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VARIATIONAL ANALYSIS FOR GRADIENT-TYPE SYSTEMS ON THE SIERPIŃSKI GASKET

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Abstract. In this paper, we continue the study of the multiple solutions to parametric quasi-linear systems of the gradient-type on the Sierpiński gasket. We give some new criteria to guarantee that the systems have at least three weak solutions by using a variational method and a critical point theorem. Finally, we give an example to illustrate our main results.

Keywords. Critical point theory; Gradient-Type systems; Nonlinear elliptic equation; Three solutions; Variational methods.

1. Introduction

In this paper, we are interested in Dirichlet gradient type system of the form:

$$\begin{cases} -\Delta u_i(x) + a_i(x)u_i(x) \\ = \nu\left(\int_{\mathscr{V}} F(u_1(x),\ldots,u_n(x))\mathrm{d}x - \lambda\right) F_{u_i}(u_1,\ldots,u_n) = 0, \quad x \in \mathscr{V}/\mathscr{V}_0, \\ u_i|_{\mathscr{V}_0} = 0 \end{cases} \qquad (\Delta^F_{\nu,\lambda})$$

for $i = 1,\ldots,n$, where $\mathscr{V}$ stands for the Sierpiński gasket, $\mathscr{V}_0$ is its intrinsic boundary, Δ denotes the weak Laplacian on $\mathscr{V}$, λ and ν are positive real parameters, $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in $(x_1,\ldots,x_n)$ and $F(0,\ldots,0) = 0$, F_{u_i} denotes the partial derivative of F with respect to u_i, and the variable potentials $a_i : \mathscr{V} \to \mathbb{R}$, $i = 1,\ldots,n$ satisfy the following conditions:

(h_1) $a_i \in \mathrm{L}^1(\mathscr{V},\mu)$ and $a_i \geq 0$ $(i = 1,\ldots,n)$ almost everywhere in $\mathscr{V}$.

Here μ denotes the restriction to $\mathscr{V}$ of the normalized $\frac{\log N}{\log 2}$−dimentional Hausdorff measure on $\mathscr{V}$ so that $\mu(\mathscr{V}) = 1$ (see [1] for more details).

The term 'fractal' derives from the Latin word fractus, i.e., broken, and has its modern meaning from Mandelbrot in 1975. A fractal is a well-shaped structure, which is repeated at arbitrary small scales. It is also self-similar and has a Hausdorff dimension greater than its topological dimension. A simple example is the Sierpiński gasket (triangle), introduced by Waclaw Sierpiński

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[2] in 1915, which plays an important role in the theory of curves. As a basic example of post-critically finite fractals, its shape constitutes a union of triangles (see [3, 4]).

The basic differential operator in the theory of fractals is the Laplacian. Thus, discussing differential equations on fractals or fractal differential equations involving the Laplacian. Many physical problems on fractal domains lead to the nonlinear models, which have reaction-diffusion equations, the problems on elastic fractal media or fluid flow through fractal regions, etc. In recent years, much interest has emerged in studying nonlinear partial differential equations on fractals. One of the problems in studying the PDEs on fractal domains is how to define differential operators, such as the Laplacian, on the fractal domains. No concept of a generalized derivative of a function exists, and thus we should initially clarify the idea of differential operators, such as the Laplacian on fractal domains. We cannot, therefore, expect the solutions of PDEs on fractal domains to act like the solutions of their Euclidean analogues. For instance, Barlow and Kigami [5] proved that many fractals have Laplacian eigenfunctions disappearing identically on large open sets, whereas the eigenfunctions of the Laplace operator are analytic in $\mathbb{R}^n$. Meanwhile, many researchers used the variational method and the critical point theory to investigate the nonlinear elliptic equations of fractals; see, e.g., [6, 7, 8] for more details.

The Sierpiński Gasket has been extensively used in showing roughness in science and nature. We refer to [9] for an elementary introduction to this subject and to [10] for an important application of fractals through their utility physics, chemistry or biology. In addition, the study of the Laplacian on fractals originated in physics, where the so-called spectral decimation method was developed in [11, 12]. To be completed, the Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process (see, e.g., [13, 14]). In [15], Teplyaev proved that the Laplacian with the Neumann boundary condition has pure point spectrum. In addition, the set of eigenfunctions with compact support is complete. The same is true if the infinite Sierpiński gasket has no boundary while it is false for the Laplacian with the Dirichlet boundary condition. In all these cases, he described the spectrum of the Laplacian and all the eigenfunctions with compact support. He also proved that the spectrum of the discrete Laplacian on a Sierpiński lattice is a pure point, and the eigenfunctions are localized.

