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THE AVERAGING PRINCIPLE FOR HILFER FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY TIME-CHANGED LÉVY NOISE

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Abstract. We consider an averaging principle for Hilfer fractional stochastic differential equations driven by time-changed Lévy noise with variable delays. Under certain assumptions, we prove that the solutions of fractional stochastic differential delay equations with time-changed Lévy noise can be approximated by solutions of the associated averaged stochastic differential equations in mean square convergence and probability, respectively. Finally, an example is given to illustrate the theoretical result.

Keywords. Averaging principle; Hilfer fractional stochastic differential equations; Time-changed Lévy noise; Variable delays.

1. Introduction

In the contemporary world, many systems, which are interrupted by various random environmental effects, are described by fractional stochastic differential equations (FSDEs) with Brownian motion, the fractional Brownian motion, the Lévy processes, and so on. In [1, 2], Zhu studied the asymptotic stability in the pth moment for stochastic differential equations with Lévy noise and the stability analysis of stochastic delay differential equations with Lévy noise. In [3], Zhu studied the well-known Razumikhin-type theorem for a class of stochastic functional differential equations with Lévy noise and Markov switching. However, we are even more in need of the fractional stochastic differential equations with delays driven by time-changed Lévy processes. On one hand, Non-Gaussian type Lévy processes allow their trajectories to change continuously most of the time. On the other hand, it allows jump discontinuities occurring at random times. Stochastic averaging provides a vigorous tool in order to strike a balance between realistically complex models and comparably simpler modelsthis, which was initiated by Khasminskii in the groundbreaking work (see [4]). In recent years, the stochastic averaging principle has been evolved for a great diversity of stochastic differential equations; see, e.g., [5, 6, 7, 8, 9, 10] and the references therein.

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In [11], Xu et al. studied an averaging principle for fractional stochastic differential equations with Lévy noise:

$$\begin{cases} D_t^{\beta} X(t) = b(t, X(t)) + \sigma(t, X(t)) \frac{dL(t)}{dt}, \\ X(0) = X_0, \end{cases}$$

where D_t^{β} is Caputo fractional derivative, $\beta \in (\frac{1}{2}, 1)$ initial value $\mathbb{E}|X_0|^2 < \infty$, function $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are measurable, and $L(t)$ is a r-dimensional Lévy motion.

In [12], Ahmed and Zhu investigated the averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps:

$$\begin{cases} D_{0^+}^{\aleph,\hbar} x(t) = \mathfrak{R}(t, x(t), x(t-\tau)) + \sigma(t, , x(t), x(t-\tau)) \dfrac{dB(t)}{dt} \\ + \displaystyle\int_V h(t, x(t), x(t-\tau), v) \tilde{N}(dt, dv), \qquad t \in J = (0, T], \\ x(t) = \phi(t), \qquad -\tau \le t \le 0, \\ I_{0^+}^{(1-\aleph)(1-\hbar)} x(0) = \phi(0), \end{cases}$$

where $D_{0^+}^{\aleph,\hbar}$ is the Hilfer fractional derivative with $0 \le \aleph \le 1, \frac{1}{2} < \hbar < 1, \mathfrak{R} : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : J \times \mathbb{R}^n \times \mathbb{R}^n \times V \to \mathbb{R}^n$.

Define $\tilde{N}(dt, dz) := N(t, dv) - t\lambda(dv)$. The Poisson martingale measure generated by p_t, $B(t)$ is m-dimensional Brownian motion defined on the complete probability space, and $\phi : [-\tau, 0] \to \mathbb{R}^n$ is a continuous function, satisfying $E|\phi(t)|^2 < \infty$. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R})$ be the family of continuous $\mathbb{R}$-valued functions. Suppose that the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is equipped with a filtration $\{\mathscr{F}_t\}_{t\ge 0}$ satisfying the usual hypotheses of completeness and right continuity. Fix $m, n \in \mathbb{N}$, and let $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ be an m-dimensional $\{F_t\}_{t\ge 0}$-Brownian motion. Let $B = (B_t)_{t\ge 0}$ be a standard Brownian motion, and let $E = (E_t)_{t\ge 0}$ be a stochastic process defined by the inverse of a subordinator $D = (D_t)_{t\ge 0}$ with infinite Lévy measure, independent of B. The time change E_t is continuous and nondecreasing. E_t is not Markovian. The composition $B \circ E = (B_{E_t})_{t\ge 0}$ is called a time-changed Brownian motion. According to [13], we define the Lévy measure on $\mathbb{R}^n \backslash \{0\}$ by $\nu(dy) := \frac{dy}{|y|^{n+1}}$, and let N be the Poisson random measure of $\{F_t\}_{t\ge 0}$. Let $\tilde{N}(dt, dy) := N(dt, dy) - \frac{dtdy}{|y|^{n+1}}$ be the compensated $\{F_t\}_{t\ge 0}$-martingale measure. Both N and $\tilde{N}$ are independent of Brownian motion B.

