J. Nonlinear Funct. Anal. 2022 (2022) 35 https://doi.org/10.23952/jnfa.2022.35

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR THE SPLIT EQUALITY PROBLEM IN BANACH SPACES

MEIYING WANG[1], TONGXIN XU[2], LUOYI SHI[3,*]

[1]School of Mathematical Sciences, Tiangong University, Tianjin 300387, China
[2]School of Mathematics and Statistics, Xidian University, Xi'an 710071, China
[3]School of Software, Tiangong University, Tianjin 300387, China

Abstract. In this paper, based on the Bregman distance, we introduce a self-adaptive method for solving a split equality problem in p-uniformly convex uniformly smooth Banach spaces. The advantage of the proposed algorithm is that the stepsize selection is self-adaptive and no prior estimation of operator norm is required. Under relatively mild conditions, we prove the strong convergence of the proposed algorithm. Finally, numerical examples are provided to verify the convergence of the algorithm.
Keywords. Bregman distance; Nonexpansive mapping; Self-adaptive method; Split equality problem.

1. Introduction

Let H_1, H_2, and H_3 be real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be two nonempty closed convex sets. In 1994, the split feasibility problem (SFP) was first proposed by Censor and Elfving [4] to solve some real-world problem caused by phase restoration and medical image reconstruction [4]. Their split feasibility problem is stated as follows:

$$\text{Find } x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_2$ is a linear operator, which is also bounded.

Many of real problems that arise in image restoration can be attributed to the SFP. In order to approximate a solution of the SFP, various algorithms were proposed and studied; see, e.g., [2, 11, 13, 14, 15, 17] and the references therein.

In 2013, Moudafi [11] proposed the following split equality problem (SEP): Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. His SEP is to find $x \in C$, $y \in Q$ with $Ax = By$. If $B = I$, then the SEP reduces to the SFP.

*Corresponding author.
E-mail address: shiluoyi@tiangong.edu.cn (L. Shi).
Received November 26, 2021; Accepted July 7, 2022.

The SEP has attracted much interest from many authors due to its real applications in signal processing, intensity-modulated radiation therapy, and image restoration [3, 4].

In 2013, Moudafi in [11] proposed the Alternating CQ-algorithm (ACQA) and Relaxed alternating CQ-algorithm (RACQA) to solve the SEP

$$(ACQA)\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases}$$

and

$$(RACQA)\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma B^*(Ax_{k+1} - By_k)). \end{cases}$$

But the above algorithms converge weakly to a solution of SEP only.

In order to achieve some strong convergence results, Shi et al. [19] proposed a modification of Moudafi's ACQA and RACQA algorithms in Hilbert spaces:

$$w_{n+1} = P_S\{(1-\alpha_n)[I - \gamma G^* G]w_n\},$$

i.e.,

$$\begin{cases} x_{n+1} = P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, \quad n \geq 0; \\ y_{k+1} = P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, \quad n \geq 0. \end{cases}$$

Since a Banach space setting sometimes allows a more realistic modelling of problems arising in applications from industry and natural sciences. Hence, the SFP and SEP in Banach spaces are interesting not only from a theoretical point of view but also to tackle real world problems. In order to solve the SFP in Banach spaces, Schöpfer et al. [17] proposed the following algorithm:

$$x_{n+1} = \Pi_C J_q^{E^*}[J_p(x_n) - t_n A^* J(Ax_n - P_Q(Ax_n))], \tag{1.1}$$

where Π_C denotes the Bregman projection, J_p, $J_q^{E^*}$, J are duality mappings, and P_Q denotes the metric projection. For more details, we refer to [17]. Indeed, they proved the result of weak convergence of algorithm (1.1).

In 2015, Shehu [18] proposed an iterative algorithm for solving the SFP in Banach spaces by using a Bregman projection method:

$$\begin{cases} y_n = J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_2}(Ax_n - P_Q Ax_n)], \\ x_{n+1} = \Pi_C J_q^{E_1^*}[\alpha_n J_p^{E_1} u + (1-\alpha_n)\beta_n J_p^{E_1}(y_n)], \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, and t_n satisfies $0 < a \leq t_n \leq b < (\frac{q}{C_q ||A||^q})^{\frac{1}{q-1}}$ for $a, b > 0$. The strong convergence of the algorithm was proved.

In [10], Mewomo and Ogbuisi proved the strong convergence of an iterative algorithm in Banach spaces for the multiple-sets split feasibility problem(MSSFP): find a point $x^* \in E_1$ such that

$$x^* \in \bigcap_{i=1}^{t} C_i, \; Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.2}$$

where E_1, E_2 are Banach spaces, and $\{C_i\}_{i=1}^t$, $\{Q_j\}_{j=1}^r$ are nonempty, closed, and convex subsets of E_1 and E_2, respectively. If $t = r = 1$ in (1.2), then the MSSFP reduces to SFP. The authors proposed the following method:

$$\begin{cases} x_n = J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_{\bigcap_{j=1}^r Q_j})Au_n], \\ u_{n+1} = \Pi_C J_q^{E_1^*}[\alpha_n J_p^{E_1} u + (1-\alpha_n)(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^t \gamma_{i,n} J_p^{E_1}(\Pi_{C_i} x_n))], \end{cases}$$

where $0 \leq t \leq t_n \leq k \leq (\frac{q}{C_q||A||^q})^{\frac{1}{q-1}}$.

