u'(s)ds + \int_{t_1}^{t_2} u'(s)ds\right)\right| \\ &= \left|(p(t_1)-p(t_2))\int_0^{t_1} u'(s)ds - p(t_2)\int_{t_1}^{t_2} u'(s)ds\right| \\ &\le \sqrt{t_1}\,|p(t_1)-p(t_2)|\left(\int_0^{t_1} u'^2(s)ds\right)^{\frac{1}{2}} \\ &+ \left(\sqrt{t_2-t_1}\right)p(t_2)\left(\int_{t_1}^{t_2} u'^2(s)ds\right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\left(\int_{t_1}^{t_2} u'^2(s)ds\right)^{\frac{1}{2}} \le |u| \le 1,$$

we deduce that

$$|(pu)(t_1)-(pu)(t_2)| \le \sqrt{t_1}\,|p(t_1)-p(t_2)| + \left(\sqrt{t_2-t_1}\right)p(t_2).$$

Since p is bounded and uniformly continuous on $\mathscr{I}$, we immediately deduce the p-equicontinuity of B.

(b) B is p-equiconvergent at $+\infty$. Indeed, for $t\in\mathbb{R}^+$ and $u\in B$, using the fact that $(pu)(+\infty)=0$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(pu)(t)-(pu)(+\infty)| &= |p(t)u(t)| = p(t)\left|\int_0^t u'(s)ds\right| \\ &\le p(t)\sqrt{t}\left(\int_0^t u'^2(s)ds\right)^{\frac{1}{2}} \le p(t)\sqrt{t}. \end{aligned}$$

This, since $p(t)\sqrt{t}\to 0$ as $t\to+\infty$, shows that B is p-equiconvergent at $+\infty$.

Hence B is relatively compact in $C_{l,p}(\mathbb{R}^+)$, and the embedding $H_0^1(\mathbb{R}^+)\subset C_{l,p}(\mathbb{R}^+)$ is compact. This completes the proof.

Lemma 2.4. *For every* $0<T<+\infty$, *the embedding* $C_{l,p}(\mathbb{R}^+)\subset L^2(0,T)$ *is continuous.*

Proof. For $u\in C_{l,p}(\mathbb{R}^+)$, we have

$$\begin{aligned} \|u\|_{2,T}^2 &= \int_0^T |u^2(t)|dt = \int_0^T \frac{1}{p^2(t)}|p(t)u(t)|^2dt \\ &\le \left(\max_{t\in[0,T]} p(t)|u(t)|\right)^2 \int_0^T \frac{1}{p^2(t)}dt \le \|u\|_{\infty,p}^2 \|\frac{1}{p}\|_{2,T}^2. \end{aligned}$$

Hence

$$\|u\|_{2,T} \le c_2\|u\|_{\infty,p},$$

where $c_2=\|1/p\|_{2,T}$.

Theorem 2.5. *For every* $0<T<+\infty$, *the embedding* $H_0^1(\mathbb{R}^+)\subset L^2(0,T)$ *is compact.*

Proof. The result follows from Lemma 2.3 and Lemma 2.4.

2.4. The Nemytskii operator in $C_{l,p}(\mathbb{R}^+)$

Lemma 2.6. *Assume that the growth condition (1.3) holds. Then the Nemytskii operator,*

$$(N_f u)(t) = f(t,u(t))$$

from $C_{l,p}(\mathbb{R}^+)$ *to* $L^2(\mathbb{R}^+)$, *is well-defined, continuous and bounded (i.e., maps bounded sets into bounded sets).*

Proof. Step 1 : N_f is well-defined from $C_{l,p}(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$, and bounded. Let $u \in C_{l,p}(\mathbb{R}^+)$. Using the growth condition (1.3) and Minkowski inequality we have

$$\begin{aligned} \|N_f u\|_{2,\infty} \leq \left\| a + b|u|^\theta \right\|_{2,\infty} &\leq \|a\|_{2,\infty} + \left\| b|u|^\theta \right\|_{2,\infty} \\ &= \|a\|_{2,\infty} + \left\| \frac{b}{p^\theta} |pu|^\theta \right\|_{2,\infty} \\ &\leq \|a\|_{2,\infty} + \|u\|_{\infty,p}^\theta \|\frac{b}{p^\theta}\|_{2,\infty}. \end{aligned}$$

This first shows that $\|N_f u\|_{2,\infty} < +\infty$, and so $N_f u \in L^2(\mathbb{R}^+)$, and then that the operator N_f is bounded.

Step 2 : N_f is continuous. Let (u_n) be a sequence in $C_{l,p}(\mathbb{R}^+)$ such that $u_n \to u$ in the $C_{l,p}(\mathbb{R}^+)$-norm. We have to show that $N_f u_n \to N_f u$ in the $L^2(\mathbb{R}^+)$-norm. From $u_n \to u$, we have that there exist $R > 0$, such that $\|u_n\|_{\infty,p} \leq R$ for every n, $\|u\|_{\infty,p} \leq R$ and $p(t)u_n(t) \to p(t)u(t)$ for every $t \in \mathbb{R}^+$. Since $p(t) \neq 0$ for every $t \in \mathbb{R}^+$, then $u_n(t) \to u(t)$ for every $t \in \mathbb{R}^+$, and from the continuity of f, we have $f(t,u_n(t)) \to f(t,u(t))$ for every $t \in \mathbb{R}^+$. Therefore $N_f u_n$ converges to $N_f u$ pointwise. Furthermore, using (1.3) we find

$$\begin{aligned} |N_f u_n(t) - N_f u(t)|^2 &\leq \left(|N_f u_n(t)| + |N_f u(t)|\right)^2 \leq 2\left(|N_f u_n(t)|^2 + |N_f u(t)|^2\right) \\ &\leq 2\left\{ \left(a(t) + b(t)|u_n(t)|^\theta\right)^2 + \left(a(t) + b(t)|u(t)|^\theta\right)^2 \right\} \\ &\leq 4\left\{ a^2(t) + b^2(t)|u_n(t)|^{2\theta} + a^2(t) + b^2(t)|u(t)|^{2\theta} \right\} \\ &= 8a^2(t) + 4b^2(t)\left(|u_n(t)|^{2\theta} + |u(t)|^{2\theta}\right). \end{aligned}$$

Next

$$\begin{aligned} b^2(t)\left(|u_n(t)|^{2\theta} + |u(t)|^{2\theta}\right) &\leq \left(\frac{b(t)}{p^\theta(t)}\right)^2 \left(\|u_n\|_{\infty,p}^{2\theta} + \|u\|_{\infty,p}^{2\theta}\right) \\ &\leq 2\left(\frac{b(t)}{p^\theta(t)}\right)^2 R^{2\theta}. \end{aligned}$$

Consequently, we have

$$|N_f u_n(t) - N_f u(t)|^2 \leq 8a^2(t) + 8\left(\frac{b(t)}{p^\theta(t)}\right)^2 R^{2\theta} =: g(t).$$

Since a and $bp^{-\theta} \in L^2(\mathbb{R}^+)$, it follows that $g \in L^1(\mathbb{R}^+)$. Then from the Lebesgue dominated convergence theorem, we have

$$\int_0^{+\infty} |N_f u_n(t) - N_f u(t)|^2 dt \to 0 \quad \text{as } n \to +\infty.$$

Hence $N_f u_n \to N_f u$ in $L^2(\mathbb{R}^+)$ proving the continuity of N_f.