Nonlinear problem ($\Delta^F_{\nu,\lambda}$) is closely associated with physical phenomena, such as reaction-diffusion problems and the elastic properties of fractal media and flow-through fractal regions. There is an extensive theory for the study of nonlinear elliptic equations ($\Delta^F_{\nu,\lambda}$) on classical domains or the open sets of $\mathbb{R}^N$, using Sobolev spacess and Sobolev embedding theorems, etc; see, e.g., [16, 17, 18, 19, 20] and the references therein. In Breckner et al. [21], by extending a method initiated by Faraci and Kristály in the framework of Sobolev spaces to the case of function spaces on fractal domains, they established the existence of infinitely many weak solutions for problem $\Delta u(x)+a(x)u(x)=g(x)f(u(x))$ in $\mathscr{V}/\mathscr{V}_0$ and $u|_{\mathscr{V}_0}=0$, where $a:\mathscr{V}\to\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathscr{V}\to\mathbb{R}$ are continuous functions with appropriate properties. In additon, in [22], Stancu-Dumitru studied the Dirichlet problem involving the weak Laplacian on the Sierpiński gasket $-\Delta u(x)=f(x)|u(x)|^{p-2}u(x)+(1-g(x))|u(x)|^{q-2}u(x)$ in $\mathscr{V}/\mathscr{V}_0$ and $u|_{\mathscr{V}_0}=0$, where Δ is the Laplacian on $\mathscr{V}$, $1<p<2<q$ are real numbers, $f,g\in \mathrm{C}(\mathscr{V})$ satisfy $f^+=\max\{f,0\}\neq 0$ and $0\leq g(x)<1$ for all $x\in\mathscr{V}$, and established the existence of at least two distinct nontrivial weak solutions with the aid of Ekeland's Variational Principle and standard tools in critical point theory combined with corresponding variational techniques. In [23], Bonanno et al. established the existence of infinitely many solutions for a system of gradient type ($\Delta^F_{\nu,\lambda}$) via variational methods . Under a proper oscillating behavior either at zero or at infinity of the nonlinear data, the existence of a sequence of weak solutions for parametric quasilinear systems of the gradient-type on the Sierpiński gasket was exactly established. In addition, by adopting the same hypotheses on the potential in presence of suitable small perturbations, the same conclusion holds. We also refer to [24], in which the existence of multiple solutions for parametric quasi-linear systems of the gradient-type on the Sierpiński gasket was investigated, and by using variational methods and some critical points theorems [25, 26], some new criteria to guarantee that these systems have at least three weak solutions were established.

Following the facts above, in this paper, we investigate the existence of at least three weak solutions of problem ($\Delta^F_{\nu,\lambda}$) for the proper values of the parameters λ and ν which belong to real intervals. Using variational methods and a three critical point theorem ([27, Theorem 3.1]), which is a consequence of [28, Theorem 1.6], we establish an existence results for problem ($\Delta^F_{\nu,\lambda}$). An example is given to illustrate our main results.

2. Preliminaries

Our main tool for proving the main result of this paper is a consequence of [28, Theorem 1.6] obtained by Cammaroto and Vilasi as follows.

Theorem 2.1 ([27, Theorem 3.1])**.** *Let $(X, \|.\|)$ be a separable and reflexive real Banach space, $\Psi : X \to \mathbb{R}$ a non-constant C^1-functional with compact derivative such that $\Psi(0_X) = 0$, and $\Phi : X \to [0,+\infty)$ a sequentially weakly lower semicontinuous and coercive C^1- functional with $\Psi(0_X) = 0$ and whose derivative admits a continuous inverse on the topological dual X^*. Set*

$$\theta^* := 2 \inf_{u\in J^{-1}(\mathbb{R}\setminus\{0\})} \frac{\Phi(u)}{|\Psi(u)|^2}.$$

Then, for each $\nu > \theta^*$ satisfying

$$\limsup_{\|u\|\to+\infty} \frac{|\Psi(u)|^2}{\Phi(u)} \leq \frac{2}{\nu}, \tag{2.1}$$

there exists an open interval $\Lambda \subseteq (\inf_X \Psi, \sup_X \Psi)$ such that, for each $\lambda \in \Lambda$, the equation $\Phi'(u) = \nu(\Psi(u) - \lambda)\Psi'(u)$ has at least three distinct solutions.

Remark 2.2 ([29, Remark 2.2])**.** In [28], Ricceri established a theorem, which is tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. The main novelty obtained in the most recent paper [28] is that the abstract energy functional does not depend on the parameter λ in an affine way.

We refer the reader to the paper [29] in which Theorem 2.1 were successfully employed to ensure the multiple solutions of fractional equations with bounded primitive.

In this paper, we denote by $\mathbb{N}$ the set of natural numbers $\{0,1,2,\ldots\}$, by $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ the set of positive naturals, and by $|.|$ the Euclidian norm on the spaces $\mathbb{R}^n, n \in \mathbb{N}^*$. Now, we give two remarks on the Sierpiński gasket.

Remark 2.3. The Sierpiński gasket is the connected subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of a quarter ($\frac{1}{4}$) of the area. Removing the corresponding open triangle from each of the three constituent triangles, and continuing this way, the gasket can also be obtained as the closure of the set of vertices arising in this construction.

Remark 2.4. Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ such that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, ..., N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by $S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i$. Obviously, every S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathscr{S} := \{S_1, \dots, S_N\}$ and set $F : \mathscr{P}(\mathbb{R}^{N-1}) \to \mathscr{P}(\mathbb{R}^{N-1})$ with $F(A) = \bigcup_{i=1}^{N} S_i(A)$. It is known (see [8, Theorem 9.1]) that there is a unique non-empty compact subset $\mathscr{V}$ of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathscr{S}$ such that $F(\mathscr{V}) = \mathscr{V}$. The set $\mathscr{V}$ is called the Sierpiński gasket (SG, for short) in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows: Put $\mathscr{V}_0 := \{p_1, ..., p_N\}, \mathscr{V}_m := F(\mathscr{V}_{m-1})$, for $m \geq 1$, and $\mathscr{V}_* := \cup_{m \geq 0} \mathscr{V}_m$. Since $p_i = S_i(p_i)$ for $i = 1, ..., N$, we have $\mathscr{V}_0 \subseteq \mathscr{V}_1$ and $F(\mathscr{V}_*) = \mathscr{V}_*$. Taking into account that the maps $S_i, i = 1, ..., N$, are homeomorphisms, we conclude that $\overline{\mathscr{V}_*}$ is a fixed point of F. On the other hand, $\overline{\mathscr{V}_*}$ is non-empty, compact, and $\mathscr{V} = \overline{\mathscr{V}_*}$. The set $\mathscr{V}_0$ is called the intrinsic boundary of the SG. The Hausdorff dimension d of $\mathscr{V}$ satisfies the equality $\sum_{i=1}^{N} (\frac{1}{2})^d = 1$ (see [8, Theorem 9.3]). Hence $d = \frac{\ln N}{\ln 2}$, and $0 < \mathscr{H}^d(\mathscr{V}) < \infty$, where $\mathscr{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalized restriction of $\mathscr{H}^d$ to the subsets of $\mathscr{V}$, so $\mu(\mathscr{V}) = 1$. Moreover $\mu(B) > 0$ for every nonempty open subset B of $\mathscr{V}$. In other words, the support of μ coincides with $\mathscr{V}$.