Based on the ideas of solving the SFP in Banach space, the following question is naturally proposed.

Question 1.1. *Can one construct an iterative algorithm to solve the SEP without prior knowledge of operator norms in p-uniformly convex and uniformly smooth real Banach spaces?*

In this paper, we propose a self-adaptive method that answers the question raised above. The paper is organized as follows: In Section 2, we introduce some definitions and lemmas for our results. In Section 3, we present our algorithm and prove that the iterative sequence formed by the proposed algorithm converges strongly. In Section 4, the last section, we give some numerical examples to verify the convergence of the algorithm.

2. Preliminaries

In this section, we introduce some basic definitions and lemmas that will be used in this paper.

Let E be a real Banach space with dual E^*. Let C be nonempty, closed, and convex subset of E. Let $\langle x^*, x\rangle$ denote the duality pairing between E *and* E^*, that is, $\langle x^*, x\rangle := x^*(x)$, where $x^* \in E^*$, $x \in E$. The notion " $\to$ " denotes strong convergence and " $\rightharpoonup$ " denotes weak convergence. Let $S_E = \{x \in E : ||x|| = 1\}$ and $B_E = \{x \in E : ||x|| \leq 1\}$ be the unit sphere and unit ball of E, respectively. A point $x \in E$ is called a fixed point of T iff $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

Let $1 \leq q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_E(\varepsilon) : [0,2] \to [0,1]$ is defined as

$$\delta_E(\varepsilon) = \inf\{1 - \frac{||x+y||}{2} : x, y \in B_E, ||x-y|| \geq \varepsilon\},$$

E is called uniformly convex iff $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$; and strictly convex iff $\delta_E(2) = 1$. If there exist $p \geq 2$ and a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$, $\forall \varepsilon \in (0,2]$, then E is called p-uniformly convex. The modulus of smoothness $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \sup\{\frac{||x+\tau y|| + ||x-\tau y||}{2} - 1 : x, y \in S_E\}.$$

E is called uniformly smooth iff $\lim_{\tau\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$, and q-uniformly smooth iff there exist $C > 0$ so that $\rho_E(\tau) \leq C\tau^q$ for any $\tau > 0$. It is know that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth [8].

The duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x^*, x\rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\right\}, \tag{2.1}$$

for every $x \in E$. If $p = 2$, then (2.1) becomes

$$J_2(x) = \{x^* \in E^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||^2\},$$

for every $x \in E$. J_2 is also called the normalized duality mapping.

It is known that the duality mapping J_p possesses the following properties (see, e.g., [5, 8, 22])
(i) J_p is surjective iff E is reflexive.
(ii) J_p is injective iff E is strictly convex.
(iii) J_p is single-valued iff E is smooth.
(iv) If E is reflexive, smooth, and strictly convex, then J_p is one-to-one single-valued and $J_p^{-1} = J_q^*$, where J_q^* is the duality mapping of E^*.
The following inequality holds in q-uniformly smooth spaces.

Lemma 2.1. [21] *Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists a $C_q > 0$ such that $||x-y||^q \leq ||x||^q - q\left\langle J_q^*(x), y\right\rangle + C_q||y||^q$.*

Recall that a function $f : E \to \mathbb{R}$ is said to be proper ([6]) if the domain of f, $domf = \{x \in E : f(x) \leq \infty\}$, is nonempty. The Fenchel conjugate of f is the function $f^* : E^* \to \mathbb{R}$ defined by

$$f^*(x^*) = \sup\{\langle x, x^*\rangle \; - f(x) : x \in E\},$$

for any $x^* \in E^*$.
Given a *Gâteaux* differentiable function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined as:

$$\Delta_f(x,y) = f(x) - f(y) - \langle \nabla f(y), x-y\rangle, \; \forall x, y \in E.$$

In particular, let $f(x) = \frac{1}{p}||x||^p$. In this case, the duality mapping J_p is the derivative of f. The Bregman distance

$$\begin{aligned}\Delta_p(x,y) :&= \frac{||x||^p}{p} - \frac{||y||^p}{p} - \left\langle J_p(y), x-y\right\rangle \\ &= \frac{||x||^p}{p} + \frac{||y||^p}{q} - \left\langle J_p(y), x\right\rangle.\end{aligned}$$

In general, the Bregman distance is not symmetric and does not satisfy the triangle inequality. However, it is possesses some distance-like properties and it has the following important properties:

$$\Delta_p(x,y) + \Delta_p(y,z) - \Delta_p(x,z) = \left\langle J_p(z) - J_p(y), x-y\right\rangle, \; \forall x, y, z \in E$$

and

$$\Delta_p(x,y) + \Delta_p(y,x) = \left\langle J_p(x) - J_p(y), x-y\right\rangle, \; \forall x, y \in E.$$

For p-uniformly convex spaces, the metric and Bregman distance has the following relation:

$$\tau||x-y||^p \leq \Delta_p(x,y) \leq \left\langle J_p(x) - J_p(y), x-y\right\rangle,$$

where $\tau > 0$ is some fixed number.
The metric projection $P_C x := \arg\min_{y \in C} ||x-y||$, $\forall x \in E$ is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\left\langle J_p(x - P_C x), z - P_C x\right\rangle \leq 0, \; \forall z \in C.$$