Theorem 2.7. *If the growth condition (1.3) is satisfied, then the Nemytskii operator* $N_f : H_0^1(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ *is compact (i.e., continuous and maps bounded sets into relatively compact sets).*

Proof. From Lemma 2.3, the embedding $H_0^1(\mathbb{R}^+) \subset C_{l,p}(\mathbb{R}^+)$ is compact, while from Lemma 2.6, N_f is continuous and bounded from $C_{l,p}(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$. Consequently, N_f is compact from $H_0^1(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$.

2.5. Critical point theorems in conical shells

In [11], starting from the bounded critical point theory of Schechter [12], [13], the third author developed a critical point theory in conical shells defined by means of two norms. This represents the main tool of the present paper. We now present shortly this theory.

Consider a real Hilbert space X with inner product and norm $(.,.), |.|$, and a Banach space H with norm $\|.\|$, and assume that there exists a linear continuous map $\mathscr{S} : X \to H$ by which we shall identify X with the linear subspace $\mathscr{S}(X)$ of H, and any element $u \in X$ with its image $\mathscr{S}u \in H$. Thus we shall say that $X \subset H$ and the embedding is continuous. When $\mathscr{S}$ is compact we say that the embedding is compact.

By $\langle .,. \rangle$ we denote the naturel duality mapping between X and X', that is $\langle u^*, u \rangle = u^*(u)$ for $u \in X$, $u^* \in X'$, and also the natural duality mapping between H and H'. We consider a C^1 normalization function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, i.e., continuously differentiable, with $\varphi(0) = 0, \varphi'(\tau) > 0$ for every $\tau > 0$ and $\varphi(\tau) \to \infty$ as $\tau \to \infty$, and we denote by P the duality map on H associated with φ, assumed to be continuous single-valued map, namely $P : H \to H'$

$$\langle Pu, u \rangle = \|Pu\|_{H'} \|u\|, \quad \|Pu\|_{H'} = \varphi(\|u\|) \quad \text{for all}\, u \in H.$$

Note that if $u \in X$, then by Pu it is understood $P\mathscr{S}u$.

In [11], it was assumed that H has the additional property that for each function $u\in C^1(\mathbb{R}^+,H)$, function $Pu(t)$ belong to $C^1(\mathbb{R}^+,H')$ and

$$\langle \frac{d}{dt}(Pu(t)),u(t)\rangle = \varkappa\langle Pu(t),u'(t)\rangle \quad \text{for all}\, t\in\mathbb{R}^+$$

and some constant $\varkappa\in\mathbb{R}^+$. The raison of such a condition was to guarantee that $\langle Pu(t),u'(t)\rangle$ and $d\|u(t)\|/dt$ have the same sign. However, this assumption is not necessary as shown recently in the paper [14].

Let L be the continuous linear operator from X to X' (the canonical isomorphism of X into X') defined by

$$\langle Lu,v\rangle = (u,v), \quad \text{for all}\, u,\, v\in X, \tag{2.3}$$

and let J from X' to X be the inverse of L. Then

$$(Ju,v)=\langle u,v\rangle, \quad \text{for all } u\in X',\, v\in X.$$

Let K be a wedge in X, i.e. a convex closed nonempty set K, $K\neq\{0\}$, with $\lambda\, u\in K$ for every $u\in K$ and $\lambda\geq 0$. Notice that K can be a cone, i.e., $K\cap(-K)=\{0\}$; a closed semi-space; and also can be the whole space X.

For any two positive numbers R_0 and R_1, we denote by $K_{R_0R_1}$ the *conical shell*

$$K_{R_0R_1} := \{u\in K \ : \ \|u\|\geq R_0 \ \text{ and } \ |u|\leq R_1\}.$$

Such a set may be empty (even if $R_0<R_1$) and may be disconnected. However, if $\phi\in K\setminus\{0\}$ is a fixed element with $|\phi|=1$, and $R_0<\|\phi\|R_1$, then $\mu\phi\in K_{R_0R_1}$ for every $\mu\in[R_0/\|\phi\|,R_1]$, and $\mu\phi$ is an interior point of $K_{R_0R_1}$, in the sense that $\|\mu\phi\|>R_0$ and $|\mu\phi|<R_1$, for $\mu\in(R_0/\|\phi\|,R_1)$. In particular, any two elements of $K_{R_0R_1}$ of the form $\mu\phi$, with $\mu\in[R_0/\|\phi\|,R_1]$, belong to the same connected component of $K_{R_0R_1}$.

Let E be a C^1 functional defined on X. We say that E has a *mountain pass geometry* in $K_{R_0R_1}$ if there exist u_0 and u_1 in the same connected component of $K_{R_0R_1}$, and $r>0$ such that $|u_0|<r<|u_1|$ and

$$\max\{E(u_0),E(u_1)\}<\inf\{E(u):\ u\in K_{R_0R_1},\ |u|=r\}.$$

In this case we consider the set

$$\Gamma=\{\gamma\in C([0,1];K_{R_0R_1}) \ : \ \gamma(0)=u_0,\ \gamma(1)=u_1\}, \tag{2.4}$$

and the number

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}E(\gamma(t)). \tag{2.5}$$

Finally, we say that E is *bounded from below* in $K_{R_0R_1}$ if

$$m:=\inf_{u\in K_{R_0R_1}}E(u)>-\infty. \tag{2.6}$$

We assume that the following conditions are satisfied:

(c1): The *invariance* condition:

$$JP(K)\subset K \text{ and } \left(I-JE'\right)(K)\subset K, \quad (I \text{ is the identity map on } X). \tag{2.7}$$

(c2): The *boundedness* condition on the shell boundary: there exists a constant $\nu_0>0$ such that

$$(JE'(u),JPu)\leq\nu_0 \text{ for all } u\in K \text{ with } \|u\|=R_0; \tag{2.8}$$

$$(JE'(u),u)\geq-\nu_0 \text{ for all } u\in K \text{ with } |u|=R_1. \tag{2.9}$$

(c3): The *compactness* condition:

the maps JP and $N=I-JE'$ are compact from X to itself.

(c4): The *compression* condition:

$$JE'(u)-\lambda JPu\neq 0 \text{ for } u\in K_{R_0R_1},\ \|u\|=R_0,\ \lambda>0; \tag{2.10}$$

$$JE'(u)+\lambda u\neq 0 \text{ for } u\in K_{R_0R_1},\ |u|=R_1,\ \lambda>0. \tag{2.11}$$

Remark 2.8. Let $N(u)=u-JE'(u)$. Then conditions (2.10) and (2.11) can be written in the form

$$N(u)-\lambda JPu\neq 0 \textit{ for } u\in K_{R_0R_1},\ \|u\|=R_0,\ \lambda>0; \tag{2.12}$$

$$N(u)+(1+\lambda)u\neq 0 \textit{ for } u\in K_{R_0R_1},\ |u|=R_1,\ \lambda>0. \tag{2.13}$$

According to [11], we have the following theorems of localization of critical points in a conical shell.