Now, denote by $\mathrm{C}(\mathscr{V})$ the space of real-valued continuous functions on $\mathscr{V}$ and

$$\mathrm{C}_0(\mathscr{V}) := \{u \in \mathrm{C}(\mathscr{V}); u|_{\mathscr{V}_0} = 0\}.$$

The space $\mathrm{C}(\mathscr{V})$ and $\mathrm{C}_0(\mathscr{V})$ endowed with the usual supremum norm $\|.\|_\infty$. For a function $u : \mathscr{V} \to \mathbb{R}$ and for $m \in N$, let

$$W_m = (\frac{N+2}{N})^m \sum_{x,y \in \mathscr{V}_m, |x-y| = 2^{-m}} [u(x) - u(y)]^2. \tag{2.2}$$

We have $W_m(u) \leq W_{m+1}(u)$ for every natural m. So we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{2.3}$$

Define

$$H_0^1(\mathscr{V}) := \{u \in \mathrm{C}_0(\mathscr{V}); W(u) < \infty\}.$$

It turns out that $H_0^1(\mathscr{V})$ is a dense linear subset of $L^2(V, \mu)$ equipped with the $\|.\|_2$ norm. We endow $H_0^1(\mathscr{V})$ with the norm $\|u\| = \sqrt{W(u)}$. In fact, there is an inner product defining this norm: for $u, v \in H_0^1(\mathscr{V})$ and $m \in N$, let

$$\mathscr{W}_m = \left(\frac{N+2}{N}\right)^m \sum_{x,y \in \mathscr{V}_m, |x-y| = 2^{-m}} (u(x) - u(y))(v(x) - v(y)).$$

Put

$$\mathscr{W}(u, v) = \lim_{m \to \infty} \mathscr{W}_m(u, v).$$

Then, $W(u, v) \in \mathbb{R}$ and $H_0^1(\mathscr{V})$, equipped with the inner product W (which obviously induces the norm $\|.\|$) becomes real Hilbert space. Moreover,

$$\|u\|_\infty \leq (2N+3)\|u\|, \quad \text{for all}\, u \in H_0^1(\mathscr{V}), \tag{2.4}$$

and the embedding

$$(H_0^1(\mathscr{V}), \|.\|) \hookrightarrow (C_0(\mathscr{V}), \|.\|_\infty), \tag{2.5}$$

is compact. We refer to [30] for further details.

Lemma 2.5. *Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with the Lipschitz constant $L \geq 0$ such that $h(0) = 0$. Then, for every $u \in H_0^1(\mathscr{V})$, $h \circ u \in H_0^1(\mathscr{V})$ and $\|h \circ u\| \leq L\|u\|$.*

Now, we define the Laplacian on the Sierpiński gasket $\mathscr{V}$. Let $H^{-1}(\mathscr{V})$ be the closure of $L^2(\mathscr{V})$ with respect to the pre-norm

$$\|w\|_{-1} = \sup_{w \in H_0^1(\mathscr{V}), \|g\|=1} |\langle w, g \rangle|,$$

where

$$\langle w, g \rangle = \int_{\mathscr{V}} wg \mathrm{d}\mu,$$

for $w \in L^2(\mathscr{V})$ and $g \in H_0^1(\mathscr{V})$. Then $H^{-1}(\mathscr{V})$ is a Hilbert space. Let $W(u, v)$ be the inner product of $u, v \in H_0^1(\mathscr{V})$. Then the relation

$$-W(u, v) = \langle \Delta u, v \rangle, \quad \text{for all}\, v \in H_0^1(\mathscr{V}),$$

uniquely defines a function $\Delta u \in H^{-1}(\mathscr{V})$ for all $u \in H_0^1(\mathscr{V})$. We term Δ the (weak) Laplacian on $\mathscr{V}$ (see [31]).

Remark 2.6. As pointed out by Falconer and Hu [8], we observe that if $a \in L^1(\mathscr{V})$ and $a \leq 0$ in $\mathscr{V}$, then the norm

$$\|u\|_* := \left(W(u, u) - \int_{\mathscr{V}} a(x) u^2(x) \mathrm{d}\mu \right)^{\frac{1}{2}},$$

is equivalent to $\sqrt{W(u)}$ in $H_0^1(\mathscr{V})$ from (2.4).

Remark 2.7. If $a_1, a_2 \in C(\mathscr{V})$, arguing as [8, Lemma 2.16], we conclude that every weak solution of problem $(\Delta^F_{\nu,\lambda})$ is also a strong solution.