Similar to the metric projections, the Bregman projection is defined as

$$\Pi_C x := \arg\min_{y \in C} \Delta_p(x,y), \; \forall x \in E$$

is the unique minimizer of the Bregman distance. It can be characterized by a variational inequality:

$$\left\langle J_p(x) - J_p(\Pi_C x), z - \Pi_C x\right\rangle \leq 0, \; \forall z \in C, \tag{2.2}$$

from which one has $\Delta_p(z,\Pi_C x) \leq \Delta_p(z,x) - \Delta_p(\Pi_C x,x)$, $\forall z \in C$. We make use of the function $V_p : E \times E^* \to [0,+\infty)$, which is defined by

$$V_p(x,x^*) = \frac{1}{p}||x||^p - \langle x,x^* \rangle + \frac{1}{q}||x^*||^q,\ \forall x \in E, x^* \in E^*$$

to find

$$V_p(x,x^*) = \Delta_p(x,J_q^*(x^*)), \tag{2.3}$$

for all $x \in E, x^* \in E^*$. In addition, V_p satisfies the following inequality:

$$V_p(x,x^*) + \left\langle y^*, J_q^*(x^*) - x \right\rangle \leq V_p(x,x^* + y^*),\ \forall x \in E, x^*, y^* \in E^*. \tag{2.4}$$

Let $T : C \to C$ be a mapping. A point $x \in E$ is called an asymptotic fixed point ([7]) of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of asymptotic fixed point of T by $\hat{F}(T)$.

From [7, 16], mapping $T : C \to C$ is called

(i) Bregman firmly nonexpansive (BFNE) if

$$\langle J_p(x) - J_p(Ty), Tx - Ty \rangle \leq \langle J_p(x) - J_p(y), Tx - Ty \rangle,\ \forall x,y \in C;$$

(ii) Bregman strongly nonexpansive (BSNE) with respect to $\hat{F}(T)$ if $\Delta_p(z,Tx) \leq \Delta_p(z,x)$ for all $z \in \hat{F}(T)$ and $x \in C$, and if whenever $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty}(\Delta_p(z,x_n) - \Delta_p(z,Tx_n)) = 0$, it follows that $\lim_{n\to\infty} \Delta_p(Tx_n,x_n) = 0$;

(iii) Bregman quasi-nonexpansive (BQNE) if $F(T) \neq \varnothing$ and

$$\Delta_p(z,Tx) \leq \Delta_p(z,x),\ \forall z \in F(T), x \in C.$$

Furthermore, Π_C is an example of Bregman firmly nonexpansive mapping and Bregman strongly nonexpansive mapping. In the case that $F(T) = \hat{F}(T)$, it is easy to see that the following inclusions hold (see [7]): $BFNE \Rightarrow BSNE \Rightarrow BQNE$.

The following lemmas are used in the sequel.

Lemma 2.2. [7] *Let $f : E \to R$ be a Legendre function which is uniformly Fréchet differentiable and on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E, and let $T : C \to E$ be a Bregman firmly nonexpansive operator. Then $F(T) = \hat{F}(T)$.*

Lemma 2.3. [12] *If f is a proper lower semicontinuous and convex function, and f^* is a weak* lower semicontinous and convex function, then, for all $z \in E$, $\Delta_p(z, J_q^*(\sum_{i=1}^N t_i J_p x_i)) \leq \sum_{i=1}^N t_i \Delta_p(z,x_i)$, where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.*

Lemma 2.4. [1] *Let E be a uniformly convex Banach space, and let $\{x_n\}$ $\{y_n\}$ be two sequences in E such that the first one is bound. If $\lim_{n\to\infty} \Delta_p(y_n,x_n) = 0$, then $\lim_{n\to\infty} ||y_n - x_n|| = 0$.*

Lemma 2.5. [20] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n$, $n \geq 0$, where (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; and (iii) $\gamma_n \geq 0, \sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 2.6. [9] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Consider the integer $\{m_k\}$ defined by $m_k = \max\{j \leq k : a_j < a_{j+1}\}$. Then $\{m_k\}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} m_n = \infty$, and for all $k \in \mathbb{N}$, the following estimates hold: $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$.*

3. Main Results

In this section, we introduce the iterative algorithm for the SEP in p-uniformly convex and uniformly smooth Banach spaces.

Let $1 \leq q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. To better give our results and convergence analysis, we assume that E_1, E_2, and E_3 are p-uniformly convex real Banach spaces which are also uniformly smooth, and E_1^*, E_2^*, E_3^* are dual spaces, respectively. Let C and Q be two nonempty closed convex subsets of E_1 and E_2. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be two bounded linear operators, respectively. We use Ω to denote the solution set of SEP, i.e,

$$\Omega = \{(x,y) \in E_1 \times E_2, Ax = By, x \in C, y \in Q\}.$$

Let $S = C \times Q$ in $E = E_1 \times E_2$ and $w = (x,y) \in S$, and define $G : E \to E_3$ by $G = [A, -B]$. The original SEP is to find $w = (x,y) \in S$ with $Gw = 0$. Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfy the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$; and (iii) $0 < a \leq \beta_n \leq b < 1$ for $a, b \in (0,1)$.