Theorem 2.9. *Let the conditions* (c1)-(c4) *hold. If in addition E is bounded from below in $K_{R_0R_1}$ and that there is a $\rho>0$ with*

$$E(u)\geq m+\rho \tag{2.14}$$

(m *is given in* (2.6)) *for all* $u \in K_{R_0R_1}$ *which simultaneously satisfy* $|u| = R_1,\ \|u\| = R_0$, *then there exists* $u \in K_{R_0R_1}$ *such that*

$$E'(u) = 0 \quad and \quad E(u) = m.$$

Theorem 2.10. *Let conditions* (c1)-(c4) *hold. If in addition* E *has the mountain pass geometry in* $K_{R_0R_1}$ *and that there is a* $\rho > 0$ *with*

$$|E(u) - c| \geq \rho \tag{2.15}$$

(c *is given in* (2.5)) *for all* $u \in K_{R_0R_1}$ *which simultaneously satisfy* $|u| = R_1,\ \|u\| = R_0$, *then there exists* $u \in K_{R_0R_1}$ *such that*

$$E'(u) = 0 \quad and \quad E(u) = c.$$

Remark 2.11. If the assumptions of both Theorems 2.9 and 2.10 are satisfied, since $m < c$, then E has two distinct critical points in $K_{R_0R_1}$.

2.6. The variational setting

Let us now give the variational setting for our boundary value problem on semi-line. In order to apply the results from Section 2.5, we consider the spaces $X = H_0^1(\mathbb{R}^+)$ and $H = L^2(0,T)$ with inner products and norms as defined in Section 2.1. Since $L^2(0,T)$ is a Hilbert space, we may identify it to its dual and so in this case, the duality map P (with respect to the normalization function $\varphi(\tau) = \tau$) is the identity map of $L^2(0,T)$, i.e., $Pu = u$ for every $u \in L^2(0,T)$. We have

$$H_0^1\left(\mathbb{R}^+\right) \subset L^2\left(\mathbb{R}^+\right) \subset H^{-1}\left(\mathbb{R}^+\right)$$

with continuous embeddings, and

$$\langle u, v\rangle = (u,v)_{2,\infty} = \int_0^{+\infty} u(t)v(t)dt, \quad \text{for all } u \in L^2\left(\mathbb{R}^+\right) \text{ and } v \in H_0^1\left(\mathbb{R}^+\right).$$

We denote by c_1 the embedding constant for $H_0^1(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^+) \subset H^{-1}(\mathbb{R}^+)$, i.e.,

$$\|u\|_{2,\infty} \leq c_1|u| \ \left(u \in H_0^1\left(\mathbb{R}^+\right)\right) \ \text{and} \ \|u\|_{H^{-1}(0,+\infty)} \leq c_1\|u\|_{2,\infty} \left(u \in L^2\left(\mathbb{R}^+\right)\right).$$

Also, in our case, $L : H_0^1(\mathbb{R}^+) \to H^{-1}(\mathbb{R}^+)$ is given by

$$\langle Lu, v\rangle = \langle -u'' + ku, v\rangle = (u,v), \quad \text{for } u,v \in H_0^1\left(\mathbb{R}^+\right).$$

The inverse of L is the operator $J : H^{-1}(\mathbb{R}^+) \to H_0^1(\mathbb{R}^+)$ defined by

$$(Jv, w) = \langle v, w \rangle, \quad \text{for } v \in H^{-1}\left(\mathbb{R}^+\right), \, w \in H_0^1\left(\mathbb{R}^+\right).$$

By direct computation we can give the expression of the restriction of J to $L^2(\mathbb{R}^+)$ in terms of Green's function. Also, some good properties of Green's function are essential in what follows.

Let G be the Green's function (see [5]) associated to the linear problem

$$\begin{cases} -u''(t) + ku(t) & = v(t), \quad t \in (0, +\infty), \\ u(0) = u(+\infty) & = 0, \end{cases} \tag{2.16}$$

namely

$$G(t,s) = \frac{1}{2\sqrt{k}} \begin{cases} e^{-\sqrt{k}s}(e^{\sqrt{k}t} - e^{-\sqrt{k}t}), & \text{if } 0 \le t \le s, \\ e^{-\sqrt{k}t}(e^{\sqrt{k}s} - e^{-\sqrt{k}s}), & \text{if } 0 \le s \le t, \end{cases}$$

and let G_t be its partial derivative with respect to t,

$$G_t(t,s) = \frac{1}{2} \begin{cases} e^{-\sqrt{k}s}(e^{\sqrt{k}t} + e^{-\sqrt{k}t}), & \text{if } 0 \le t < s, \\ -e^{-\sqrt{k}t}(e^{\sqrt{k}s} - e^{-\sqrt{k}s}), & \text{if } 0 \le s < t. \end{cases}$$

It is easy to see that the following properties hold.

Lemma 2.12. (a) $0 < G(t,s)e^{-\sqrt{k}t} \le G(s,s)e^{-\sqrt{k}s}$, *for all* $t, s \in (0, +\infty)$.
(b) *For every* γ *and* δ *with* $0 < \gamma < \delta$, *one has* $G(t,s) \ge \Lambda G(s,s)e^{-\sqrt{k}s}$, *for all* $t \in [\gamma, \delta]$ *and* $s \in \mathbb{R}^+$, *where* $0 < \Lambda = \min\{e^{-\sqrt{k}\delta}, e^{\sqrt{k}\gamma} - e^{-\sqrt{k}\gamma}\}$.

(c) *For all* $t, s \in \mathbb{R}^+$,

$$\int_0^{+\infty} G(t,s)ds \le \frac{1}{k}, \quad \int_0^{+\infty} G(t,s)dt \le \frac{1}{k}, \tag{2.17}$$

$$\int_0^{+\infty} |G_t(t,s)|\, ds \le \frac{1}{\sqrt{k}} \ \text{ and } \ \int_0^{+\infty} |G_t(t,s)|\, dt \le \frac{1}{\sqrt{k}}. \tag{2.18}$$

Lemma 2.13. *For each* $v \in L^2(\mathbb{R}^+)$, *one has*

$$(Jv)(t) = \int_0^{+\infty} G(t,s)v(s)ds \quad \left(t \in \mathbb{R}^+\right). \tag{2.19}$$

Proof. Let $v \in L^2(\mathbb{R}^+)$ and denote

$$u(t) := \int_0^{+\infty} G(t,s)\, v(s)\, ds.$$

To prove this lemma, it is enough to show that u belongs to $H_0^1(\mathbb{R}^+)$ and weakly solves the problem (2.16). From the construction of Green's function we have that u solves the equation $-u''+ku=v$. Also it is clear that $u(0)=0$. It remains to prove that $u\in H^1(\mathbb{R}^+)$.