Here and in the sequel, E will denote the product space $E = H_0^1(\mathscr{V}) \times \ldots, \times H_0^1(\mathscr{V})$ endowed with the norm

$$\|u\|_E = \|(u_1, \ldots, u_n)\|_E := \sum_{i=1}^n \left(W(u_i) - \int_{\mathscr{V}} a_i(x) u_i^2(x) \mathrm{d}\mu \right)^{\frac{1}{2}}.$$

We say that a function $u = (u_1, \ldots, u_n) \in H_0^1(\mathscr{V}) \times \ldots, \times H_0^1(\mathscr{V})$ is called a weak solution of $(\Delta^F_{\nu,\lambda})$ if

$$\sum_{i=1}^n \left(W(u_i, v_i) - \int_{\mathscr{V}} a_i(x) u_i(x) v_i(x) \mathrm{d}\mu \right)$$
$$+ \nu \left(\int_{\mathscr{V}} F(u_1(x), \ldots, u_n(x)) \mathrm{d}\mu - \lambda \right) \int_{\mathscr{V}} \sum_{i=1}^n F_{u_i}(u_1(x), \ldots, u_n(x)) \mathrm{d}\mu = 0$$

for every $v = (v_1, \ldots, v_n) \in H_0^1(\mathscr{V}) \times \ldots, \times H_0^1(\mathscr{V})$.

Now, for every $u = (u_1, \dots, u_n) \in \mathrm{E}$, we define

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n}\left(\|u_i\|^2_{H^1_0(\mathscr{V})} - \int_{\mathscr{V}} a_i(x)u_i^2(x)\mathrm{d}\mu\right) \tag{2.6}$$

and

$$\Psi(u) = \int_{\mathscr{V}} F(u_1(x), \dots, u_n(x))\mathrm{d}\mu. \tag{2.7}$$

Standard arguments show that $I =: \Phi - \nu(\Psi - \lambda)^2$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u = (u_1, \dots, u_n) \in \mathrm{E}$ given by

$$\begin{aligned} I'(u)(v) = &\sum_{i=1}^{n}\left(W(u_i, v_i) - \int_{\mathscr{V}} a_i(x)u_i(x)v_i(x)\mathrm{d}\mu\right) \\ &- \nu\left(\int_{\mathscr{V}} F(u_1(x), \dots, u_n(x))\mathrm{d}\mu - \lambda\right)\int_{\mathscr{V}}\sum_{i=1}^{n} F_{u_i}(u_1(x), \dots, u_n(x))\mathrm{d}\mu \end{aligned}$$

for all $v = (v_1, \dots, v_n) \in \mathrm{E}$. We observe that a vector $u \in \mathrm{E}$ is a solution of problem ($\Delta^F_{\nu,\lambda}$) if and only if u is a critical point of the function I.

Proposition 2.8. *Let* $\mathscr{J} := \Phi' : \mathrm{E} \longrightarrow \mathrm{E}^*$ *be the operator defined by*

$$\mathscr{J}(u)(v) = \sum_{i=1}^{n}\left(W(u_i, v_i) - \int_{\mathscr{V}} a_i(x)u_i(x)v_i(x)\mathrm{d}\mu\right)$$

for every $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathrm{E}$*. Then* $\mathscr{J}$ *admits a continuous inverse on* E^*.

Proof. Since $\mathscr{J}$ is the Fréchet derivative of Φ, $\mathscr{J}$ is continuous and bounded. For all $x \in \mathscr{V}$, $u = (u_1, \dots, u_n) \in \mathrm{E}$ and $v = (v_1, \dots, v_n) \in \mathrm{E}$ with $u \neq v$, we have

$$\begin{aligned} &\langle \mathscr{J}(u_1, \dots, u_n)(u_1 - v_1, \dots, u_n - v_n) - \mathscr{J}(v_1, \dots, v_n)(u_1 - v_1, \dots, u_n - v_n)\rangle \\ &= \sum_{i=1}^{n}\left(W(u_i, u_i - v_i) - \int_{\mathscr{V}} a_i u_i(u_i - v_i)\mathrm{d}\mu\right) \\ &- \sum_{i=1}^{n}\left(W(v_i, u_i - v_i) - \int_{\mathscr{V}} a_i v_i(u_i - v_i)\mathrm{d}\mu\right) \\ &= \sum_{i=1}^{n} - \int_{\mathscr{V}} a_i(x)u_i(u_i - v_i)\mathrm{d}\mu + \sum_{i=1}^{n}\int_{\mathscr{V}} a_i(x)v_i(u_i - v_i)\mathrm{d}\mu \\ &= -\sum_{i=1}^{n}\int_{\mathscr{V}} a_i(u_i - v_i)(u_i - v_i)\mathrm{d}\mu \\ &= -\sum_{i=1}^{n}\int_{\mathscr{V}} a_i(x)(u_i - v_i)^2\mathrm{d}\mu. \end{aligned}$$

Since $a_i \in \mathrm{L}^1(\mathscr{V}, \mu)$ and $a_i \leq 0$, for $i = 1, \dots, n$, then

$$\begin{aligned} &\langle \mathscr{J}(u_1, \dots, u_n)(u_1 - v_1, \dots, u_n - v_n) - \mathscr{J}(v_1, \dots, v_2)(u_1 - v_1, \dots, u_n - v_n)\rangle \\ &\geq \tau\left(\int_{\mathscr{V}}(u_1 - v_1)^2\mathrm{d}\mu + \dots + \int_{\mathscr{V}}(u_n - v_n)^2\mathrm{d}\mu\right) = \tau\|u - v\|^2, \end{aligned}$$

where $\tau = \sum_{i=1}^{n} \int_{\mathscr{V}} a_i(x)\mathrm{d}\mu$. So, $\mathscr{J}$ is a strictly monotone, and $\mathscr{J}$ is injective. Moreover, $\mathscr{J}$ is a coercive operator. Indeed, for each $u = (u_1, \dots, u_n) \in \mathrm{E}$ with $\|u\| \geq 1$, we have

$$\frac{\langle \mathscr{J}(u), u\rangle}{\|u\|} = \frac{\Phi(u)}{\|u\|} \geq \frac{\|u\|}{2} \to \infty \quad \text{as } \|u\| \to \infty.$$

