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EXISTENCE AND ATTRACTIVITY THEOREMS FOR NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH ANTICIPATION AND RETARDATION

BAPURAO C. DHAGE

Kasubai, Gurukul Colony, Thodga Road, Ahmedpur, Latur, 413 515, Maharashtra, India

Abstract. In this paper, we establish an existence result and a global attractivity result for the mild solutions of a nonlinear quadratic hybrid fractional differential equation with the Caputo derivative on the unbounded intervals of the real line with the mixed arguments of anticipation and retardation. The hybrid fixed point theorem of Dhage is used in the analysis of our nonlinear differential problem. A positivity result is also obtained under some usual conditions.

Keywords. Hybrid differential equation; Dhage fixed point theorem; Existence theorem; Attractivity of solutions.

1. Introduction

Let $t_0 \in \mathbb{R}$ be a fixed real number and let $J_\infty = [t_0, \infty)$ be a closed but unbounded interval in $\mathbb{R}$. Let $\mathscr{CRB}(J_\infty)$ denote the class of pulling functions $a : J_\infty \to (0, \infty)$ satisfying the following properties:

(i) a is continuous, and
(ii) $\lim\limits_{t\to\infty} a(t) = \infty$.

The notion of the pulling function was introduced in Dhage [1, 2] and Dhage, Dhage and Sarkate [3]. There do exist functions $a : J_\infty \to (0, \infty)$ satisfying the above two conditions. In fact, if $a_1(t) = |t| + 1$, and $a_2(t) = e^{|t|}$, then $a_1, a_2 \in \mathscr{CRB}(J_\infty)$. Again, the class of continuous and strictly monotone functions $a : J_\infty \to (0, \infty)$ going to ∞ satisfy the above criteria. Note that if $a \in \mathscr{CRB}(J_\infty)$, then the reciprocal function $\overline{a} : J_\infty \to \mathbb{R}_+$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim\limits_{t\to\infty} \overline{a}(t) = 0$. It was shown in Dhage [1, 2] and Dhage, Dhage and Sarkate [3] that the pulling functions are useful in proving different asymptotic characterizations of the solutions of nonlinear differential and integral equations on unbounded interval of the real line $\mathbb{R}$.

E-mail address: bcdhage@gmail.com.

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In this paper, we employ the pulling functions for characterizing the solutions of a nonlinear hybrid fractional differential equation when the value of independent variable is large. The frequent use of fractional calculus in the resolution of some anomalous problems of engineering and physical sciences (see, e.g., [4, 5, 6] and references therein) prompted the development of the subject of fractional differential equations a lot. Nowadays, this is the vast growing most active area of research in the subject of nonlinear analysis. Therefore, the study of fractional versions of ordinary differential equations is vigorously started with a good pace. The fractional versions of ordinary hybrid differential equations is also desirable for different aspects of the solutions. Sometimes, it may happen that a dynamic system runs for a long period of time and one is interested in the behavior of the system in the long duration of time. These and other similar problems form the motivation for this paper. We discuss the behavior of the solution of a certain hybrid fractional differential equation on the unbounded interval of the real line in this paper.

We need the following fundamental definitions from fractional calculus (see [4, 5] and the references therein) in what follows.

Definition 1.1. Let $J_\infty = [t_0, \infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$. Then, for any $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of fractional order $q > 0$ is defined as

$$I_{t_0}^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{x(s)}{(t-s)^{1-q}}\, ds,\ t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞), where Γ is the Euler's gamma function defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1}\, dt$.

Definition 1.2. Let $x \in C^n(J_\infty, \mathbb{R})$. Then the Caputo fractional derivative ${}^C D_{t_0}^q x$ of x of fractional order q is defined as

$${}^C D_{t_0}^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} x^{(n)}(s)\, ds,\ t \in J_\infty,$$

where $n-1 < q < n$, $n = [q]+1$, $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $C^n(J_\infty, \mathbb{R})$ denotes the space of real valued functions $x(t)$ which have continuous derivatives up to order $(n-1)$ on J_∞ and that $x^{(n)} \in C(J_\infty, \mathbb{R})$.

Given a pulling function $a \in \mathscr{CRB}(J_\infty) \bigcap C^1(J_\infty, \mathbb{R})$, we consider the following nonlinear hybrid fractional differential equation (in short HFRDE) involving the Caputo fractional derivative,

$$\left.\begin{aligned} {}^C D_{t_0}^q \left[\frac{a(t)x(t)}{f(t, x(t), x(\theta(t)))}\right] &= g(t, x(t), x(\gamma(t))),\ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$, Γ is a Euler's gamma function, $f : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous, $g : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and $\theta, \gamma : J_\infty \to J_\infty$ are continuous functions, which are, respectively, anticipatory and retardatory, that is, $\theta(t) \geq t$ and $\gamma(t) \leq t$ for all $t \in J_\infty$ with $\theta(t_0) = t_0$.

Definition 1.3. By a solution for the functional differential equation (1.1), we mean a function $x \in C^1(J_\infty, \mathbb{R})$ such that

(i) the map $(x,y) \mapsto \dfrac{a(t)x}{f(t,x,y)}$ is well defined for each $t \in J_\infty$,

(ii) the map $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t),x(\theta(t)))} = z(t)$ is differentiable on J_∞ and $z' \in C(J_\infty,\mathbb{R})$, and

(iii) x satisfies the equations in (1.1) on J_∞,

where $C^1(J_\infty,\mathbb{R})$ is the space of continuous real-valued functions defined on J_∞ whose first derivative x' exist and $x' \in C(J_\infty,\mathbb{R})$.

As the functions θ and γ in the HFRDE (1.1) are respectively anticipatory and retardatory, the arguments in problem (1.1) are deviating over the unbounded interval J_∞. Therefore, the behaviour of the dynamic system modelled on the HFRDE (1.1) depends both on back history and future data. As a result, the existence analysis of HFRDE (1.1) involves both anticipation and retardation information of the state variable.