Step 1: $u\in L^2(\mathbb{R}^+)$. Indeed, using the Cauchy-Schwarz inequality and the property (2.17) of Lemma 2.10, we have

$$\begin{aligned}|u(t)| &\leq \int_0^{+\infty} G(t,s)|v(s)|ds = \int_0^{+\infty} G(t,s)^{\frac{1}{2}}G(t,s)^{\frac{1}{2}}|v(s)|ds\\ &\leq \left(\int_0^{+\infty} G(t,s)ds\right)^{\frac{1}{2}}\left(\int_0^{+\infty} G(t,s)|v(s)|^2ds\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{k}}\left(\int_0^{+\infty} G(t,s)|v(s)|^2ds\right)^{\frac{1}{2}}.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_0^{+\infty}|u(t)|^2\quad dt &\leq \int_0^{+\infty}\left(\frac{1}{k}\int_0^{+\infty} G(t,s)|v(s)|^2ds\right)dt\\ &= \frac{1}{k}\int_0^{+\infty}|v(s)|^2\left(\int_0^{+\infty} G(t,s)dt\right)ds\\ &\leq \frac{1}{k^2}\int_0^{+\infty}|v(s)|^2ds<+\infty.\end{aligned}$$

Hence $u\in L^2(\mathbb{R}^+)$ and $\|u\|_{2,\infty}\leq k^{-1}\|v\|_{2,\infty}$.

Step 2: $u'\in L^2(\mathbb{R}^+)$. One has

$$u'(t)=\int_0^{+\infty} G_t(t,s)\,v(s)\,ds.$$

Then using a similar reasoning based this time on property (2.18) from Lemma 2.10, we deduce that $\|u'\|_{2,\infty}\leq k^{-1/2}\|v\|_{2,\infty}$.

Thus $u\in H^1(\mathbb{R}^+)$ as claimed.

The next result is concerning with the energy functional (1.4) associated to the boundary value problem.

Lemma 2.14 *Assume that the growth condition (1.3) holds. Then the energy functional E defined by (1.4) is of class C^1, bounded from bellow on each bounded subset of $H_0^1(\mathbb{R}^+)$, and*

$$E'(u)=Lu-N_f u \quad \textit{in } H^{-1}\left(\mathbb{R}^+\right).$$

Proof. Step 1 : The functional E is well-defined. Indeed, for $u \in H_0^1(\mathbb{R}^+)$, the growth condition (1.3) implies that

$$\begin{aligned} |F(t,u(t))| &= \left|\int_0^{u(t)} f(t,s)ds\right| \leq \left|\int_0^{u(t)} \left(a(t)+b(t)|s|^\theta\right) ds\right| \\ &\leq a(t)|u(t)| + \frac{b(t)}{\theta+1}|u(t)|^{\theta+1}. \end{aligned}$$

Then

$$\begin{aligned} \left|\int_0^{+\infty} F(t,u(t))dt\right| &\leq \int_0^{+\infty} |F(t,u(t))|\,dt \leq \int_0^{+\infty} \left(a(t)|u(t)| + \frac{b(t)}{\theta+1}|u(t)|^{\theta+1}\right) dt \\ &\leq \int_0^{+\infty} a(t)|u(t)|dt + \frac{1}{\theta+1}\int_0^{+\infty} \frac{b(t)}{p^\theta(t)}|u(t)||p(t)u(t)|^\theta dt \\ &\leq \int_0^{+\infty} a(t)|u(t)|dt + \frac{1}{\theta+1}\|u\|_{\infty,p}^\theta \int_0^{+\infty} \frac{b(t)}{p^\theta(t)}|u(t)|dt, \end{aligned}$$

and using the Cauchy-Schwarz inequality we obtain

$$\left|\int_0^{+\infty} F(t,u(t))dt\right| \leq \|a\|_{2,\infty}\|u\|_{2,\infty} + \frac{1}{\theta+1}\|u\|_{\infty,p}^\theta \|\frac{b}{p^\theta}\|_{2,\infty}\|u\|_{2,\infty}.$$

Since $H_0^1(\mathbb{R}^+)$ embeds continuously in $L^2(\mathbb{R}^+)$ and in $C_{l,p}(\mathbb{R}^+)$, then

$$\left|\int_0^{+\infty} F(t,u(t))dt\right| \leq \left(c_1\|a\|_{2,\infty} + \frac{c_1 c_{\infty,p}^\theta}{\theta+1}\|\frac{b}{p^\theta}\|_{2,\infty}|u|^\theta\right)|u|.$$

Hence E is well-defined.

Step 2 : We prove that E is Gâteaux differentiable on $H_0^1(\mathbb{R}^+)$. To this end it suffices to prove that only for the functional

$$\Phi(u) = \int_0^{+\infty} F(t,u(t))\,dt.$$

More exactly we show that for any $u, v \in H_0^1(\mathbb{R}^+)$,

$$\lim_{\tau\to 0}\frac{1}{\tau}\left(\Phi(u+\tau v) - \Phi(u)\right) = \int_0^{+\infty} f(t,u(t))v(t)dt,$$

or equivalently

$$\lim_{\tau\to 0}\int_0^{+\infty}\left[\frac{1}{\tau}\left(F(t,u(t)+\tau v(t)) - F(t,u(t))\right) - f(t,u(t))v(t)\right]dt = 0. \tag{2.20}$$

Since $F(t,.)$ is a primitive of $f(t,.)$, the function under the integral tends to zero as $\tau \to 0$ pointwise. In view of Lebesgue dominated convergence theorem, to have (2.20) it is sufficient that the integrant is dominated on $\mathbb{R}^+$ by an L^1 function independent of τ. By the mean value theorem, for each $t\in\mathbb{R}^+$ and $0<|\tau|<1$ there exists a number $\eta(t,\tau)$ such that $0\le\eta(t,\tau)\le 1$ and

$$\frac{1}{\tau}\left[F(t,u(t)+\tau v(t))-F(t,u(t))\right]-f(t,u(t))\,v(t)=\left[f(t,u(t)+\tau\eta(t,\tau)v(t))-f(t,u(t))\right]v(t).$$

Then

$$\begin{aligned}&\left|\frac{1}{\tau}\left[F(t,u(t)+\tau v(t))-F(t,u(t))\right]-f(t,u(t))\,v(t)\right|\\&\le\left|f(t,u(t)+\tau\eta(t,\tau)v(t))v(t)\right|+\left|f(t,u(t))\,v(t)\right|.\end{aligned}\tag{2.21}$$