Now we present our strong convergence algorithm to solve the SEP as follows:

Algorithm 1

Step 1: Choose $\varepsilon > 0$, $z_1 \in E$, pick a fixed point $u \in E$, and set $n = 1$.
Step 2: Given the iterates $\{z_n\}$, and calculate the step size as shown below:

$$t_n^{q-1} \in \left(\varepsilon, \frac{q||Gz_n||^p}{C_q||G^* J_p^{E_3} Gz_n||^q} - \varepsilon\right), \quad n \in \Gamma$$

where $\Gamma := \{n \in \mathbb{N} : Gz_n \neq 0\}$, otherwise $t_n = t$, t is nonnegative real number.
Step 3: Compute $w_n = J_q^{E^*}[J_p^E z_n - t_n G^* J_p^{E_3} Gz_n]$.
Step 4: If $w_n = z_n$ stop. Otherwise, calculate z_{n+1} via

$$z_{n+1} = J_q^{E^*}[\alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n))], \quad \forall n \geq 1.$$

Step 5: Set $n = n+1$ and go to *Step 2*.

Theorem 3.1. *Let $\{z_n\}$ and $\{w_n\}$ be generated by Algorithm 3.1. Then $\{z_n\}$ and $\{w_n\}$ converge strongly to a solution $\hat{w}$ of the SEP, where $\hat{w} = \Pi_\Omega u$.*

Proof. First, we prove that the sequence $\{w_n\}$ is bounded. Let $w^* \in \Omega$. From Lemma 2.1, we obtain

$$\begin{aligned}
&\Delta_p(w^*, w_n) \\
&= \frac{||w^*||^p}{p} - \left\langle J_p^E z_n - t_n G^* J_p^{E_3} Gz_n, w^* \right\rangle + \frac{||J_p^E z_n - t_n G^* J_p^{E_3} Gz_n||^q}{q} \\
&\leq \frac{||w^*||^p}{p} - \left\langle J_p^E z_n - t_n G^* J_p^{E_3} Gz_n, w^* \right\rangle + \frac{||J_p^E z_n||^q}{q} - t_n \left\langle J_p^{E_3} Gz_n, Gz_n \right\rangle + \frac{C_q t_n^q}{q} ||G^* J_p^{E_3} Gz_n||^q \\
&= \frac{||w^*||^p}{p} - \left\langle J_p^E z_n, w^* \right\rangle + \frac{1}{q} ||J_p^E z_n||^q - t_n \left\langle J_p^{E_3} Gz_n, Gz_n - Gw^* \right\rangle + \frac{C_q t_n^q}{q} ||G^* J_p^{E_3} Gz_n||^q \\
&= V_p(w^*, J_p^E z_n) - t_n \left\langle J_p^{E_3} Gz_n, Gz_n - Gw^* \right\rangle + \frac{C_q t_n^q}{q} ||G^* J_p^{E_3} Gz_n||^q
\end{aligned}$$

$$
\begin{aligned}
&= \Delta_p(w^*, z_n) - t_n \left\langle J_p^{E_3} G z_n, G z_n - G w^* \right\rangle + \frac{C_q t_n^q}{q} ||G^* J_p^{E_3} G z_n||^q \\
&= \Delta_p(w^*, z_n) - t_n (||G z_n||^p - \frac{C_q t_n^{q-1}}{q} ||G^* J_p^{E_3} G z_n||^q).
\end{aligned} \tag{3.1}
$$

Using the range of values of $\left\{ t_n^{q-1} \right\}$, we have

$$
\Delta_p(w^*, w_n) \leq \Delta_p(w^*, z_n). \tag{3.2}
$$

Furthermore, using (3.2) and Lemma 2.3, we have that

$$
\begin{aligned}
\Delta_p(w^*, z_{n+1}) &= \Delta_p(w^*, J_q^{E^*}[\alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n))]) \\
&\leq \alpha_n \Delta_p(w^*, u) + (1-\alpha_n)\beta_n \Delta_p(w^*, w_n) + (1-\alpha_n)(1-\beta_n)\Delta_p(w^*, \Pi_S w_n) \\
&\leq \alpha_n \Delta_p(w^*, u) + (1-\alpha_n)\Delta_p(w^*, w_n) \\
&\leq \alpha_n \Delta_p(w^*, u) + (1-\alpha_n)\Delta_p(w^*, z_n) \\
&\leq \max \left\{ \Delta_p(w^*, u), \Delta_p(w^*, z_n) \right\} \\
&\;\vdots \\
&\leq \max \left\{ \Delta_p(w^*, u), \Delta_p(w^*, z_0) \right\}.
\end{aligned} \tag{3.3}
$$

Hence, $\{\Delta_p(w^*, z_n)\}$ is bounded. It then follows that $\{z_n\}$, and $\{w_n\}$ are also bounded.