The HFRDE (1.1) is a mixed quadratic perturbation of second type obtained by multiplying the unknown function under the Caputo derivative with a scalar function a and dividing by a nonlinearity f. The classification of the different types of perturbations of a differential equation was given in Dhage [7]. If $f(t,x,y) = 1$ and $g(t,x,y) = g(t,x)$ for all $(t,x,y) \in J_\infty \times \mathbb{R} \times \mathbb{R}$, then the following fractional differential equation holds

$$\left.\begin{aligned} {}^C D^q_{t_0}[a(t)x(t)] &= g(t,x(t)),\ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.2}$$

This equation was studied in Dhage [2] for the existence and the asymptotic attractivity, and the stability of solutions. If $q = 1$, then HFRDE (1.2) reduces to the nonlinear functional ordinary differential equation

$$\left.\begin{aligned} \frac{d}{dt}[a(t)x(t)] &= g(t,x(t)),\ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.3}$$

which was studied in Dhage [1] for different characterizations of the solutions on J_∞. If $q = 1$, then HFRDE (1.1) reduces to the nonlinear quadratic functional ordinary differential equation

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t,x(t),x(\theta(t)))}\right] &= g(t,x(t),x(\gamma(t))),\ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.4}$$

which again includes the class of the nonlinear quadratic differential equations

$$\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + k\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x(t)),\ t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \tag{1.5}$$

as special cases, where $a(t) = e^{kt}$, $k > 0$. HFDE (1.4) was studied in Dhage, Dhage and Sarkate [3] whereas the equation (1.5) was studied in Dhage [8] for the existence and the attractivity of the solutions on the unbounded interval J_∞ of $\mathbb{R}$.

Now we state a useful lemma, which is helpful in transforming the fractional Caputo differential equations into the Riemann-Liouville fractional integral equations.

Lemma 1.4. [4, p. 96] *Let $x \in C^n(J,\mathbb{R})$ and $q>0$. Then,*

$$I_{t_0}^q\left({}^C D_{t_0}^q x(t)\right) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(t_0)}{k\,!}(t-t_0)^k = x(t) + \sum_{k=0}^{n-1} c_k (t-t_0)^k$$

for all $t \in J=[a,b]$, where $n-1<q\le n,\ \ n=[q]+1$ and $c_0,\dots,c_{n-1}$ are constants.

The converse of the above lemma is not true. It was mentioned in Kilbas, Srivastava and Trujillo [4, page 95] that if $q>0$ and $x\in C(J,\mathbb{R})$, then ${}^C D_{t_0}^q\left(I_{t_0}^q x(t)\right) = x(t)$ for all $t\in J=[a,b]$. It has been proved recently in Cohen and Salem [9, 10] that this is not true for any continuous functions on J.

Remark 1.5. The conclusion of Lemma 1.4 also remains true if we replace the function spaces $C^n([a,b],\mathbb{R})$ and $C([a,b],\mathbb{R})$ with the function spaces $BC^n(J_\infty,\mathbb{R})$ and $BC(J_\infty,\mathbb{R})$, respectively.

2. Auxiliary Results

Let X be a non-empty set and let $\mathscr{T} : X\to X$. An invariant point under $\mathscr{T}$ in X is called a fixed point of $\mathscr{T}$, that is, the fixed points are the solutions of the functional equation $\mathscr{T}x=x$. Any statement asserting the existence of fixed points of a mapping $\mathscr{T}$ is called a fixed point theorem for the mapping $\mathscr{T}$ in X. The fixed point theorems are obtained by imposing the conditions on T or on X or on both $\mathscr{T}$ and X. Usually, if spaces or mappings are better, then we have better fixed point principles. As we go on adding more structures to the non-empty set, we derive more fixed point theorems, which are useful in applications to nonlinear differential and integral equations. Below, we give some fixed point theorems, which are useful in establishing the attractivity and ultimate positivity of the solutions for HFRDE (1.1) on unbounded intervals. Before stating these results, we give some tools.

Definition 2.1. Let X be an infinite dimensional Banach space with norm $\|\cdot\|$. A mapping $\mathscr{T} : X\to X$ is called $\mathscr{D}$-Lipschitz if there is an upper semi-continuous and nondecreasing function $\psi_{\mathscr{T}}:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$\|\mathscr{T}x-\mathscr{T}y\| \le \psi_{\mathscr{T}}(\|x-y\|) \tag{2.1}$$

for all $x,y\in X$, where $\psi_{\mathscr{T}}(0)=0$. The function $\psi_{\mathscr{T}}$ is called a $\mathscr{D}$-function of $\mathscr{T}$ on X.

If $\psi_{\mathscr{T}}(r)=kr,\ k>0$, then $\mathscr{T}$ is said to be Lipschitz with the Lipschitz constant k. In particular, if $k<1$, then $\mathscr{T}$ is said to be a contraction on X with the contraction constant k. Further, if $\psi_{\mathscr{T}}(r)<r$ for $r>0$, then $\mathscr{T}$ is called a nonlinear $\mathscr{D}$-contraction on X. There do exist several $\mathscr{D}$-functions in the literature and the commonly used $\mathscr{D}$-functions are $\psi_{\mathscr{T}}(r)=kr$ and $\psi_{\mathscr{T}}(r)=\dfrac{r}{1+r}$, etc. (see Banas and Dhage [11] and the references therein).

Definition 2.2. An operator $\mathscr{T}$ on a Banach space X into itself is said to be totally bounded if, for any bounded subset S of X, $\mathscr{T}(S)$ is a relatively compact subset of X. If $\mathscr{T}$ is continuous and totally bounded, then it is said to be completely continuous on X.

The fixed point technique is a powerful method often used in the study of nonlinear equations. Our essential tool used in this paper is the following hybrid fixed point theorem of Dhage [12, 13] for a quadratic operator equation involving two operators in a Banach algebra X. We refer to Dhage [7, 12, 13, 14] for some related results and applications.

Theorem 2.3 (Dhage fixed point theorem [7, 12, 13])**.** *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $\mathscr{A} : X \to X$ and $\mathscr{B} : S \to X$ be two operators such that*

(a) $\mathscr{A}$ is $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{A}}$,
(b) $\mathscr{B}$ is completely continuous,
(c) $M_{\mathscr{B}}\psi_{\mathscr{A}}(r) < r$,where $M_{\mathscr{B}} = \|\mathscr{B}(S)\| = \sup\{\|\mathscr{B}x\| : x \in S\}$, and
(d) $x = \mathscr{A}x\mathscr{B}y \Longrightarrow x \in S$ for all $y \in S$.