By using the growth condition (1.3) again, the inequality $(x+y)^\theta\le c_0(x^\theta+y^\theta)$, $(x,y\in\mathbb{R}^+)$ where $c_0=\max\left\{1,2^{\theta-1}\right\}$, and $0\le\eta(t,\tau)\le 1$, we have

$$\begin{aligned}|f(t,u(t)+\tau\eta(t)v(t))v(t)|&\le\left\{a(t)+b(t)|u(t)+\tau\eta(t)v(t)|^\theta\right\}|v(t)|\\&\le\left\{a(t)+c_0b(t)\left(|u(t)|^\theta+|\tau\eta(t)|^\theta|v(t)|^\theta\right)\right\}|v(t)|\\&\le\left\{a(t)+c_0b(t)\left(|u(t)|^\theta+|v(t)|^\theta\right)\right\}|v(t)|.\end{aligned}\tag{2.22}$$

Similarly, we have

$$|f(t,u(t))\,v(t)|\le\left\{a(t)+c_0b(t)\left(|u(t)|^\theta+|v(t)|^\theta\right)\right\}|v(t)|.\tag{2.23}$$

Also

$$\begin{aligned}&\left\{a(t)+c_0b(t)\left(|u(t)|^\theta+|v(t)|^\theta\right)\right\}|v(t)|\\&=\left\{a(t)+c_0\frac{b(t)}{p^\theta(t)}\left(|p(t)u(t)|^\theta+|p(t)v(t)|^\theta\right)\right\}|v(t)|\\&\le\left\{a(t)+c_0\frac{b(t)}{p^\theta(t)}\left(\|u\|_{\infty,p}^\theta+\|v\|_{\infty,p}^\theta\right)\right\}|v(t)|\\&\le\frac{1}{2}\left\{a(t)+c_0\frac{b(t)}{p^\theta(t)}\left(\|u\|_{\infty,p}^\theta+\|v\|_{\infty,p}^\theta\right)\right\}^2+\frac{1}{2}|v(t)|^2\\&\le a^2(t)+c_0^2\left(\frac{b(t)}{p^\theta(t)}\right)^2\left(\|u\|_{\infty,p}^\theta+\|v\|_{\infty,p}^\theta\right)^2+\frac{1}{2}|v(t)|^2\\&\le a^2(t)+c_0^2c_{\infty,p}^\theta\left(\frac{b(t)}{p^\theta(t)}\right)^2\left(\|u\|^\theta+\|v\|^\theta\right)^2+\frac{1}{2}|v(t)|^2\equiv g(t).\end{aligned}\tag{2.24}$$

From (2.21)-(2.24), we have

$$\left|\frac{1}{\tau}\left[F(t,u(t)+\tau v(t))-F(t,u(t))\right]-f(t,u(t))\,v(t)\right|\le 2g(t).$$

Since $a,\ bp^{-\theta} \in L^2(\mathbb{R}^+)$, we have $g \in L^1(\mathbb{R}^+)$ as desired. Therefore E is Gâteaux differentiable on $H_0^1(\mathbb{R}^+)$ and

$$\langle E'(u), v\rangle = (u,v) - \int_0^{+\infty} f(t,u(t))v(t)dt = \left\langle Lu - N_f u, v\right\rangle.$$

Hence

$$E'(u) = Lu - N_f u \quad \text{in } H^{-1}\left(\mathbb{R}^+\right).$$

Since L is continuous from $H_0^1(\mathbb{R}^+)$ to $H^{-1}(\mathbb{R}^+)$, N_f is continuous from $H_0^1(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$ (Theorem 2.7), and $L^2(\mathbb{R}^+)$ embeds continuously in $H^{-1}(\mathbb{R}^+)$, we have that E' is continuous, so E is of class C^1.

Step 3: The functional E is bounded from below on each bounded subset of $H_0^1(\mathbb{R}^+)$. Indeed, if $u \in H_0^1(\mathbb{R}^+)$, then by using the Cauchy-Schwarz inequality we have

(1)
$$\begin{aligned} E(u) &= \frac{1}{2}|u|^2 - \int_0^{+\infty} F(t,u(t))dt \geq -\int_0^{+\infty} F(t,u(t))dt \\ &\geq -\int_0^{+\infty}\left(a(t)\,|u(t)| + \frac{b(t)}{\theta+1}\,|u(t)|^{\theta+1}\right)dt \\ &\geq -\|a\|_{2,\infty}\|u\|_{2,\infty} - \frac{1}{\theta+1}\|\frac{b}{p^\theta}\|_{2,\infty}\|u\|_{p,\infty}^{\theta}\|u\|_{2,\infty} \\ &\geq -c_1\|a\|_{2,\infty}|u| - \frac{c_1 c_{\infty,p}^{\theta}}{\theta+1}\|\frac{b}{p^\theta}\|_{2,\infty}|u|^{\theta+1}. \end{aligned}$$

Hence, if $|u| \leq C$, then $E(u) \geq -c_1\|a\|_{2,\infty}C - \frac{c_1 c_{\infty,p}^{\theta}}{\theta+1}\|\frac{b}{p^\theta}\|_{2,\infty}C^{\theta+1}$.

3. Main results

Our first result is a Harnack type inequality which is an essential tool for the subsequent estimations from below.

3.1. A Harnack type inequality

Lemma 3.1. *For given numbers $0 < \gamma < \delta$ and $0 < T < +\infty$, there is a constant $M > 0$ such that the following local Harnack inequality*

$$(Jv)(t) \geq M\|Jv\|_{2,T}, \quad t \in [\gamma, \delta] \tag{3.1}$$

holds for every $v \in L^2(\mathbb{R}^+)$ with $v \geq 0$ on $\mathbb{R}^+$.

Proof. Let v be such a function and denote $u(t) = (Jv)(t)$. Then

$$u(t) = \int_0^{+\infty} G(t,s)\, v(s)\, ds,$$

and for any $t \in [\gamma, \delta]$ and $\tau \in [0,T]$, using the properties of the Green's function, we have

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)v(s)ds \geq \Lambda \int_0^{+\infty} G(s,s)e^{-\sqrt{k}s}v(s)ds \\ &\geq \Lambda \int_0^{+\infty} G(\tau,s)e^{-\sqrt{k}\tau}v(s)ds = \Lambda e^{-\sqrt{k}\tau}u(\tau) \geq \Lambda e^{-\sqrt{k}T}u(\tau). \end{aligned}$$

Taking the power two and integrating in τ on the interval $[0,T]$ yields

$$u(t)\sqrt{T} \geq \Lambda e^{-\sqrt{k}T} \|u\|_{2,T}.$$

Hence (3.1) holds with $M = \Lambda e^{-\sqrt{k}T}/\sqrt{T}$.

3.2. Main existence and localization result

In connection with the Harnack type inequality (3.1), we fix the numbers γ, δ and T as required by Lemma 3.1 and we define a cone K in $H_0^1(\mathbb{R}^+)$, by

$$K = \{u \in H_0^1\left(\mathbb{R}^+\right) : u \geq 0 \text{ on } \mathbb{R}^+ \text{ and } u(t) \geq M\|u\|_{2,T} \text{ for } t \in [\gamma, \delta]\}.$$

As element $\phi \in K \setminus \{0\}$ we may take $\phi = Jv$, where $v \in L^2(\mathbb{R}^+)$ is any nonzero function such that $v \geq 0$ on $\mathbb{R}^+$. Then in virtue of Lemma 3.1, $\phi \in K$. Up to a positive multiplicative constant, we may assume that $|\phi| = 1$. From now on, ϕ is such a fixed function.