In [13], Shen et al. studied an averaging principle for stochastic differential delay equations driven by time-changed Lévy noise:

$$\begin{aligned} dx(t) = \; & f(t, E_t, x(t-), x(t-\delta(t)) dE_t + g(t, E_t, x(t-), x(t-\delta(t))) dB_{E_t} \\ & + \int_{|z|<c} h(t, E_t, x(t-), x(t-\delta(t)), z) \tilde{N}(dE_t, dz), \qquad t \in [0, T], \end{aligned}$$

with the initial value $x(0) = \xi = \xi(\theta) : -\tau \le \theta \le 0 \in C([-\tau, 0]; \mathbb{R}^n)$ fulfilling $\xi(0) \in \mathbb{R}^n$ and $\mathbb{E}\|\xi\|^2 < \infty$, where the functions $f : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, g : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}, h : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}^n$ are measurable continuous functions, $\delta : [0,T] \to [0,\tau]$, and the constant $c > 0$ is the maximum allowable jump size. However, the averaging principle for Hilfer fractional stochastic equations driven by time-changed Lévy noise with variable delays have not yet been considered in the literature.

Motivated by the above discussion, in this paper, we generalize Shen's work to Hilfer type fractional stochastic equations driven by time-changed Lévy noise with variable delays:

$$\begin{cases} D_{0^+}^{\aleph,\hbar}x(t) = f(t,E_t,x(t-),x(t-\delta(t))dE_t + g(t,E_t,x(t-),x(t-\delta(t)))dB_{E_t} \\ + \int_{|z|<c} h(t,E_t,x(t-),x(t-\delta(t)),z)\tilde{N}(dE_t,dz), \qquad t\in J, \\ x(t) = \phi(t), \qquad -\tau \le t \le 0, \\ I_{0^+}^{(1-\aleph)(1-\hbar)}x(0) = \phi(0), \end{cases} \tag{1.1}$$

where $J=(0,T], D_{0^+}^{\aleph,\hbar}$ is the Hilfer fractional derivative with $0\le\aleph\le 1, \frac{1}{2}<\hbar<1$, where the functions $f:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n, g:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}, h:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\})\to\mathbb{R}^n$ are measurable continuous functions, $\delta:[0,T]\to[0,\tau]$, and the constant $c>0$ is the maximum allowable jump size, and $\phi:[\tau,0]\to\mathbb{R}^n$ is continuous function.

The novelties and contributions of this paper are as follows:

(i) One of the goals of this paper is to discuss the averaging principle for Hilfer fractional stochastic differential equations driven by time-changed Lévy noise with variable delays. For all we know, there are only few papers about this issue.

(ii) The results we obtained can be applied to the stochastic differential equations with standard Riemann-Liouville (R-L for short) fractional derivative, Caputo fractional derivative, and integral order stochastic differential equations,respectively.

The rest of this paper is organised as follows. In the next section, we present appropriate conditions to the relevant FSDEs (1.1). Section 3 is devoted to our main results and their proofs. Section 4, the last section, presents an example to illustrate the obtained averaging principle.

2. Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.

Definition 2.1. [14] The fractional integral operator of order $\aleph>0$ for a function f with the lower bound 0 is defined as

$$I_{0^+}^{\aleph}f(t) = \frac{1}{\Gamma(\aleph)}\int_0^t \frac{f(s)}{(t-s)^{1-\aleph}}ds, t>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. [15] The R-L derivative of order $\aleph>0, n-1<\aleph<n, n\in\mathbb{N}$ for a function f is defined as

$${}^{L}D_{0^+}^{\aleph}f(t) = \frac{1}{\Gamma(n-\aleph)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\aleph+1-n}}ds, t>0, s\in\mathbb{R}^+.$$

Definition 2.3. [15] The Caputo derivative of order $\aleph>0, n-1<\aleph<n, n\in\mathbb{N}$ for a function f is defined as

$${}^{C}D_{0^+}^{\aleph}f(t) = \frac{1}{\Gamma(n-\aleph)}\int_0^t \frac{f^n(s)}{(t-s)^{\aleph+1-n}}ds = I_{0^+}^{n-\aleph}f^n(t), t>0, s\in\mathbb{R}^+.$$

Definition 2.4. [16] The Hilfer fractional derivative of order $0 \leq \aleph \leq 1$ and $0 \leq \hbar \leq 1$ is defined as

$$D_{0^+}^{\aleph,\hbar} f(t) = I_{0^+}^{\hbar(1-\aleph)} \frac{d}{dt} I_{0^+}^{(1-\aleph)(1-\hbar)} f(t).$$