Then the operator equation $\mathscr{A}x\mathscr{B}x = x$ has a solution in S.

Recently, the above hybrid fixed point theorem of Dhage played an important role in the subject of nonlinear analysis. The nonlinear alternatives related to Dhage fixed point theorem, Theorem 2.3 on the lines of Leray-Schauder and Schafer are also available in the literature (see Dhage [15] and the references therein). However, the present version is more convenient to apply in the theory of nonlinear hybrid differential equations. A collection of applicable fixed point theorems can be found in the monographs of Granas and Dugundji [16], Deimling [17], Dhage [13] and the references therein. Next, we give different types of characterizations of the solutions for nonlinear fractional differential equations on the unbounded intervals of the real line.

3. Characterizations of Solutions

We seek solutions of the HFRDE in the space $BC(J_\infty,\mathbb{R})$ of continuous and bounded real-valued functions defined on J_∞. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(J_\infty,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J_\infty} |x(t)|$$

and

$$(x\cdot y)(t) = (xy)(t) = x(t)y(t),\ t \in J_\infty.$$

Clearly, $BC(J_\infty,\mathbb{R})$ becomes a Banach algebra w.r.t. the above norm and the multiplication in it. Let $\mathscr{A},\mathscr{B} : BC(J_\infty,\mathbb{R}) \to BC(J_\infty,\mathbb{R})$ be two continuous operators and consider the following operator equation in the Banach algebra $BC(J_\infty,\mathbb{R})$

$$\mathscr{A}x(t)\,\mathscr{B}x(t) = x(t) \tag{3.1}$$

for all $t \in J_\infty$. Below we give different characterizations of the solutions for operator equation (3.1) in space $BC(J_\infty,\mathbb{R})$.

Definition 3.1. We say that the solutions of operator equation (3.1) are locally attractive if there exists a closed ball $\overline{B}_r(x_0)$ in space $BC(J_\infty,\mathbb{R})$ for some $x_0 \in BC(J_\infty,\mathbb{R})$ such that, for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $\overline{B}_r(x_0)$,

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{3.2}$$

In the case when limit (3.2) is uniform with respect to set $\overline{B}_r(x_0)$, i.e., for each $\varepsilon > 0$, there exists $T > 0$ such that

$$|x(t) - y(t)| \le \varepsilon \tag{3.3}$$

for all $x, y \in \overline{B}_r(x_0)$ being the solutions of (3.1) and for $t \ge T$, we say that the solutions of equation (3.1) uniformly locally attractive on J_∞.

Definition 3.2. A solution $x = x(t)$ of equation (3.1) is said to be globally attractive if (3.2) holds for each solution $y = y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$. In other words, we say that the solutions of equation (3.1) are globally attractive if, for arbitrary solutions $x(t)$ and $y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$, condition (3.2) s satisfied. In the case that condition (3.2) is satisfied uniformly with respect to space $BC(J_\infty, \mathbb{R})$, i.e., if, for every $\varepsilon > 0$, there exists $T > 0$ such that inequality (3.2) is satisfied for all $x, y \in BC(J_\infty, \mathbb{R})$ being the solutions of (3.1) and for $t \geq T$, we say that the solutions of equation (3.1) are uniformly globally attractive on J_∞.

Remark 3.3. We mention here that the details of the global attractivity of solutions can be found in a recent paper of Hu and Yan [18] while the concepts of uniform local and global attractivity (in the above sense) can be found in Banas and Dhage [11], Dhage [8, 19] and the references therein.

Now, we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (3.1) in space $BC(J_\infty, \mathbb{R})$.

Definition 3.4. [20] A solution x of equation (3.1) is said to be locally ultimately positive if there exists a closed ball $\overline{B}_r(x_0)$ in e $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that $x \in \overline{B}_r(0)$ and

$$\lim_{t\to\infty} \big[|x(t)| - x(t)\big] = 0. \tag{3.4}$$

In the case that (3.4) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., for each $\varepsilon > 0$, there exists $T > 0$ such that

$$\big|\,|x(t)| - x(t)\big| \leq \varepsilon \tag{3.5}$$

for all x being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we say that the solutions of equation (3.1) are uniformly locally ultimately positive on J_∞.

Definition 3.5. [20] A solution $x \in BC(J_\infty, \mathbb{R})$ of equation (3.1) is said to be globally ultimately positive if (3.4) is satisfied. In the case that (3.5) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., for each $\varepsilon > 0$, there exists $T > 0$ such that (3.5) is satisfied for all x being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we say that the solutions of equation (3.1) are uniformly globally ultimately positive on J_∞.

Finally, we have the following characterization of the asymptotic stability of the solutions of the equation (3.1) on J_∞.

Definition 3.6. A solution of equation (3.1) is said to be asymptotically stable to t-axis or zero if $\lim_{t\to} x(t) = 0$. Again, x is said to be uniformly asymptotically stable to zero if, for $\varepsilon > 0$, there exists a real number $T \geq t_0$ such that $|x(t)| \leq \varepsilon$ for all $t \geq T$.

Remark 3.7. We remark here that global attractivity implies the local attractivity and the uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (3.1) on J_∞. Similarly, the global ultimate positivity implies the local ultimate positivity of the solutions for the operator equation (3.1) on unbounded intervals. However, the inverse of the above two statements may not be true.

4. Attractivity and Positivity Results

In this section, we discuss the attractivity results for the ordinary hybrid fractional differential equation (1.1) on J_∞. We need the following definition in the sequel.

Definition 4.1. A function $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if

(i) the map $t \mapsto \beta(t,x,y)$ is measurable for each $x,y \in \mathbb{R}$, and

(ii) the map $(x,y) \mapsto \beta(t,x,y)$ is jointly continuous for each $t \in J_\infty$.

The following lemma is often used in the study of nonlinear differential equations (see Granas, Guenther and Lee [21] and the references therein).