Also, for any two positive numbers, we define the conical shell

$$K_{R_0R_1} = \{u \in K : \quad \|u\|_{2,T} \geq R_0,\ |u| \leq R_1\}.$$

Let us denote

$$\omega(t) = \inf_{\tau \in \left[MR_0, \sqrt{\delta}R_1\right]} f(t,\tau),$$

$$\Phi_{R_0R_1}(t) = \int_\gamma^\delta G(t,s)\,\omega(s)\,ds, \quad \Psi_{R_1}(t) = c_1 \sup_{0 \leq \tau \leq \sqrt{t}R_1} f(t,\tau).$$

Our assumptions are as follows:

(H1): There exist R_0, R_1 with $0 < R_0 < \|\phi\|_{2,T} R_1$ such that

$$R_0 \leq \|\Phi_{R_0R_1}\|_{2,T}, \tag{3.2}$$

$$R_1 \geq \|\Psi_{R_1}\|_{2,\infty}. \tag{3.3}$$

(H2): There exists $\rho > 0$ such that

$$E(u) \geq m+\rho \quad \text{for all} \quad u \in K_{R_0R_1} \text{ satisfying simultaneously } |u| = R_1,\ \|u\|_{2,T} = R_0.$$

(H3): There exist u_0, u_1 in the same connected component of $K_{R_0R_1}$ and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in K_{R_0R_1},\ |u| = r\}.$$

(H4): There exists $\rho > 0$ such that

$$|E(u) - c| \geq \rho \quad \text{for all} \quad u \in K_{R_0R_1} \text{ satisfying simultaneously } |u| = R_1,\ \|u\|_{2,T} = R_0.$$

Note that the numbers m and c in (H2) and (H4) are defined by (2.6) and (2.5), respectively.

We have the following principles of existence and localization of solutions.

Theorem 3.2. *Assume that the conditions* (H1) *and* (H2) *hold. Then the problem (1.1) has at least one positive solution u_m in $K_{R_0R_1}$ such that*

$$E(u_m) = m.$$

If in addition (H3) *and* (H4) *hold, then a second positive solution u_c exists in $K_{R_0R_1}$ with*

$$E(u_c) = c.$$

Proof. We shall apply Theorems 2.8 and 2.9 to the functional E on $H_0^1(\mathbb{R}^+)$. First note that

from $E'(u) = Lu - N_f(u)$ we have

$$N(u) = u - JE'(u) = JN_f(u),$$

where

$$(JN_f(u))(t) = \int_0^{+\infty} G(t,s) f(s,u(s)) ds.$$

Check of the invariance condition (c1): The relation $JP(K) \subset K$ is immediate from Lemma 3.1. To prove that $(I - JE')(K) \subset K$, let $u \in K$ be arbitrary. Then $u \in H_0^1(\mathbb{R}^+)$, $u \geq 0$ on $\mathbb{R}^+$, and since N_f sends $H_0^1(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ and $f(\mathbb{R}^+ \times \mathbb{R}^+) \subset \mathbb{R}^+$, one has $N_f(u) \in L^2(\mathbb{R}^+)$ and $N_f(u) \geq 0$ on $\mathbb{R}^+$. Then Lemma 3.1 guarantees $JN_f(u) \in K$ as desired.

The boundedness condition on the shell boundary (c2), is satisfied since E' maps bounded sets into bounded sets.

Check of the compactness condition (c3): The operators JP and N are compact from $H_0^1(\mathbb{R}^+)$ to itself. The compactness of JP is a consequence of Theorem 2.5, while the compactness of $N = JN_f$ follows from Theorem 2.7.

Check of the compression condition (c4): (a) The condition (2.11) holds. Assume the contrary. Then there exist $u \in K_{R_0R_1}$ with $|u| = R_1$ and $\lambda > 0$ such that $JE'(u) + \lambda u = 0$, i.e., $N(u) = (1+\lambda)u$. Then

$$\begin{aligned} R_1^2 = |u|^2 &= (1+\lambda)^{-1}(N(u), u) < (JN_f(u), u) = \langle N_f(u), u\rangle \\ &\leq \|N_f(u)\|_{2,\infty}\|u\|_{2,\infty} \leq c_1|u|\|N_f(u)\|_{2,\infty} \qquad (3.4)\\ &= c_1R_1\|N_f(u)\|_{2,\infty}. \end{aligned}$$

From $u(t) \leq \sqrt{t}|u|$, we have $0 \leq u(t) \leq \sqrt{t}R_1$ for every $t \in \mathbb{R}^+$. Then

$$N_f(u)(t) = f(t, u(t)) \leq \sup_{0\leq\tau\leq\sqrt{t}R_1} f(t,\tau) = \frac{1}{c_1}\Psi_{R_1}(t).$$

Hence (3.4) yields

$$R_1 < \|\Psi_{R_1}\|_{2,\infty},$$

a contradiction to (3.3).

(b) The condition (2.10) holds. Otherwise, there exist $u \in K_{R_0R_1}$, with $\|u\|_{2,T} = R_0$ and $\lambda > 0$ such that $JE'(u) - \lambda JPu = 0$, i.e., $N(u) + \lambda JPu = u$. Then $J(N_fu + \lambda Pu) = u$. Since $u \in K_{R_0R_1}$ and $\|u\|_{2,T} = R_0$, for $t \in [\gamma, \delta]$, we have

$$MR_0 = M\|u\|_{2,T} \leq u(t) \leq \sqrt{t}|u| \leq \sqrt{\delta}R_1.$$

Then, we have

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)\left[f(s,u(s)) + \lambda (Pu)(s)\right] ds \\ &> \int_0^{+\infty} G(t,s) f(s,u(s))\, ds \geq \int_\gamma^\delta G(t,s) f(s,u(s))\, ds \\ &\geq \int_\gamma^\delta G(t,s)\,\omega(s)\, ds = \Phi_{R_0R_1}(t). \end{aligned}$$

Consequently

$$R_0 = \|u\|_{2,T} > \|\Phi_{R_0R_1}\|_{2,T},$$

which contradicts (3.2).

3.3. The case of nonlinearities with separated variables

It is worthwhile to see how the compression condition looks like in the particular case of functions f of the form

$$f(t,s) = g(t)\, h(s).$$

Here g is nonnegative on $\mathbb{R}^+$ with $gp^{-\theta} \in L^2(\mathbb{R}^+)$, and $h : \mathbb{R} \to \mathbb{R}$ satisfies $h(\mathbb{R}^+) \subset \mathbb{R}^+$ and

$$|h(s)| \leq a_0 + b_0 |s|^{\theta} \quad \text{for all } s \in \mathbb{R}. \tag{3.5}$$

Then

$$|f(t,s)| = g(t)\,|h(s)| \leq a_0 g(t) + b_0 g(t)\,|s|^{\theta}.$$

Since $gp^{-\theta} \in L^2(\mathbb{R}^+)$ and $p^{\theta} \in L^{\infty}(\mathbb{R}^+)$, one has $g \in L^2(\mathbb{R}^+)$. Hence f satisfies the growth condition (1.3) with $a = a_0 g$ and $b = b_0 g$.