Second, we prove that the following inequalities holds:

$$
\text{(i)} \;\; \Delta_p(w^*, w_{n+1}) \leq (1-\alpha_n)\Delta_p(w^*, w_n) + \alpha_n \left\langle J_p^E u - J_p^E w^*, z_{n+1} - w^* \right\rangle,
$$

and

$$
\text{(ii)} \;\; -1 \leq \limsup_{n\to\infty} \left\langle J_p^E u - J_p^E w^*, z_{n+1} - w^* \right\rangle < \infty.
$$

By (2.3) and (2.4), we have

$$
\begin{aligned}
&\Delta_p(w^*, w_{n+1}) \\
&\leq \Delta_p(w^*, J_q^{E^*}[\alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n))]) \\
&= V_p(w^*, \alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n)) \\
&\leq V_p(w^*, \alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n)) - \alpha_n (J_p^E u - J_p^E w^*)) \\
&\quad - \left\langle -\alpha_n (J_p^E u - J_p^E w^*), J_q^{E^*}[\alpha_n J_p^E(u) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n))] - w^* \right\rangle \\
&= V_p(w^*, \alpha_n J_p^E(w^*) + (1-\alpha_n)(\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n))) + \alpha_n \left\langle J_p^E u - J_p^E w^*, z_{n+1} - w^* \right\rangle \\
&\leq \alpha_n \Delta_p(w^*, w^*) + (1-\alpha_n)\beta_n \Delta_p(w^*, w_n) + (1-\alpha_n)(1-\beta_n)\Delta_p(w^*, \Pi_S w_n) \\
&\quad + \alpha_n \left\langle J_p^E u - J_p^E w^*, z_{n+1} - w^* \right\rangle \\
&\leq (1-\alpha_n)\Delta_p(w^*, w_n) + \alpha_n \left\langle J_p^E u - J_p^E w^*, z_{n+1} - w^* \right\rangle.
\end{aligned} \tag{3.4}
$$

This establishes (i).

Next, we prove (ii). Since $\{z_n\}$ is bounded, then

$$\sup_{n\geq 0}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle \leq \sup_{n\geq 0}||J_p^E u - J_p^E w^*||\cdot||z_{n+1}-w^*|| < \infty,$$

which means that $\limsup_{n\to\infty}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle < \infty$. Now we prove that

$$\limsup_{n\to\infty}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle \geq -1.$$

We suppose $\limsup_{n\to\infty}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle < -1$. Then we can choose $n_0 \in \mathbb{N}$ such that $\limsup_{n\to\infty}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle < -1$ for all $n \geq n_0$. Then, for all $n \geq n_0$, it follows from (3.2) and (3.4) that

$$\begin{aligned}\Delta_p(w^*, z_{n+1}) &\leq (1-\alpha_n)\Delta_p(w^*, z_n) + \alpha_n\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle\\ &< (1-\alpha_n)\Delta_p(w^*, z_n) - \alpha_n\\ &= \Delta_p(w^*, z_n) - \alpha_n(\Delta_p(w^*, z_n)+1)\\ &< \Delta_p(w^*, z_n) - \alpha_n.\end{aligned}$$

Taking $\limsup$ of the last inequality, we have $\limsup_{n\to\infty}\Delta_p(w^*, z_{n+1}) \leq \Delta_p(w^*, z_{n_0}) - \lim_{n\to\infty}\sum_{i=n_0}^{\infty}\alpha_n = -\infty$, which is a nonnegative sequence contradiction with $\{\Delta_p(w^*, z_n)\}$. Thus

$$\limsup_{n\to\infty}\left\langle J_p^E u - J_p^E w^*, z_{n+1}-w^*\right\rangle \geq -1.$$

We now consider the following two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\left\{\Delta_p(w^*, w_n)\right\}_{n=n_0}^{\infty}$ is non-increasing. Then $\left\{\Delta_p(w^*, w_n)\right\}_{n=n_0}^{\infty}$ converges and $\lim_{n\to\infty}(\Delta_p(w^*, w_{n+1}) - \Delta_p(w^*, w_n)) = 0$. From (3.1), we can obtain

$$t_n(||Gz_n||^p - \frac{C_q t_n^q}{q}||G^* J_p^{E_3} G z_n||^q) \leq \Delta_p(w^*, z_n) - \Delta_p(w^*, w_n).$$

From (3.3), we have

$$\begin{aligned}0 &\leq \Delta_p(w^*, z_{n+1}) - \Delta_p(w^*, w_{n+1})\\ &\leq \alpha_n\Delta_p(w^*, u) + (\Delta_p(w^*, w_n) - \Delta_p(w^*, w_{n+1})) - \alpha_n\Delta_p(w^*, w_n) \to 0, n\to\infty.\end{aligned}$$

Thus $\lim_{n\to\infty}(\Delta_p(w^*, z_n) - \Delta_p(w^*, w_n)) = 0$ and then

$$\lim_{n\to\infty}(||Gz_n||^p - \frac{C_q t_n^q}{q}||G^* J_p^{E_3} G z_n||^q) = 0. \tag{3.5}$$

By the choice of the step size, it holds that

$$t_n^{q-1} < \frac{q||Gz_n||^p}{C_q||G^* J_p^{E_3} G z_n||^q} - \varepsilon,$$

that is,

$$\begin{aligned}\frac{\varepsilon C_q}{q}||G^* J_p^{E_3} G z_n||^q &< ||Gz_n||^p - \frac{C_q t_n^{q-1}}{q}||G^* J_p^{E_3} G z_n||^q\\ &\leq \Delta_p(w^*, z_n) - \Delta_p(w^*, w_n) \to 0\end{aligned}$$

as $n \to \infty$. Hence, $\lim_{n\to\infty} ||G^* J_p^{E_3} G z_n|| = 0$. From (3.5), we obtain $\lim_{n\to\infty} ||Gz_n|| = 0$. Let $u_n = J_q^*[\beta_n J_p^E(w_n) + (1-\beta_n) J_p^E(\Pi_S w_n)]$. It follows that

$$\Delta_p(w^*, u_n) \le \beta_n \Delta_p(w^*, w_n) + (1-\beta_n)\Delta_p(w^*, \Pi_S w_n) \le \Delta_p(w^*, w_n).$$