Lemma 4.2 (Carathéodory)**.** *Let $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the map $(t,x,y) \mapsto \beta(t,x,y)$ is jointly measurable. In particular, the map $t \mapsto \beta(t,x(t),y(t))$ is measurable on J_∞ for all $x,y \in C(J_\infty,\mathbb{R})$.*

We need the following hypotheses in what follows.

(A_1) The function $t \mapsto f(t_0,0,0)$ is continuous and bounded on J_∞ with bound F_0.

(A_2) The function f is continuous and there exist a function $\ell \in BC(J_\infty,\mathbb{R}_+)$ and a constant $K>0$ such that
$$\big|f(t,x_1,x_2)-f(t,x_1,x_2)\big| \le \frac{\ell(t)\max\{|x_1-x_2|,|x_2-y_2|\}}{K+\max\{|x_1-x_2|,|x_2-y_2|\}}$$
for all $t \in J_\infty$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$. Moreover, $\sup\limits_{t\in J_\infty} \ell(t) = L$.

(B_1) The function g is bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with the bound M_g.

(B_2) The function g is Carathéodory on $J_\infty \times \mathbb{R} \times \mathbb{R}$.

(B_3) The pulling function a satisfies the condition $\lim\limits_{t\to\infty} \overline{a}(t)\,t^q = 0$.

Remark 4.3. If $a \in \mathscr{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty,\mathbb{R}_+)$. Hence, the number $\|\overline{a}\| = \sup_{t\in J_\infty} \overline{a}(t)$ exists. Again, since hypothesis (H_3) holds, the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the expression $w(t) = \overline{a}(t)\,t^q$ is continuous on J_∞ and satisfies the relation $\lim\limits_{t\to\infty} w(t) = 0$. So, the number $W = \sup_{t\ge t_0} w(t)$ exists.

The following lemma is useful in the sequel.

Lemma 4.4. *Assume that hypothesis (A_1) is satisfied. Furthermore, for any integrable function $h \in L^1(J_\infty,\mathbb{R})$, if function $x \in BC(J_\infty,\mathbb{R})$ is a solution of the HFRDE*
$$ {}^{C}D^{q}_{t_0}\left[\frac{a(t)x(t)}{f(t,x(t),x(\theta(t)))}\right] = h(t),\ t \in J_\infty, \tag{4.1}$$
and
$$x(0) = x_0, \tag{4.2}$$
then x satisfies the hybrid fractional integral equation (HFRIE)
$$x(t) = \big[f(t,x(t),x(\theta(t)))\big]\left(c_0\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds\right) \tag{4.3}$$
for all $t \in J_\infty$, where $c_0 = \dfrac{a(t_0)x_0}{f(t_0,x_0,x_0)}$ and $x_0 \neq 0$.

Proof. Let $h \in L^1(J_\infty, \mathbb{R}_+)$. Assume that x is a solution of the HFRDE (4.1)-(4.2). We apply the Reimann-Liouville fractional integration $I^q_{t_0}$ of fractional order q from t_0 to t to both sides of (4.1). Then, by using Lemma 1.4, we obtain the HFRIE (4.3) on J_∞. □

Definition 4.5. A solution $x \in BC(J_\infty, \mathbb{R})$ of the FRIE (4.3) is called a mild solution of the HFRDE (4.1)-(4.2) defined on J_∞.

In the following, we deal with the mild solution of the HFRDE (1.1) on unbounded interval J_∞ of the real line. Our main existence and global attractivity result is as follows.

Theorem 4.6. *Assume that the hypotheses* (A_1) - (A_2) *and* (B_1) - (B_3) *hold. Further, assume that*

$$L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) \leq K. \tag{4.4}$$

Then the HFRDE (1.1) has a mild solution and mild solutions are uniformly globally attractive defined on J_∞.

Proof. Now, using hypotheses (B_1) and (B_2), it can be shown that the mild solution x of the HFRDE (1.1) is equivalent to the functional fractional integral equation

$$x(t) = \left[f(t,x(t),x(\theta(t)))\right]\left(c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right), \tag{4.5}$$

for all $t \in J_\infty$. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at origin of radius r given by

$$r = \left(L + F_0\right)\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right).$$

Define two operators $\mathscr{A}$ on X and $\mathscr{B}$ on $\overline{B}_r(0)$ by

$$\mathscr{A}x(t) = f(t,x(t),x(\theta(t))),\ t \in J_\infty \tag{4.6}$$

and

$$\mathscr{B}x(t) = c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds,\ t \in J_\infty. \tag{4.7}$$

Then the FIE ((4.5)) is transformed into the operator equation as

$$\mathscr{A}x(t)\,\mathscr{B}x(t) = x(t),\ t \in J_\infty. \tag{4.8}$$

We show that $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.3 on $BC(J_\infty, \mathbb{R})$. First, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ define the mappings $\mathscr{A} : X \to X$ and $\mathscr{B} : \overline{B}_r(0) \to X$. Let $x \in X$ be arbitrary. Obviously, $\mathscr{A}x$ is a continuous function on J_∞. We show that $\mathscr{A}x$ is bounded on J_∞. Thus, if $t \in J_\infty$, then

$$\begin{aligned}
|\mathscr{A}x(t)| &= |f(t,x(t),x(\theta(t)))| \\
&\leq |f(t,x(t),x(\theta(t))) - f(t,0,0)| + |f(t,0,0)| \\
&\leq \ell(t)\frac{\max\{|x(t)|,|x(\theta(t))|\}}{K + \max\{|x(t)|,|x(\theta(t))|\}} + F_0 \\
&\leq L + F_0.
\end{aligned}$$

Therefore, taking the supremum over t, $\|\mathscr{A}x\| \leq L + F_0 = N$. Thus, $\mathscr{A}x$ is continuous and bounded on J_∞. As a result, $\mathscr{A}x \in X$. Similarly, it can be shown that $\mathscr{B}x \in X$. In particular,