Also assume that h is nondecreasing in $\mathbb{R}^+$. Then $\omega(t) = g(t)\, h(MR_0)$,

$$\Phi_{R_0 R_1}(t) = h(MR_0) \int_{\gamma}^{\delta} G(t,s)\, g(s)\, ds, \quad \Psi_{R_1}(t) = c_1 g(t)\, h\left(\sqrt{t} R_1\right),$$

and the compression condition reads as follows:

$$R_0 \leq h(MR_0) \left\| \int_{\gamma}^{\delta} G(t,s)\, g(s)\, ds \right\|_{2,T}, \quad R_1 \geq c_1 \left\| g(t)\, h\left(\sqrt{t} R_1\right) \right\|_{2,\infty}. \tag{3.6}$$

Notice that $\Psi_{R_1}(t) \leq c_1 g(t) \left(a_0 + b_0 \sqrt{t}^{\theta} R_1^{\theta}\right)$ and since $\left\| g(t) \left(a_0 + b_0 \sqrt{t}^{\theta} R_1^{\theta}\right) \right\|_{2,\infty} \leq a_0 \|g\|_{2,\infty}$ $+ b_0 R_1^{\theta} \left\| \sqrt{t}^{\theta} g(t) \right\|_{2,\infty}$, a sufficient condition for (3.3) to hold is

$$R_1 \geq a_0 c_1 \|g\|_{2,\infty} + b_0 c_1 R_1^{\theta} \left\| \sqrt{t}^{\theta} g(t) \right\|_{2,\infty}. \tag{3.7}$$

3.4. Example

Let $f(t,s) = g(t)h(s)$. Assume that $h : \mathbb{R} \to \mathbb{R}$ is an even function with

$$h(s) = \begin{cases} \alpha \sqrt{s}, & \text{if } 0 \leq s \leq 1, \\ \alpha s^2, & \text{if } 1 < s \leq \beta, \\ 4\alpha\beta \sqrt{(\beta-1)(s-1)} + \sqrt{s} + 4\alpha\beta - 3\alpha\beta^2, & \text{if } s > \beta, \end{cases} \tag{3.8}$$

where $\alpha > 0$ and $\beta > 1$ are chosen below. Also assume that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $g(t) > 0$ for every $t > 0$ and $gp^{-2} \in L^2(\mathbb{R}^+)$ for some p as above. It is easy to see that

$$|h(s)| \leq a_0 + b_0\sqrt{|s|}, \ \ s \in \mathbb{R},$$

for some constants a_0 and b_0 depending on α and β. Hence condition (3.5) holds with $\theta = 1/2$.

Note that from $gp^{-2} \in L^2(\mathbb{R}^+)$ we have $gp^{-\sigma} \in L^2(\mathbb{R}^+)$ for every $\sigma < 2$. To prove this, denote $w := gp^{-2} \in L^2(\mathbb{R}^+)$. Then

$$gp^{-\sigma} = gp^{-2}p^{2-\sigma} = wp^{2-\sigma}.$$

For $\sigma < 2$, one has $p^{2-\sigma} \in L^\infty(\mathbb{R}^+)$, hence $wp^{2-\sigma} \in L^2(\mathbb{R}^+)$ as claimed. In particular $gp^{-1/2} \in L^2(\mathbb{R}^+)$.

Since h is concave on $[\beta, +\infty]$ and $h'(\beta - 0) = h'(\beta + 0) = 2\alpha\beta$, one has $h(s) \leq \alpha s^2$ for every $s \geq \beta$. Hence

$$h(s) \leq \alpha\sqrt{s} + \alpha s^2 \quad \text{for all } s \in \mathbb{R}^+. \tag{3.9}$$

Note that h is continuous on $\mathbb{R}$, nondecreasing on $\mathbb{R}^+$, and a primitive F of f is $F(t,s) = g(t)H(s)$, where

$$H(s) = \begin{cases} \frac{2\alpha}{3}s^{\frac{3}{2}}, & \text{if } 0 \leq s \leq 1, \\ \frac{\alpha}{3}(s^3+1), & \text{if } 1 < s \leq \beta, \\ \frac{8\alpha\beta}{3}\sqrt{(\beta-1)}(s-1)^{\frac{3}{2}} + (4\alpha\beta - 3\alpha\beta^2)s + \frac{\alpha}{3}(2\beta^3 + 4\beta^2 - 8\beta + 1), & \text{if } s > \beta. \end{cases}$$

From (3.9), we have $H(s) \leq \alpha\left(2s^{3/2}/3 + s^3/3\right)$. Hence

$$F(t,s) \leq \alpha g(t)\,(\frac{2}{3}s^{\frac{3}{2}} + \frac{1}{3}s^3) \quad \text{for all } s \in \mathbb{R}^+. \tag{3.10}$$

(1) First we check condition (H3). Choose $r = 2$. For $u \in K$ and $|u| = 2$, using the estimate from (3.10), one obtains

$$\begin{aligned} E(u) &\geq \frac{1}{2}|u|^2 - \int_0^{+\infty} F(t,u(t))dt \\ &= 2 - \int_0^{+\infty} F(t,u(t))dt \geq 2 - \frac{\alpha}{3}\int_0^{+\infty} g(t)\left(2u^{\frac{3}{2}} + u^3\right)dt. \end{aligned} \tag{3.11}$$

Since $u(t) \leq \sqrt{t}|u| = 2\sqrt{t}$ and $\|u\|_{2,\infty} \leq k^{-1/2}|u| = 2k^{-1/2}$, we have

$$\begin{aligned}\int_0^{+\infty} g(t)u(t)^{\frac{3}{2}}dt &= \int_0^{+\infty} g(t)u(t)u(t)^{\frac{1}{2}}dt \leq \sqrt{2}\int_0^{+\infty} g(t)u(t)t^{\frac{1}{4}}dt \\ &= \sqrt{2}\int_0^{+\infty} \frac{g(t)}{p(t)^{\frac{1}{2}}}u(t)\left(p(t)\sqrt{t}\right)^{\frac{1}{2}}dt \\ &\leq \sqrt{2}\sqrt{c_{\infty,p}}\left\|gp^{-\frac{1}{2}}\right\|_{2,\infty}\|u\|_{2,\infty} \leq \frac{2\sqrt{2}\sqrt{c_{\infty,p}}}{\sqrt{k}}\left\|gp^{-\frac{1}{2}}\right\|_{2,\infty}.\end{aligned} \tag{3.12}$$