Hence, we have

$$\begin{aligned}
0 &\le \Delta_p(w^*, w_n) - \Delta_p(w^*, u_n) \\
&\le \Delta_p(w^*, w_n) - \Delta_p(w^*, w_{n+1}) + \Delta_p(w^*, z_{n+1}) - \Delta_p(w^*, u_n) \\
&\le \Delta_p(w^*, w_n) - \Delta_p(w^*, w_{n+1}) + \alpha_n((\Delta_p(w^*, u) - \Delta_p(w^*, u_n))) \to 0, n \to \infty.
\end{aligned}$$

Observe that

$$\begin{aligned}
\Delta_p(w^*, u_n) &\le \beta_n \Delta_p(w^*, w_n) + (1-\beta_n)\Delta_p(w^*, \Pi_S w_n) \\
&= \Delta_p(w^*, w_n) - (1-\beta_n)\Delta_p(w^*, w_n) + (1-\beta_n)\Delta_p(w^*, \Pi_S w_n) \\
&= \Delta_p(w^*, w_n) + (1-\beta_n)(\Delta_p(w^*, \Pi_S w_n) - \Delta_p(w^*, w_n)).
\end{aligned}$$

Thus

$$(1-\beta_n)(\Delta_p(w^*, w_n) - \Delta_p(w^*, \Pi_S w_n)) \le \Delta_p(w^*, w_n) - \Delta_p(w^*, u_n) \to 0, n \to \infty.$$

Since Π_S is Bregman strongly nonexpansive, we have that $\lim_{n\to\infty} \Delta_p(\Pi_S w_n, w_n) = 0$, which implies by Lemma 2.4 that $\lim_{n\to\infty} ||\Pi_S w_n - w_n|| = 0$. Since the sequence $\{w_n\}$ is bounded, there exists a sequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup w \in E$. From $\lim_{n\to\infty} ||\Pi_S w_n - w_n|| = 0$, it follows that $w \in \hat{F}(\Pi_S) = F(\Pi_S)$. Thus $w \in S$. We obtain from the definition of w_n that

$$\begin{aligned}
0 &\le ||J_p^E w_n - J_p^E z_n|| \\
&= t_n ||G^*|| ||J_p^{E_3} G z_n|| \\
&= t_n ||G^*|| ||G z_n|| \to 0, n \to \infty.
\end{aligned}$$

Since $J_q^{E^*}$ is norm-to-norm uniformly continuous, we have $\lim_{n\to\infty} ||w_n - z_n|| = 0$. There exists a sequence $\{z_{n_j}\}$ of $\{z_n\}$. It follows that $\lim_{j\to\infty} ||Gz_{n_j}|| = 0$. By the continuity of G, $Gw_{n_j} \rightharpoonup Gw$, as $j \to \infty$, and $||Gw_{n_j}|| - ||Gz_{n_j}|| \le ||G(w_n - z_n)|| \le ||G|| ||w_{n_j} - z_{n_j}|| \to 0$ as $j \to \infty$. Hence, $||Gw_{n_j}|| \to 0$ and

$$\begin{aligned}
0 &\le ||Gw||^p = \left\langle J_p^{E_3} Gw, Gw \right\rangle \\
&= \lim_{n\to\infty} \left\langle J_p^{E_3} Gw, Gw_{n_j} \right\rangle \\
&\le \lim_{n\to\infty} ||J_p^{E_3} Gw|| ||Gw_{n_j}|| = 0.
\end{aligned}$$

Thus $Gw = 0$. We now prove that $\{w_n\}$ converges strongly to a point $\hat{w} = \Pi_\Omega u$. To do this, it suffices to show that $\limsup_{n\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, z_{n+1} - \hat{w} \right\rangle \le 0$. Choose $\{w_{n_j}\}$ of $\{w_n\}$. Without loss of generality, we suppose $w_{n_{j+1}} \rightharpoonup w, j \to \infty$ with

$$\limsup_{n\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, w_{n+1} - \hat{w} \right\rangle = \limsup_{j\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, w_{n_j+1} - \hat{w} \right\rangle.$$

Let $\hat{w} = \Pi_\Omega u$. From (2.2), we have that

$$\begin{aligned}\limsup_{n\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, w_{n+1} - \hat{w} \right\rangle &= \limsup_{j\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, w_{n_j+1} - \hat{w} \right\rangle \\ &= \left\langle J_p^E u - J_p^E \hat{w}, w - \hat{w} \right\rangle \le 0.\end{aligned}$$

Since $||w_n - z_n|| \to 0, n \to \infty$, we have

$$\limsup_{n\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, z_{n+1} - \hat{w} \right\rangle = \limsup_{n\to\infty} \left\langle J_p^E u - J_p^E \hat{w}, w_{n+1} - \hat{w} \right\rangle \le 0.$$

Putting $w^* = \hat{w}$ in (3.4). Using Lemma 2.5, we obtain $\lim_{n\to\infty} \Delta_p(\hat{w}, w_n) = 0$. This implies that $||w_n - \hat{w}|| \to 0, n \to \infty$. Thus $\{w_n\}$ converges strongly to a point $\hat{w} = \Pi_\Omega u$.