$\mathscr{A} : X \to X$ and $\mathscr{B} : \overline{B}_r(0) \to X$. We show that $\mathscr{A}$ is a Lipschitz on X. Let $x, y \in X$ be arbitrary. Then, by hypothesis (H_3), we have

$$\begin{aligned}
\|\mathscr{A}x - \mathscr{A}y\| &= \sup_{t\in J_\infty} |\mathscr{A}x(t) - \mathscr{A}y(t)| \\
&\le \sup_{t\in J_\infty} \ell(t) \frac{\max\{|x(t)-y(t)|, |x(\theta(t)) - y(\theta(t))|\}}{K + \max\{|x(t)-y(t)|, |x(\theta(t)) - y(\theta(t))|\}} \\
&\le \frac{L\|x-y\|}{K + \|x-y\|} \\
&= \psi_{\mathscr{A}}(\|x-y\|)
\end{aligned}$$

for all $x, y \in X$. This shows that $\mathscr{A}$ is a $\mathscr{D}$-Lipschitz on X with $\mathscr{D}$-function $\psi_{\mathscr{A}}(r) = \dfrac{Lr}{K+r}$.

Next we show that $\mathscr{B}$ is a completely continuous operator on $\overline{B}_r(0)$. First, we show that $\mathscr{B}$ is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\varepsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then

$$\begin{aligned}
&|(\mathscr{B}x_n)(t) - (\mathscr{B}x)(t)| \\
&\le \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} |g(s, x_n(s), x_n(\gamma(s))) - g(s, x(s), x(\gamma(s)))| ds \\
&\le \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} [|g(s, x_n(s), x_n(\gamma(s)))| + |g(s, x(s), x(\gamma(s)))|] ds \\
&\le 2M_g \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1}\, ds \\
&= \frac{2M_g}{\Gamma q} \cdot w(t), \qquad (4.9)
\end{aligned}$$

where, $w(t) = \overline{a}(t) t^q$. Hence, by virtue of hypothesis (H_3), we infer that there exists a $T > 0$ such that $w(t) \le \varepsilon$ for $t \ge T$. Thus, for $t \ge T$, from estimate (3.3), we derive that

$$|(\mathscr{B}x_n)(t) - (\mathscr{B}x)(t)| \le \frac{2M_g}{\Gamma q} \varepsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, from dominated convergence theorem, we obtain the following estimate

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{B}x_n(t) &= \lim_{n\to\infty} \left[c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x_n(s), x_n(\gamma(s)))\, ds \right] \\
&= c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \left[\lim_{n\to\infty} g(s, x_n(s), x_n(\gamma(s))) \right] ds \\
&= c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s), x(\gamma(s)))\, ds = \mathscr{B}x(t) \qquad (4.10)
\end{aligned}$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathscr{B}x_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas and Dugundji [16], it is proved that $\mathscr{B}$ is a a continuous operator on $\overline{B}_r(0)$.

Next, we show that $\mathscr{B}$ is a compact operator on $\overline{B}_r(0)$. To this end, it is enough to show that every sequence $\{\mathscr{B}x_n\}$ in $\mathscr{B}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, proceeding with the earlier arguments, we prove $\|\mathscr{B}x_n\| \leq |c_0|\,\|\overline{a}\| + \frac{M_f W}{\Gamma q} = r$ for all $n \in \mathbb{N}$. This shows that $\{\mathscr{B}x_n\}$ is a uniformly bounded sequence in $\mathscr{B}(\overline{B}_r(0))$.

Next, we show that $\{\mathscr{B}x_n\}$ is also a equicontinuous sequence in $\mathscr{B}(\overline{B}_r(0))$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T_1 > t_0 \geq 0$ such that $|w(t)| < \dfrac{\varepsilon}{8M_f/\Gamma(q)}$ for all $t \geq T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\varepsilon > 0$, there is a real number $T_2 > t_0 \geq 0$ such that $|\overline{a}(t)| < \dfrac{\varepsilon}{8|c_0|}$ for all $t \geq T_2$. Thus, if $T = \max\{T_1, T_2\}$, then

$$|w(t)| < \frac{\varepsilon}{8M_f/\Gamma(q)} \quad \text{and} \quad |\overline{a}(t)| < \frac{\varepsilon}{8|c_0|} \tag{4.11}$$

for all $t \geq T$. Let $t, \tau \in J_\infty$ be arbitrary. If $t, \tau \in [t_0, T]$, then

$$\begin{aligned}
&|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| \\
&\leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&+ \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s,x(s))\,ds \right| \\
&\leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&+ \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{t} (\tau-s)^{q-1} f(s,x(s))\,ds \right| \\
&+ \left| \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{t} (\tau-s)^{q-1} f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s,x(s))\,ds \right| \\
&\leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| + \frac{M_f}{\Gamma q} \int_{t_0}^{t} \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| ds \\
&+ \frac{M_f}{\Gamma q} \left| \int_{\tau}^{t} \left| \overline{a}(\tau)(\tau-s)^{q-1} \right| ds \right| \\
&\leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| + \frac{M_f}{\Gamma q} \int_{t_0}^{T} \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| ds + \frac{M_f\,\|\overline{a}\|}{\Gamma q} \left|(\tau-t)^q\right|.
\end{aligned}$$

Since the functions $t \mapsto \overline{a}(t)$ and $t \mapsto \overline{a}(t)(t-s)^{q-1}$ are continuous on compact $[t_0, T]$, they are uniformly continuous. Therefore, by the uniform continuity, for the above ε, we have the real numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending on ε only such that

$$|t-\tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\varepsilon}{9|c_0|}$$

and

$$|t-\tau| < \delta_2 \Longrightarrow \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| < \frac{\varepsilon}{9M_f T/\Gamma q}.$$

Similarly, choose the real number $\delta_3 = \left(\frac{\varepsilon}{9M_f\|\overline{a}\|/\Gamma(q)}\right)^{1/q} > 0$ such that

$$|t-\tau| < \delta_3 \Longrightarrow |(t-\tau)^q| < \frac{\varepsilon}{9M_f\|\overline{a}\|/\Gamma(q)}.$$