Similarly, we have

$$\begin{aligned}\int_0^{+\infty} g(t)u(t)^3dt &= \int_0^{+\infty} g(t)u(t)u(t)^2dt \leq 4\int_0^{+\infty} g(t)u(t)tdt \\ &= 4\int_0^{+\infty} \frac{g(t)}{p(t)^2}u(t)\left(p(t)\sqrt{t}\right)^2 \\ &\leq 4c_{\infty,p}^2\left\|gp^{-2}\right\|_{2,\infty}\|u\|_{2,\infty} \leq \frac{8c_{\infty,p}^2}{\sqrt{k}}\left\|gp^{-2}\right\|_{2,\infty}.\end{aligned} \tag{3.13}$$

Using (3.11), (3.12) and (3.13), we have $E(u) \geq 2 - \alpha C_0$, where

$$C_0 = \frac{2\sqrt{c_{\infty,p}}}{3\sqrt{k}}\left(\sqrt{2}\left\|gp^{-\frac{1}{2}}\right\|_{2,\infty} + 4c_{\infty,p}^{3/2}\left\|gp^{-2}\right\|_{2,\infty}\right).$$

Hence, if $\alpha > 0$ is sufficiently small, then

$$E(u) \geq \frac{1}{2} \quad \text{for all } u \in K \text{ with } |u| = 2.$$

Let $u_0 \in K$ such that $|u_0| = 1$. Clearly $|u_0| < r = 2$, and

$$E(u_0) = \frac{1}{2} - \int_0^{+\infty} F(t,u_0)dt < \frac{1}{2}.$$

Next, we choose u_1 in the form

$$u_1 = \kappa\frac{\phi}{|\phi|},$$

where $\kappa > 2$ and ϕ is the fixed function considered at the beginning of Section 3.2. Then $|u_1| = \kappa > r = 2$. Next

$$\begin{aligned}E(u_1) &= \frac{\kappa^2}{2} - \int_0^{+\infty} g(s)H(u_1(s))ds \\ &\leq \frac{\kappa^2}{2} - \int_{(u_1>1)} g(s)H(u_1(s))ds.\end{aligned}$$

One has $(u_1 > 1) = (\kappa\phi > |\phi|) \supset (\kappa_0\phi > |\phi|)$ for $\kappa > \kappa_0$, and we may assume that the measure of $(\kappa_0\phi > |\phi|)$ is strictly positive. Hence there exists an interval $[\mu_1, \mu_2] \subset (\kappa_0\phi > |\phi|)$ with $0 < \mu_1 < \mu_2$. Let $\mu = \min_{s\in[\mu_1,\mu_2]} g(s)$. Since $g > 0$ in $(0,\infty)$, we have $\mu > 0$. Then

$$E(u_1) \leq \frac{\kappa^2}{2} - \int_{\mu_1}^{\mu_2} g(s) H(u_1(s))\, ds \leq \frac{\kappa^2}{2} - \mu \int_{\mu_1}^{\mu_2} H(u_1(s))\, ds.$$

For $t \in [\mu_1, \mu_2]$, one has $1 \leq u_1(t) \leq \sqrt{t}|u_1| \leq \sqrt{\mu_2}\kappa$. Now chose $\kappa = \beta/\sqrt{\mu_2}$, where it is assume that $\beta > 2\sqrt{\mu_2}$. Then $1 \leq u_1(t) \leq \beta$ for all $t \in [\mu_1, \mu_2]$, consequently, on the interval $[\mu_1, \mu_2]$,

$$H(u_1(s)) = \frac{\alpha}{3}(u_1(s)^3 + 1) \geq \frac{\alpha}{3} u_1(s)^3 = \frac{\alpha\beta^3}{3\mu_2^{\frac{3}{2}}|\phi|^3}\phi(s)^3.$$

Hence

$$E(u_1) \leq \frac{\beta^2}{2\mu_2} - \frac{\mu\alpha\beta^3}{3\mu_2^{\frac{3}{2}}|\phi|^3}\int_{\mu_1}^{\mu_2} \phi(s)^3\, ds.$$

Since the right-hand side of this inequality goes to $-\infty$ as $\beta \to +\infty$, we may choose a β sufficiently large such that $E(u_1) < 1/2$. Therefore, the mountain pass condition (H3) is satisfied.

(2) Now we check (H1). First we observe that since in our case $\theta = 1/2$, condition (3.7) holds for each large enough R_1. Next, since

$$\lim_{\tau\to 0} \frac{h(\tau)}{\tau} = \alpha \lim_{\tau\to 0} \frac{\sqrt{\tau}}{\tau} = +\infty,$$

for any sufficiently small $R_0 > 0$ one has

$$\frac{h(MR_0)}{MR_0} \geq \frac{1}{M\left\|\int_\gamma^\delta G(t,s)g(s)ds\right\|_{2,T}},$$

that is the first inequality in (3.6) holds. Hence (H1) holds for any choice of R_0 sufficiently small and any R_1 sufficiently large.

(3) It remains to guarantee the conditions (H2) and (H4). Since $c > m$, it is enough to prove that for a given $\rho > 0$, there exists R_1 as large as desired such that

$$E(u) \geq c + \rho, \quad \text{for all } u \in K \text{ with } |u| = R_1. \tag{3.14}$$

Let us fix R_0, $\overline{R}_1$, α and β such that (H1) and (H3) hold for every $R_1 \geq \overline{R}_1$. For any $R_1 \geq \overline{R}_1$, denote Γ_{R_1} and c_{R_1} the corresponding Γ and c as given by (2.4) and (2.5), respectively. Clearly, $\Gamma_{\overline{R}_1} \subset \Gamma_{R_1}$ and $c_{R_1} \leq c_{\overline{R}_1}$. Hence to have (3.14) it suffices to find an $R_1 \geq \overline{R}_1$ such that

$$E(u) \geq c_{\overline{R}_1} + \rho \quad \text{for all } u \in K \text{ with } |u| = R_1. \tag{3.15}$$

Using the estimate from (2.25), for every $u \in K$ with $|u| = R_1$, one has

$$E(u) \geq \frac{R_1^2}{2} - \left(C_1 R_1 + C_2 R_1^{\frac{3}{2}}\right),$$

where C_1 and C_2 are constants independent of R_1. Since the right-hand side tends to $+\infty$ as $R_1 \to +\infty$, we may find R_1 large enough that

$$R_1^2/2 - \left(C_1 R_1 + C_2 R_1^{\frac{3}{2}}\right) \geq c_{\overline{R}_1} + \rho,$$

and consequently (3.15) holds.

Therefore, based on Theorem 3.2, we have the following result.

Theorem 3.3. *Assume that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $g(t) > 0$ for every $t > 0$ and $gp^{-2} \in L^2(\mathbb{R}^+)$ for some continuous function $p : \mathbb{R}^+ \longrightarrow (0, +\infty)$ satisfying (1.2). If $\alpha > 0$ is sufficiently small and $\beta > 1$ is sufficiently large, then the boundary value problem (1.1) on the half-line, for $f(t,s) = g(t)h(s)$ and h given by (3.8), has at least two positive solutions.*

An example of a function g with the above properties is $g(t) = e^{-mt}$, where $m > 1/2$, for which $p(t) = e^{-t/4}$.

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