Consequently, thanks to Minty-Browder theorem [32], the operator $\mathscr{J}$ is an surjection and admits an inverse mapping. Thus it is sufficient to show that $\mathscr{J}^{-1}$ is continuous. For this, let $(v_m)_{m=1}^{\infty}$ be a sequence in E^* such that $v_m \to v$ in E^*. Let $(u_m)_{m=1}^{\infty} = (u_{m_1}, \dots, u_{m_n})_{m=1}^{\infty}$ and $u = (u_1, \dots, u_n)$ in E such that $\mathscr{J}^{-1}(v_m) = u_m$ and $\mathscr{J}^{-1}(v) = u$. By the coercivity of $\mathscr{J}$, we conclude that the sequence $(u_m)_{m=1}^{\infty}$ is bounded in the reflexive space E. For a subsequence, we have $u_m \rightharpoonup \widehat{u} = (\widehat{u}_1, \dots, \widehat{u}_n)$ in E, which implies

$$\lim_{n\to+\infty} \langle \mathscr{J}(u_m) - \mathscr{J}(u), u_m - \widehat{u}\rangle = \lim_{m\to+\infty} \langle u_m - \widehat{u}, u_m - \widehat{u}\rangle = 0.$$

Therefore, by the continuity of $\mathscr{J}$, we have

$$u_n \to \widehat{u} \quad \text{in E} \quad \text{and} \quad \mathscr{J}(u_n) \to \mathscr{J}(\widehat{u}) = \mathscr{J}(u) \quad \text{in } \mathrm{E}^*.$$

Moreover, since $\mathscr{J}$ is an injection, we conclude that $u = \widehat{u}$. This completes the proof. □

3. Main Results

In this section, we formulate our main results. Let us denote by $\mathscr{F}$ the class of all functions $F : \mathbb{R}^n \to \mathbb{R}$ that are continuously differentiable in ξ satisfy the standard summability condition

$$\sup_{|\xi|\leq\rho_1} \left\{ \max\{|F(\xi)|,\ |G(\xi)|,\ |F_{\xi_i}(\xi)|,\ |G_{\xi_i}(\xi)|,\ i = 1, \dots, n\} \right\} \in \mathrm{L}^1(\mathscr{V}, \mu) \tag{3.1}$$

for any $\rho_1 > 0$ with $\xi = (\xi_1, \dots, \xi_n)$.

Theorem 3.1. *Let $F \in \mathscr{F}$ and*

$$\limsup_{|\xi|\to+\infty} \frac{F(\xi_1, \dots, \xi_n)}{|\xi|} = 0, \tag{3.2}$$

where $|\xi| = \sqrt{\sum_{i=1}^{n} \xi_i^2}$. Then, for each ν satisfying

$$\nu > \inf_{u\in\mathrm{E}} \left\{ \frac{\sum_{i=1}^{n} \left(\|u_i\|^2_{H_0^1(\mathscr{V})} - \int_{\mathscr{V}} a_i(x) u_i^2(x)\mathrm{d}\mu \right)}{2\left(\int_{\mathscr{V}} F(u_1(x), \dots, u_n(x))\mathrm{d}\mu\right)^2};\ \int_{\mathscr{V}} F(u_1(x), \dots, u_n(x))\mathrm{d}\mu \neq 0 \right\} \tag{3.3}$$

there exists an open interval

$$\Lambda \subseteq \left(\inf_{(\xi_1,\dots,\xi_n)\in\mathbb{R}^n} F(\xi_1, \dots, \xi_n),\ \sup_{(\xi_1,\dots,\xi_n)\in\mathbb{R}^n} F(\xi_1, \dots, \xi_n) \right)$$

such that, for each $\lambda \in \Lambda$, problem ($\Delta^F_{\nu,\lambda}$) has at least three distinct weak solutions.

Proof. Take $X = \mathrm{E}$. It is clear that E is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.6) and (2.7), respectively. The functional Φ is C^1, and by Proposition 2.8, its derivative admits a continuous inverse on X^*. Moreover, Φ is a coercive and sequentially weakly lower semicontinuous functional. Furthermore, Ψ is a C^1-functional with compact derivatives, and Φ has a strict local minimum 0 with $\Phi(0) = \Psi(0) = 0$. In view of (3.2), there exist $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(u_1, \dots, u_n) \leq \varepsilon |u| \tag{3.4}$$

for every $u \in \mathbb{R}^n$ with $|u| \in [0, \tau_1) \cup (\tau_2, +\infty)$, where $|u| = \sqrt{\sum_{i=1}^n u_i^2}$. By using (3.4), for each $u = (u_1, \dots, u_n) \in \mathrm{E} \setminus \{(0, \dots, 0)\}$, we obtain

$$\begin{aligned}
\frac{|\Psi(u_1, \dots, u_n)|^2}{\Phi(u_1, \dots, u_n)} &\leq \frac{\left| \int_{|u| \leq \tau_2} F(u_1, \dots, u_n) \mathrm{d}\mu \right|^2}{\Phi(u)} + \frac{\left| \int_{|u| > \tau_2} F(u_1, \dots, u_n) \mathrm{d}\mu \right|^2}{\Phi(u)} \\
&\leq \frac{\left| \sup_{|u| \in [0, \tau_2]} F(u_1, \dots, u_n) \right|^2}{\Phi(u)} + \frac{\varepsilon^2 (2N+3)^2 \|u\|_{\mathrm{E}}^2}{\Phi(u)} \\
&\leq \frac{\left| 2 \sup_{x \in \Omega, |u| \in [0, \tau_2]} F(u_1, \dots, u_n) \right|}{\|u\|_{\mathrm{E}}^2} + \frac{\varepsilon^2 (2N+3)^2}{2}.
\end{aligned}$$