Let $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. Then $|t-\tau| < \delta_4 \Longrightarrow |\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| < \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. Again, if $t, \tau > T$, then $\delta_5 > 0$ depends on ε only with

$$\begin{aligned}
|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| &\leq |c_0|\,|a(t)-a(\tau)| + \frac{\overline{a}(t)}{\Gamma q}\left|\int_{t_0}^{t}(t-s)^{q-1}f(s,x_n(s))\,ds\right| \\
&\quad + \frac{\overline{a}(\tau)}{\Gamma q}\left|\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x_n(s))\,ds\right| \\
&\leq \left|c_0\left[|\overline{a}(t)| + |\overline{a}(\tau)|\right] + \frac{M_f}{\Gamma(q)}\left[w(t)+w(\tau)\right]\right. \\
&< \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta_5$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then

$$|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| \leq |\mathscr{B}x_n(t) - \mathscr{B}x_n(T)| + |\mathscr{B}x_n(T) - \mathscr{B}x_n(\tau)|.$$

Take $\delta = \min\{\delta_4, \delta_5\} > 0$ depending only on ε. Therefore, from the above obtained estimates, it follows that $|\mathscr{B}x_n(t) - \mathscr{B}x_n(T)| < \frac{\varepsilon}{2}$ and $|\mathscr{B}x_n(T) - \mathscr{B}x_n(\tau)| < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. As a result, $|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| < \varepsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. This shows that $\{\mathscr{B}x_n\}$ is a equicontinuous sequence in X. Now an application of Arzelà-Ascoli theorem yields that $\{\mathscr{B}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of J_∞. Without loss of generality, we consider the subsequence to be the sequence itself. We show that $\{\mathscr{B}x_n\}$ is Cauchy in X. Now $|\mathscr{B}x_n(t) - \mathscr{B}x(t)| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$. Then, for given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \leq t \leq T} \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\left|f(s,x_m(s)) - f(s,x_n(s))\right|ds < \frac{\varepsilon}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then

$$\begin{aligned}
&\|\mathscr{B}x_m - \mathscr{B}x_n\| \\
&= \sup_{t_0 \leq t < \infty}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\left|f(s,x_m(s)) - f(s,x_n(s))\right|ds\right| \\
&\leq \sup_{t_0 \leq t \leq T}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\left|f(s,x_m(s)) - f(s,x_n(s))\right|ds\right| \\
&\quad + \sup_{t \geq T}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\left[\left|f(s,x_m(s))\right| + \left|f(s,x_n(s))\right|\right]ds\right| \\
&< \varepsilon.
\end{aligned}$$

This shows that $\{\mathscr{B}x_n\} \subset \mathscr{B}(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{\mathscr{B}x_n\}$ converges to a point in X. As $\mathscr{B}(\overline{B}_r(0))$ is closed, we have that $\{\mathscr{B}x_n\}$ converges to a point in $\mathscr{B}(\overline{B}_r(0))$. Hence $\mathscr{B}(\overline{B}_r(0))$ is relatively compact and consequently $\mathscr{B}$ is a continuous and compact operator on $B_r(0)$ into itself.

Next, we estimate the value of the constant $M_{\mathscr{B}}$ of the hypothesis (c) of Theorem 2.3. From definition of $M_{\mathscr{B}}$, one has

$$\begin{aligned}
&\|\mathscr{B}(B_r(0))\| \\
&= \sup\{\|\mathscr{B}x\| \ : \ x \in \overline{B}_r(0)\} \\
&= \sup\left\{\sup_{t\in J_\infty} |\mathscr{B}x(t)| \ : \ x \in \overline{B}_r(0)\right\} \\
&\leq \sup_{x\in\overline{B}_r(0)} \left\{\sup_{t\in J_\infty}\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| |\overline{a}(t)| + \frac{1}{\Gamma q}\cdot \sup_{t\in J_\infty} |\overline{a}(t)| \int_{t_0}^{t} (t-s)^{q-1} |g(s,x(s),x(\gamma(s)))|\,ds\right\} \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g}{\Gamma q}\cdot \sup_{t\in J_\infty} \overline{a}(t) \int_{t_0}^{t} (t-s)^{q-1}\,ds \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g}{\Gamma q}\cdot \sup_{t\in J_\infty} \overline{a}(t)\, t^q \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathscr{B}}.
\end{aligned}$$

Thus,

$$\|\mathscr{B}x\| \leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathscr{B}}$$

for all $x \in \overline{B}_r(0)$. Hence,

$$M_{\mathscr{B}}\psi_{\mathscr{A}}(r) \leq \frac{L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) r}{K+r} < r$$

for $r > 0$ due to

$$L\left(\left|\frac{a(t_0)x_0}{g(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) \leq K.$$

Therefore, hypothesis (c) of Theorem 2.3 is satisfied.

Next, let $x, y \in X$ be arbitrary. Then,

$$\begin{aligned}
|x(t)| &\leq |\mathscr{A}x(t)|\,|\mathscr{B}y(t)| \\
&\leq \|\mathscr{A}x\|\,\|\mathscr{B}y\| \\
&\leq \|\mathscr{A}(X)\|\,\|\mathscr{B}(\overline{B}_r(0))\| \\
&\leq \big(L+F_0\big)\, M_{\mathscr{B}} \\
&\leq \big(L+F_0\big)\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right)
\end{aligned}$$

for all $t \in J_\infty$. Therefore,

$$\|x\| \leq \big(L+F_0\big)\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) = r.$$

This shows that $x \in \overline{B}_r(0)$ and the hypothesis (c) of Theorem 2.3 is satisfied. Now, we apply Theorem 2.3 to the operator equation $\mathscr{A}x\,\mathscr{B}x = x$ to yield that the HFRDE (1.1) has a mild solution on J_∞. Moreover, the mild solutions of the HFRDE (1.1) are in $\overline{B}_r(0)$. Hence, mild solutions are global in nature.