Case 2. Suppose that $\left\{\Delta_p(w^*, w_n)\right\}_{n=n_0}^{\infty}$ is not monotonically decreasing.

From Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that, as $k \to \infty$, $\Delta_p(w^*, w_{m_k}) \le \Delta_p(w^*, w_{m_{k+1}})$ and $\Delta_p(w^*, w_k) \le \Delta_p(w^*, w_{m_{k+1}})$. Following the proof in *Case 1*, we obtain that $\lim_{k\to\infty} ||Gw_{m_k}|| = 0$, $\lim_{k\to\infty} ||w_{m_k} - \Pi_S w_{m_k}|| = 0$, and

$$\limsup_{k\to\infty} \left\langle J_p^E u - J_p^E w, z_{n_k+1} - \hat{w} \right\rangle \le 0.$$

Also, we have

$$\Delta_p(\hat{w}, w_{m_k+1}) \le (1-\alpha_{m_k})\Delta_p(\hat{w}, w_{m_k}) + \alpha_{m_k} \left\langle J_p^E u - J_p^E w, z_{m_k+1} - \hat{w} \right\rangle,$$

which implies

$$\alpha_{m_k}\Delta_p(\hat{w}, w_{m_k}) \le \Delta_p(\hat{w}, w_{m_k}) - \Delta_p(\hat{w}, w_{m_k+1}) + \alpha_{m_k} \left\langle J_p^E u - J_p^E w, z_{m_k+1} - \hat{w} \right\rangle.$$

That is,

$$\Delta_p(\hat{w}, w_{m_k}) \le \left\langle J_p^E u - J_p^E w, z_{m_k+1} - \hat{w} \right\rangle.$$

Thus $\lim_{k\to\infty} \Delta_p(\hat{w}, w_{m_k}) = 0$. Since $\Delta_p(\hat{w}, w_k) \le \Delta_p(\hat{w}, w_{m_{k+1}})$ all $k \in \mathbb{N}$, we conclude that $w_k \to \hat{w}$ as $k \to \infty$. This completes the proof. □

Remark 3.2. Theorem 3.1 holds in p-uniformly convex and uniformly smooth Banach spaces. The algorithm in Theorem 3.1 only needs to perform one projection on the feasible set per iteration. We use a self-adaptive method to select the step size, which is relatively simple because it avoids calculating the norm of operator G.

It is also important to note that our algorithm is new even in Hilbert spaces. Indeed, in Hilbert spaces, the following corollary can be drawn.

Corollary 3.3. *Let H_1, H_2, and H_3 be real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subsets of H_1 and H_2. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set Ω of the SEP is nonempty. Let $S = C \times Q$ in $H_1 \times H_2$, $w = (x,y) \in S$, and $G : H \to H_3$ by $G = [A, -B]$. For sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, and a fixed point $u \in H$, suppose that the sequences $\{z_n\}$ and $\{w_n\}$ are generated by the following iteration procedure: for $\forall z_1 \in H$,*

$$\begin{cases} w_n = z_n - t_n G^* G z_n, \\ z_{n+1} = \alpha_n u + (1-\alpha_n)(\beta_n w_n + (1-\beta_n) P_S w_n), \qquad \forall n \ge 1, \end{cases}$$

where P_S represents metric projection onto S. Assume the following conditions holds

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \le \beta_n \le b < 1$ *for* $a, b \in (0,1)$;
(iv) $0 < t_n < \frac{2}{||G||^2}$.
Then, $\{z_n\}$ *and* $\{w_n\}$ *converge strongly to a solution* $\hat{w}$ *of the SEP, where* $\hat{w} = P_\Omega u$.

4. Numerical Experiment

In this section, we provide a numerical example to prove the convergence of our proposed algorithm. We consider the example of Theorem 3.1 in $(\mathbb{R}^3, ||.||_2)$.

Take $S := \{w = (w_1, w_2, w_3) \in \mathbb{R}^3 : \langle a, w\rangle \ge b\}$, where $a = (4, -1, 3)$ and $b = -5$. Then

$$P_S(w) = \frac{b - \langle a, w\rangle}{||a||_2^2} a + w.$$

Let $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2n}$, and

$$G = \begin{pmatrix} 3 & -2 & -8 \\ -5 & 2 & -2 \\ -4 & -2 & 6 \end{pmatrix}.$$

Then Algorithm 1 becomes

$$\begin{cases} w_n = z_n - t_n G^* G z_n, \\ z_{n+1} = \frac{u}{n+1} + (1 - \frac{1}{n+1})[(1 - \frac{1}{2n}) w_n + \frac{1}{2n} P_S w_n], \quad n \ge 1. \end{cases}$$

Choosing different z_1, u, t_n, and stopping criterion $\frac{||z_{n+1} - z_n||}{||z_2 - z_1||} \le 10^{-2}$, our proposed algorithm is always convergent.

Take $u = (1, 1, -1)$ and $z_1 = (4, -1, 3)$, Case 1 $t_n = 0.01$ and Case 2 $t_n = 0.008$.