So,

$$\limsup_{\|u\|_{\mathrm{E}} \to +\infty} \frac{|\Psi^2(u_1, \dots, u_n)|}{\Phi(u_1, \dots, u_n)} \leq \frac{\varepsilon^2 (2N+3)^2}{2}. \tag{3.5}$$

Thus

$$\limsup_{\|u\|_{\mathrm{E}} \to +\infty} \frac{|\Psi^2(u_1, \dots, u_n)|}{\Phi(u_1, \dots, u_n)} = 0 \leq \frac{2}{\nu}.$$

and condition (2.1) holds as desired. Hence, the existence of three solutions to the problem is achieved. □

Remark 3.2. If $F_{t_i}(0, \dots, 0) \neq 0$ for some $i = 1, \dots, n$, then Theorem 3.1 ensures the existence of three nontrivial weak solutions for problem ($\Delta^F_{\nu,\lambda}$). If $F_{t_i}(0, \dots, 0) = 0$ for all $i = 1, \dots, n$, the second solution u_2 of problem ($\Delta^F_{\nu,\lambda}$) may be trivial.

Remark 3.3. If $F(\xi_1, \dots, \xi_n) \geq 0$ for all $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, the nontriviality of the second weak solution ensured by Theorem 3.1 can be achieved also in the case $F_{t_i}(0, \dots, 0) = 0$ for all $i = 1, \dots, n$, requiring the extra condition at zero in the form of

$$\limsup_{(\xi_1, \dots, \xi_n) \to (0^+, \dots, 0^+)} \frac{F(\xi_1, \dots, \xi_n)}{|\xi|} = \infty \tag{3.6}$$

and

$$\liminf_{(\xi_1, \dots, \xi_n) \to (0^+, \dots, 0^+)} \frac{F(\xi_1, \dots, \xi_n)}{|\xi|} > -\infty. \tag{3.7}$$

Indeed, let $\nu > 0$ and $\lambda \geq 0$, and let Φ and Ψ be as given in Section 2. Due to Theorem 3.1, $J_\nu = \Phi - \nu(\Psi - \lambda)^2$ has a critical point u_λ that is a global minimum of J_λ. We next prove that u_λ cannot be trivial. Let us show that

$$\limsup_{\|(u_1, \dots, u_n)\| \to 0^+} \frac{(\Psi(u_1, \dots, u_n) - \lambda)^2}{\Phi(u_1, \dots, u_n)} = +\infty. \tag{3.8}$$

Owing to the assumptions (3.6) and (3.7), we can consider a sequence $\{\xi_m\} = \{\xi_{1m}, \dots, \xi_{nm}\}_{m=1}^\infty \subset \mathbb{R}^+ \times \ldots \times \mathbb{R}^+$ converging to $(0, \dots, 0)$, and two constants σ, κ (with $0 < \sigma < 1$) such that

$\lim_{m\to+\infty}\frac{F(\xi_{1m},\ldots,\xi_{nm})}{|\xi_m|}=+\infty$ with $|\xi_m|=\sqrt{\sum_{i=1}^n \xi_{im}^2}$, and $F(\xi_{1m},\ldots,\xi_{nm})\geq \kappa\sqrt{\sum_{i=1}^n \xi_i^2}$ for every $(\xi_1,\ldots,\xi_n)\in[0,\sigma]\times\ldots\times[0,\sigma]$. We consider a set $\mathscr{G}\subset B$ with $\mu(\mathscr{G})>0$, and a function $v=(v_1,\ldots,v_n)\in \mathrm{E}$ such that $v(x)\in[0,1]\times\ldots\times[0,1]$ for every $x\in\mathscr{V}$, $|v(x)|=1$ for every $x\in\mathscr{G}$, and $v(x)=(0,\ldots,0)$ for every $x\in\mathscr{V}\setminus D$. Hence, we fix $N>0$ and consider a real positive number η with

$$N<\frac{2\eta^2\mu(\mathscr{G})^2+2\kappa^2\left(\int_{D\setminus\mathscr{G}}|v(x)|\mathrm{d}\mu\right)^2}{\|v\|_{\mathrm{E}}^2}.$$

Then, there is $m_0\in\mathbb{N}$ such that $\xi_{im}<\sigma$ for all $i=1,\ldots,n$ and $F(\xi_{1m},\ldots,\xi_{nm})\geq\eta\sqrt{\sum_{i=1}^n\xi_{im}^2}$ for every $m>m_0$. Now, for every $m>m_0$, by considering the properties of the function $v=(v_1,\ldots,v_n)$ (that is, $0\leq\xi_{im}v_i(t)<\sigma$ for $i=1,\ldots,n$ and m large enough), one has

$$\begin{aligned}\frac{\Psi^2(\xi_{1m}v_1,\ldots,\xi_{nm}v_n)}{\Phi(\xi_{1m}v_1,\ldots,\xi_{nm}v_n)}&\geq\frac{F^2(\xi_{1m},\ldots,\xi_{nm})\mu(\mathscr{G})^2+\left(\int_{D\setminus\mathscr{G}}F(\xi_{1m}v_1,\ldots,\xi_{nm}v_n)\mathrm{d}\mu\right)^2}{\Phi(\xi_{1m}v_1,\ldots,\xi_{nm}v_n)}\\&>\frac{2\eta^2\mu(\mathscr{G})^2+2\kappa^2\left(\int_{D\setminus\mathscr{G}}|v(x)|\mathrm{d}\mu\right)^2}{\|v\|_{\mathrm{E}}^2}>N.\end{aligned}$$