Finally, let $x, y \in \overline{B}_r(0)$ be any two mild solutions of the HFRDE (1.1) on J_∞. Then

$$
\begin{aligned}
|x(t)-y(t)| \leq \Bigg| & \left[\, f(t,x(t),x(\theta(t)))\, \right] \times \\
& \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds \right) \\
& - \left[\, f(t,y(t),y(\theta(t)))\, \right] \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s),y(\gamma(s)))\,ds \right) \Bigg| \\
\leq & \Bigg| \left[\, f(t,x(t),x(\theta(t))) - f(t,y(t),y(\theta(t)))\, \right] \\
& \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds \right) \Bigg| \\
& + \left|\, f(t,y(t),y(\theta(t)))\, \right| \times \\
& \times \left| \left(\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\left[g(s,x(s),x(\gamma(s)))\,ds - g(s,y(s),y(\gamma(s)))\right]ds \right) \right| \\
\leq & \left| f(t,x(t),x(\theta(t))) - f(t,y(t),y(\theta(t)))\, \right| \times \\
& \times \left| \left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t)| + \frac{M_g W}{\Gamma q} \right) \right| \\
& + 2\left[|f(t,y(t),y(\theta(t))) - f(t,0,0)| + |f(t,0,0)| \right] \frac{M_g}{\Gamma q} w(t) \\
\leq & \, \ell(t) \frac{|x(t)-y(t)|}{K+|x(t)-y(t)|} \left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} \right) \\
& + \frac{2M_g}{\Gamma q} \left[\frac{\ell(t)\max\{|y(t)|,\,|y(\theta(t))|\}}{K+\max\{|y(t)|,\,|y(\theta(t))|\}} + F_0 \right] w(t) \\
\leq & \, \frac{L\left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} \right)|x(t)-y(t)|}{K+|x(t)-y(t)|} + \frac{2M_g}{\Gamma q}(L+F_0)w(t).
\end{aligned}
\tag{4.12}
$$

Taking the limit superior as $t \to \infty$ in the above inequality (4.12) yields $\lim_{t\to\infty}|x(t)-y(t)| = 0$. Therefore, there is a real number $T > 0$ such that $|x(t)-y(t)| < \varepsilon$ for all $t \geq T$. Consequently, the mild solutions of HFRDE (1.1) are uniformly globally attractive on J_∞. This completes the proof. □

Theorem 4.7. *Assume that the hypotheses (A_1) - (A_2) and (B_1) - (B_2) hold. Then the HFRDE (1.1) has a mild solution and mild solutions are uniformly globally attractive and ultimately positive defined on J_∞.*

Proof. By Theorem 4.6, the HFRDE (1.1) has a global mild solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.6, and the mild solutions are uniformly globally attractive on J_∞. We know that, for any $x,y \in \mathbb{R}$,

$$\big||xy| - (xy)\big| \leq |x|\,\big||y| - y\big| + \big||x| - x\big|\,|y| \tag{4.13}$$

for all $x,y \in \mathbb{R}$. Now, for any mild solution $x \in \overline{B}_r(0)$, one has from the above inequality (4.13) that

$$\begin{aligned}
&\big||x(t)| - x(t)\big| \\
&= \Bigg|\Big|f(t,x(t),x(\theta(t)))\Big|\,\Big|\left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right)\Big| \\
&\qquad - \big[f(t,x(t),x(\theta(t)))\big]\left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right)\Bigg| \\
&\leq \left|\Big|f(t,x(t),x(\theta(t)))\Big|\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| - \frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right)\overline{a}(t)\right| \\
&\qquad + \Big|f(t,x(t),x(\theta(t)))\Big|\,\Bigg|\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right| \\
&\qquad\qquad - \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\Bigg| \\
&\qquad + \Big||f(t,x(t),x(\theta(t)))| - f(t,x(t),x(\theta(t)))\Big| \\
&\qquad\quad \times \left|\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right| \\
&\leq \left[4(L+F_0)\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right|\right]\overline{a}(t) + \left[4(L+F_0)\frac{M_g}{\Gamma q}\right]w(t),
\end{aligned} \tag{4.14}$$

for all $t \in J_\infty$.

Taking the limit superior as $t \to \infty$ in inequality (4.14), we obtain the estimate that $\lim\limits_{t\to\infty}\big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $\big||x(t)| - x(t)\big| \leq \varepsilon$ for all $t \geq T$. Hence, the mild solutions of the HFRDE (1.1) are uniformly globally attractive as well as uniformly ultimately positive defined on J_∞. This completes the proof. □

Theorem 4.8. *Assume that the hypotheses (A_1) - (A_2) and (B_1) - (B_2) hold. Then the HFRDE (1.1) has a mild solution and mild solutions are uniformly globally attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on J_∞.*

Proof. From Theorems 4.6 and 4.7, we have that the HFRDE (1.1) has a global mild solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.6, and the mild solutions are uniformly globally attractive and uniformly ultimately positive on J_∞. Now, for any mild solution $x \in \overline{B}_r(0)$, we have from (4.12) that

$$|x(t)| \le (L+F_0)\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right|\overline{a}(t) + \frac{M_g}{\Gamma q}w(t)\right).$$

Taking the limit superior as $t \to \infty$ in the above inequality yields that $\lim_{t\to\infty}|x(t)| = 0$. Therefore, for $\varepsilon > 0$, there exists a real number $T \ge t_0$ such that $|x(t)| < \varepsilon$ whenever $t \ge T$. Consequently, the mild solution x is a uniformly asymptotically stable to zero defined on J_∞. This completes the proof. □

Remark 4.9. The conclusions of Theorems 4.6, 4.7 and 4.8 also remain true under the following new modified conditions

(i) the hypothesis (A_1) is replaced with
   (A_1') the function f is bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_f, and

(ii) the hypothesis (A_2) is replaced with
   (A_2') the function f is continuous and there exist a $\mathscr{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$\big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| \le \psi_f\big(\max\{|x_1-y_1|,|x_2-y_2|\}\big)$$

   for all $t \in J_\infty$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$. Moreover,

$$\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right|\|\overline{a}\| + \frac{M_g W}{\Gamma q}\right)\psi_f(r) < r,$$

   for all $r > 0$.

Remark 4.10. We remark that the existence, attractivity, positivity and asymptotic stability results in Theorems 4.6 4.7 and 4.8 of the HFRDE (1.1) include the existence, attractivity, positivity and asymptotic stability results for the nonlinear differential equations (1.2) through (1.5) as special cases which were studied earlier in Dhage [1, 2, 13, 19] and Dhage, Dhage and Sarkate [3] via the fixed point techniques from nonlinear functional analysis.