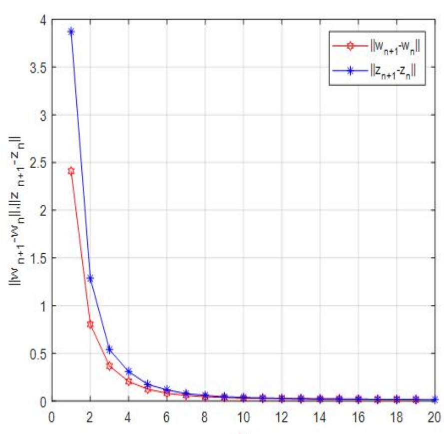

Figure 1. Case 1

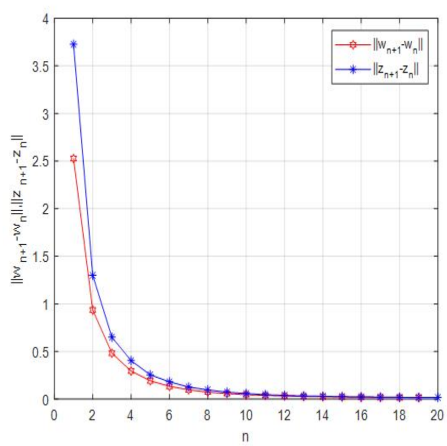

Figure 2. Case 2

Take $u = (2, 0, 1)$ and $z_1 = (5, -1, 2)$, Case 3 $t_n = 0.01$ and Case 4 $t_n = 0.008$.

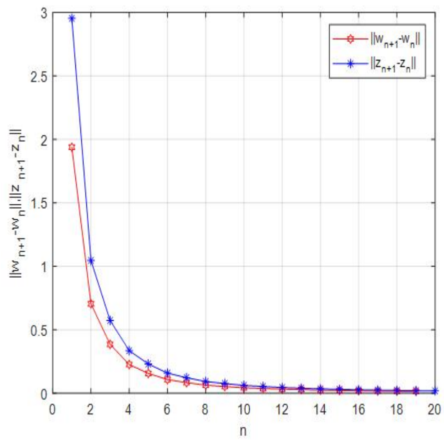

FIGURE 3. Case 3

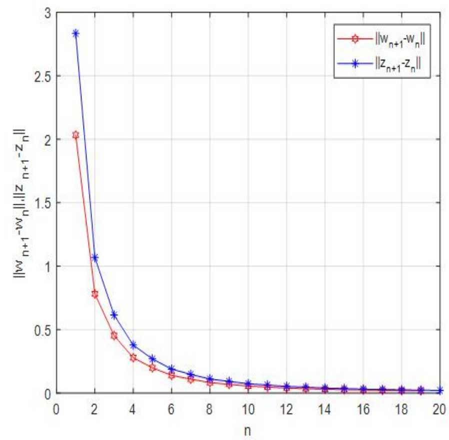

FIGURE 4. Case 4

REFERENCES

[1] D. Butnariu, I.N. Iusem, E. Resmerita, Totall convexity for powers of the norm in uniformly convex Banach spaces, J. Convex Anal. 7 (2000) 319-334.

[2] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse Probl. 18 (2002) 441-453.

[3] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2004) 103-120.

[4] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994) 221-239.

[5] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, Netherland, 1990.

[6] L.O. Jolaoso, Y. Shehu, Y.J. Cho, Convergence analysis for variational inequalities and fixed point problems in reflexive Banach spaces, J. Inequal. Appl. 2021 (2021) 44.

[7] G. Kassay, S. Reich, S. Sabach, Iterative methods for solving system of variational inequalities in reflexive Banach spaces, SIAM J. Optim. 21 (2011) 1319-1344.

[8] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II. Springer, Berlin, 1979.

[9] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008) 899-912.

[10] O.T. Mewomo, F.U. Ogbuisi, Convergence analysis of an iterative method for solving multiple-set split feasibility problems in certain Banach spaces, Quaestiones Math. 41 (2018) 129-148.

[11] A. Moudafi, A relaxed alternating CQ-algorithms for convex feasibility problem, Nonlinear Anal. 79 (2013) 117-121.

[12] R.R. Phelps, Convex Functions, Monotone Operators and Differentiablity, 2nd edn. Lecture Notes in Mathematics, vol. 1364, Spinger, Berlin, 1993.

[13] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019) 1497-1506.

[14] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018) 157-165.

[15] S. Reich, T.M. Tuyen, Projection algorithms for solving the split feasibility problem with multiple output sets, J. Optim. Theory Appl. 190 (2021) 861–878.

[16] S. Reich, S. Sabach, A strong convergence theorem for proximal-type algorithm in reflexive Banach spaces, J. Nonlinear Convex Anal. 10 (2009) 471-485.

[17] F. Schöpfer, T. Schuster, A.K. Louis, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, Inverse Probl. 24 (2008), 055008.

[18] Y. Shehu, Iterative methods for split feasibility problems in certain Banach spaces, J. Nonlinear Convex Anal. 16 (2015) 2351-2364.
[19] L. Shi, R. Chen, Y. Wu, Strong convergence of iterative algorithms for split equality problem, J. Inequal. Appl. 2014 (2014) Article ID 478.
[20] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240-256.
[21] Z.B. Xu, G.F. Roach, Characteristic inequalities of uniformly convex and uniformly smooth Banach, J. Math. Anal. Appl. 157 (1991) 189-210.
[22] H. Zhou, X. Qin, Fixed Points of Nonlinear Operators, De Gruyter, Berlin, 2020.