Since N could be arbitrarily large, we obtain $\lim_{m\to\infty}\frac{\Psi(\xi_{1m}v_1,\ldots,\xi_{nm}v_m)}{\Phi(\xi_{1m}v_1,\ldots,\xi_{nm}v_m)}=+\infty$, from which (3.8) clearly follows. So, there exists a sequence $\{\zeta_m\}=\{\zeta_{1m},\ldots,\zeta_{nm}\}\subset\mathrm{E}$ strongly converging to $(0,\ldots,0)$ such that, for m large enough, $J_\nu(\zeta_m)=\Phi(\zeta_m)-\nu(\Psi(\zeta_m)-\lambda)^2<0$. Since u_ν is a global minimum of J_ν, we obtain $J_\nu(u_\nu)<0$. Hence, u_ν is not trivial.

Remark 3.4. Define a function $h:\mathbb{R}\to\mathbb{R}$ by $h(t)=|\min\{t,1\}|$ for every $t\in\mathbb{R}$. Then $h(0)=0$, and h is a Lipschitz function whose Lipschitz constant L is equal to 1. Set $D:=\{x\in\mathscr{V}:u(x)>0\}$. Since u is a continuous function, then D is nonempty subset of $\mathscr{V}$. Applying Lemma 2.5, we can write $\varphi:=h\circ u\in H_0^1(\mathscr{V})$. Indeed, $\varphi(x)=1$ for every $x\in D$, and $0\leq\varphi(x)\leq1$ for every $x\in\mathscr{V}$. Let $\eta_1,\ldots,\eta_n$ be the positive real numbers. Thus $(\eta_1\varphi(x),\ldots,\eta_n\varphi(x))\in\mathrm{E}$ for all $x\in[0,\infty)$.

Put $\vartheta(\varphi):=\min\left\{\|\varphi\|^2-\int_{\mathscr{V}}a_1(x)\varphi(x)^2\mathrm{d}\mu,\ldots,\|\varphi\|^2-\int_{\mathscr{V}}a_n(x)\varphi(x)^2\mathrm{d}\mu\right\}$. The next theorem provides sufficient conditions for applying Theorem 3.1 which does not require to know a test function $w=(w_1,\ldots,w_n)$.

Theorem 3.5. *Let $F\in\mathscr{F}$, the assumption (3.2) in Theorem 3.1 hold, and there exist numbers $\eta_1,\ldots,\eta_n\in\mathbb{R}$ such that $\int_{\mathscr{V}}F(\eta_1\varphi,\ldots,\eta_n\varphi)\mathrm{d}\mu\neq0$, where $\varphi\in H_0^1(V)$ is the same as given in Remark 3.4. Then, for each*

$$\nu>\frac{\vartheta(\varphi)\sum_{i=1}^n\eta_i^2}{4\left(\int_{\mathscr{V}}F(\eta_1\varphi(x),\ldots,\eta_n\varphi(x))\mathrm{d}\mu\right)^2},$$

there exists an open interval

$$\Lambda\subseteq\left(\inf_{(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n}F(\xi_1,\ldots,\xi_n),\ \sup_{(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n}F(\xi_1,\ldots,\xi_n)\right)$$

such that, for each $\lambda\in\Lambda$, problem ($\Delta^F_{\nu,\lambda}$) has at least three distinct weak solutions.

Proof. We claim that all the assumptions of Theorem 3.1 are fulfilled by choosing $w(x) = (w_1(x), \ldots, w_n(x)) := (\xi_1\varphi(x), \ldots, \xi_n\varphi(x))$. According to Remark 3.4, $w \in \mathrm{E}$. Moreover,

$$\Phi(w) = \frac{1}{2}\sum_{i=1}^{n}\left(\eta_i^2\|\varphi\|^2 - \int_{\mathscr{V}} a_i(x)\varphi(x)^2 d\mu\right) \geq \frac{\vartheta(\varphi)}{2}\sum_{i=1}^{n}\eta_i^2.$$

Hence, we conclude from Theorem 3.1 the desired result immediately. □

Remark 3.6. Clearly, Theorems 3.1 and 3.5 give the result of at least three solutions to problem ($\Delta^F_{\nu,\lambda}$) with $F(u_1, \ldots, u_n)$ being of subquadratic growth.

4. Scalar Case

As an application of the results in Section 3, we consider the problem

$$\begin{cases} -\Delta u(x) + a(x)u(x) = \nu\left(\int_{\mathscr{V}}\left(\int_0^u f(\xi)\mathrm{d}\xi\right)\mathrm{d}x - \lambda\right)f(u), & x \in \mathscr{V}/\mathscr{V}_0 \\ u|_{\mathscr{V}_0} = 0 \end{cases} \qquad (\Delta^f_{\nu,\lambda})$$

where $\mathscr{V}$, $\mathscr{V}_0$, Δ, λ, and ν are as introduced in problem ($\Delta^F_{\nu,\lambda}$). We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and the variable potential $a : \mathscr{V} \to \mathbb{R}$ satisfy the conditions (h_1) in Section 1.

Set $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$ for all $t \in \mathbb{R}$. The following existence result is a consequence of Theorem 3.1.