Example 4.11. Let $J_\infty = \mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$. Given a pulling function $a(t) = e^t \in \mathscr{CRB}(\mathbb{R}_+)$, consider the following nonlinear hybrid fractional Caputo differential equation with the mixed arguments of anticipation and retardation,

$$\left.\begin{aligned} {}^C D^q_{t_0}\left[\frac{e^t x(t)}{1+\dfrac{t}{t^2+1}\left(\dfrac{|x(t)|+|x(2t)|}{2+|x(t)|+|x(2t)|}\right)}\right] &= \frac{e^{-t}\log\big(1+|x(t)|+|x(t/2)|\big)}{2+|x(t)|+|x(t/2)|}, \\ x(0) &= 0, \end{aligned}\right\} \tag{4.15}$$

for all $t \in \mathbb{R}_+$, where ${}^C D^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$. Here, $a(t) = e^t$, $\theta(t) = 2t$, $\gamma(t) = \dfrac{t}{2}$ for $t \in \mathbb{R}_+$ and the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ and $g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined by

$$f(t,x,y) = 1 + \frac{t}{t^2+1}\left[\frac{|x|+|y|}{2+|x|+|y|}\right]$$

and

$$g(t,x,y) = \frac{e^{-t}\log(|x|+|y|)}{1+|x|+|y|}.$$

Clearly, the function f is a continuous and bounded real function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$. In particular, we have $F_0 = 1$. Now, it can be shown as in Banas and Dhage [11] that the function f satisfies the hypothesis (A$_2$) with $\ell(t) = \dfrac{t}{t^2+1}$. Furthermore, $L = 1$ and $K = 1$. g satisfies the hypotheses (B$_1$)-(B$_2$) with $M_g = 1$. We also have

$$\lim_{t\to\infty} \overline{a}(t) = \lim_{t\to\infty} e^{-t} = 0 \quad \text{and} \quad \lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-t} t^q = 0.$$

So, hypothesis (B$_3$) is satisfied. Consequently, $\|\overline{a}\| = \sup_{t\in\mathbb{R}_+} e^{-t} = 1$ and $W = \sup_{t\in\mathbb{R}_+} e^{-t} t^q = 1$. Finally, it is verified that functions a, f and g satisfy the condition (4.4) of Theorem 4.6. Hence, the HFRDE (4.15) has a mild solution and mild solutions are globally uniformly attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on $\mathbb{R}_+$.

Remark 4.12. We remark that the ideas of this paper could be extended with appropriate modifications to a more general hybrid fractional differential equations with the Caputo fractional derivative,

$$\left.\begin{aligned} &{}^C D_{t_0}^q \left[\frac{a(t)x(t)}{f(t, x(\theta_1(t)), \cdots, x(\theta_n(t)))} \right] \\ &\qquad\qquad = g(t, x(\gamma_1(t)), \cdots, x(\gamma_n(t))),\ t \in J_\infty, \\ &x(t_0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{4.16}$$

where ${}^C D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$, Γ is a Euler's gamma function, $f : J_\infty \times \mathbb{R} \times \ldots (n \text{ times}) \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous, $g : J_\infty \times \mathbb{R} \times \ldots (n \text{ times}) \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and $\theta_i, \gamma_i : J_\infty \to J_\infty$ are continuous functions which are respectively anticipatory and retardatory, that is, $\theta_i(t) \ge t$ and $\gamma_i(t) \le t$ for all $t \in J_\infty$ with $\theta_i(t_0) = t_0$ for $i = 1, \ldots, n$. The results obtained are useful to prove the existence and attractivity theorems for the HFRDE of the form

$$\left.\begin{aligned} &{}^C D_{t_0}^q \left[\frac{e^t x(t)}{1 + \dfrac{t}{t^2+1} \cdot \dfrac{\sum_{i=1}^n |x(it)|}{n + \sum_{i=1}^n |x(it)|}} \right] = \frac{e^{-t} \log\left(1 + \sum_{i=1}^n |x(t/i)|\right)}{2 + \sum_{i=1}^n |x(t/i)|}, \\ &\qquad\qquad x(0) = 0. \end{aligned}\right\} \tag{4.17}$$

for all $t \in \mathbb{R}_+$.

Remark 4.13. If g is assumed to be a continuous function on $J_\infty \times \mathbb{R} \times \mathbb{R}$, then the existence and the global uniform attractivity, the uniform positivity and the uniform asymptotic stability results for the HFRDE (1.1) can be obtained via the measure of noncompactness. We refer to Banas and Dhage [11], Hu and Yan [18], Dhage [8, 19] and the references therein.

5. The Conclusion

From the foregoing discussion, we observe that the pulling functions and the hybrid fixed point theorems are useful for proving the existence theorems as well as for characterizing the mild solutions of several nonlinear hybrid fractional differential equations of the type (1.1) on unbounded intervals J_∞ of the real line. We also notice that the pulling function involved (1.1) controls the behaviour of the mild solutions under suitable conditions and the obtained results, Theorems 4.6, 4.7 and 4.8 are useful in determining the asymptotic stability and the positivity of the anomalous problems of dynamic systems modelled on (1.1). The choice of the pulling function and the fixed point theorems in the abstract algebraic spaces depend on the situations and the circumstances of the nonlinearities involved in the nonlinear problem. An appropriate selection of fixed point theorems can yield powerful existence results as well as different characterizations of the nonlinear hybrid fractional differential equations (see [22]). In this paper, we proved the existence as well as the global attractivity and the ultimate positivity of the mild solutions for the HFRDE (1.1) on the unbounded interval J_∞, however, other characterizations, such as, the monotonic global attractivity, the monotonic asymptotic attractivity and the monotonic ultimate positivity of the mild solutions can also be treated similarly with appropriate modifications. It is of interest to further discuss the global asymptotic and the monotonic attractivity of the solutions for nonlinear hybrid fractional differential equations involving three nonlinearities via classical and applicable hybrid fixed point theorems.

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