Theorem 4.1. *Let f be a continuous function and*

$$\limsup_{|\xi|\to+\infty}\frac{F(\xi)}{|\xi|} = 0. \qquad (4.1)$$

Then, for each ν satisfying

$$\nu > \inf_{u\in H_0^1(\mathscr{V})}\left\{\frac{\|u\|^2_{H_0^1(\mathscr{V})} - \int_{\mathscr{V}} a(x)u^2(x)\mathrm{d}\mu}{2\left(\int_{\mathscr{V}} F(u(x))\mathrm{d}\mu\right)^2};\ \int_{\mathscr{V}} F(u(x))\mathrm{d}\mu \neq 0\right\},$$

there exists an open interval $\Lambda \subseteq \left(\inf_{\xi\in\mathbb{R}} F(\xi), \sup_{\xi\in\mathbb{R}} F(\xi)\right)$ such that, for each $\lambda \in \Lambda$, problem ($\Delta^f_{\nu,\lambda}$) has at least three distinct weak solutions.

Remark 4.2. Due to Remark 3.2, if $f(0) \neq 0$, then Theorem 4.1 ensures the existence of three nontrivial weak solutions for problem ($\Delta^f_{\nu,\lambda}$). If $f(0) \neq 0$ does not hold, the second solution u_2 of problem ($\Delta^f_{\nu,\lambda}$) may be trivial.

Remark 4.3. According to Remark 3.3, the nontriviality of the second weak solution ensured by Theorem 4.1 can be also achieved in the case $f(0) = 0$ requiring the extra condition at zero in the form of

$$\limsup_{\xi\to 0^+}\frac{F(\xi)}{|\xi|} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+}\frac{F(\xi)}{|\xi|} > -\infty. \qquad (4.2)$$

The following existence result is a consequence of Theorem 3.5.

Theorem 4.4. *Let f be a continuous function. Assume that the assumption (4.1) in Theorem 4.1 holds, and there exist a real number η such that $F(\eta) \neq 0$. Then, for each $\nu > \frac{\eta^2\|\varphi\|}{4F^2(\eta)}$, where $\varphi \in H_0^1(\mathscr{V})$ is the same as given in Remark 3.4, there exists an open interval $\Lambda \subseteq \left(\inf_{\xi\in\mathbb{R}} F(\xi), \sup_{(\xi\in\mathbb{R}} F(\xi)\right)$ such that, for each $\lambda \in \Lambda$, problem ($\Delta^f_{\nu,\lambda}$) has at least three distinct weak solutions.*

Remark 4.5. Our results show that no asymptotic conditions on f is required, and only the algebraic conditions on f are supposed to guarantee the existence of solutions.

Now, we present the following example to illustrate Theorem 4.4.

Example 4.6. Let $a(x) = \frac{1}{1+x^2}$ for all $x \in \mathscr{V}$ and

$$f(t) = \begin{cases} 2(t+\sin t)^2, & \text{if } t < \pi, \\ 2\pi^2 + \tanh(t-\pi), & \text{if } t \geq \pi. \end{cases}$$

Thus, $F(\eta) = F(1) = \int_0^1 2(\xi + \sin\xi)^2 \mathrm{d}\xi > 0$ and $\lim_{|\xi|\to+\infty} \frac{F(\xi)}{|\xi|} = 0$. Hence, by applying Theorem 4.4, for each $\nu > \frac{1}{16}$ there exists an open interval $\Lambda \subseteq (0,+\infty)$ such that, for each $\lambda \in \Lambda$ the problem ($\Delta^f_{\nu,\lambda}$) has at least three distinct weak solutions.

Remark 4.7. From Remark 4.2, we see that the solutions obtained in Example 4.6 are non-zero because $f(0) = 1 \neq 0$.

Remark 4.8. Example 4.6 shows that our existence results to establish three solutions for the problem ($\Delta^f_{\nu,\lambda}$) in Theorem 4.1 is different from the existence results of Molica Bicsi in [1, Theorem 5.1]. Indeed, the nonlinear term of ($\Delta^f_{\nu,\lambda}$) and [1, Equation $(DP_{\lambda,\eta})$] are different, and the function f in [1, Theorem 5.1] should satisfy in

$$|f(t)| \leq m(a(x) + |t|^\alpha), \quad m \geq 0,\ \alpha \in [0,2),\ a \in \mathrm{L}^1(\mathscr{V},\mu),\ (x,t) \in \mathscr{V}\times\mathbb{R}, \tag{4.3}$$

while in Example 4.6, f does not apply to (4.3).

Finally, we give the following remark to illustrate the main results in this section.

Remark 4.9. We observe that if f is nonnegative, Theorem 4.1 is a bifurcation result in the sense that the pair $(0,0) \in \mathrm{E}^\nu_f \subset \mathrm{E}\times\mathbb{R}$ with

$$\mathrm{E}^\nu_f := \left\{(u_\nu,\nu) \in \mathrm{E}\times(0,\infty) : u_\nu \text{ is a non-trivial weak solution of } (\Delta^f_{\nu,\lambda})\right\}.$$

In particular, from Theorem 4.4, $\|u_\nu\|_{\mathrm{E}} \to 0$ as $\nu \to 0$. Hence, there exist two sequences $\{u_j\}$ in E and $\{\nu_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\nu_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to \infty$. Moreover, since f is nonnegative, $\Psi(u) < 0$ for all $u \in \mathbb{R}$. Thus, $(0,\nu^*) \ni \nu \mapsto I_\nu(u_\nu)$ is strictly decreasing. Hence, for every $\nu_1,\nu_2 \in (0,\lambda^*)$, with $\nu_1 \neq \nu_2$, the weak solutions u_{ν_1} and u_{ν_2} ensured by Theorem 4.1 are different.

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