Remark 2.5. (i) If $\hbar = 0, 0 < \aleph < 1$, then the Hilfer fractional derivative can be written as the standard R-L fractional derivative:

$$D_{0^+}^{\aleph,0} f(t) = \frac{d}{dt} I_{0^+}^{1-\aleph} f(t) =^L D_{0^+}^{\aleph} f(t).$$

(ii) If $\hbar = 1, 0 < \aleph < 1$, then the Hilfer fractional derivative can be written as the standard Caputo fractional derivative:

$$D_{0^+}^{\aleph,1} f(t) = I_{0^+}^{1-\aleph} \frac{d}{dt} f(t) =^C D_{0^+}^{\aleph} f(t).$$

In order to derive the main results of this paper, we require that the functions $f(t_1,t_2,x,y), g(t_1, t_2,x,y)$ and $h(t_1,t_2,x,y,z)$ satisfy the following assumptions:

Assumpiton 2.6. For any $x_1,x_2,y_1,y_2 \in \mathbb{R}^n$, there exists a positive bounded function $\varphi(t) \in L^2[-\tau,T]$ such that

$$|f(t,E_t,x_1,y_1) - f(t,E_t,x_2,y_2)| \vee |g(t,E_t,x_1,y_1) - g(t,E_t,x_2,y_2)| \leq \varphi(t)(|x_1 - x_2| + |y_1 - y_2|),$$

and

$$\int_{|z|<c} |h(t,E_t,x_1,y_1,z) - h(t,E_t,x_2,y_2,z)|^2 \nu(dz) \leq \varphi(t)(|x_1 - x_2|^2 + |y_1 - y_2|^2),$$

where $x \vee y = \max\{x,y\}$.

Assumpiton 2.7. For all $T_1 \in [0,T], x,y \in \mathbb{R}^n$, there exist several positive bounded functions $\lambda_i \leq C_i$ such that

$$\frac{1}{T_1} \int_0^{T_1} |f(s,E_s,x,y) - \bar{f}(x,y)| dE_s \leq \lambda_1(T_1)(|x| + |y|),$$

$$\frac{1}{T_1} \int_0^{T_1} |g(s,E_s,x,y) - \bar{g}(x,y)|^2 dE_s \leq \lambda_2(T_1)(|x|^2 + |y|^2),$$

and

$$\frac{1}{T_1} \int_0^{T_1} \int_{|z|<c} |h(s,E_s,x,y,z) - \bar{h}(x,y,z)|^2 \nu(dz) dE_s \leq \lambda_3(T_1)(|x|^2 + |y|^2),$$

where $\lim_{T_1 \to \infty} \lambda_i(T_1) = 0, i = 1,2,3$, $\bar{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \bar{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $\bar{h} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$ are measurable functions.

Lemma 2.8. *(Time-changed Gronwall's inequality [17]) Suppose that $D(t)$ is a β-stable subordinator and that E_t is the associated inverse stable subordinator. Let $T > 0$ and $x, v : \Omega \times [0,T] \to \mathbb{R}_+$ be $\mathscr{F}_t$-measurable functions which are integrable with respect to E_t. Assume that $u_0 \geq 0$ is a constant. Then, the inequality $x(t) \leq u_0 + \int_0^t v(s)x(s)dE_s$, $0 \leq t \leq T$ implies, almost surely, that $x(t) \leq u_0 exp(\int_0^t v(s)dE_s), 0 \leq t \leq T$.*

Lemma 2.9. *(Burkholder-Davis-Gundy inequality [18]) For any $p > 0$, there exists a constant $b_p > 0$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t|^p\right] \le b_p\mathbb{E}\left[[M,M]_T^{p/2}\right] \tag{2.1}$$

for any stopping time T and any continuous local martingale M with quadratic variation [M,M]. The constant b_p can be taken independently of T and M.

3. Main Results

In this section, we study the averaging principle for fractional stochastic differential equations driven by time-changed Lévy noise with variable delays. According to [19, 20], the standard form of equation (1.1) is

$$\begin{aligned} x^{\varepsilon}(t) = & \frac{t^{(\aleph-1)(1-\hbar)}}{\Gamma(\aleph(1-\hbar)+\hbar)}\phi(0) + \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} f(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dE_s \\ & + \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} g(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dB_{E_s} \\ & + \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} h(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)), z)\tilde{N}(dE_s, dz), \end{aligned} \tag{3.1}$$

with initial value $\phi(0)$; the coefficients have the same definitions and conditions as in Equation (1.1), and $\varepsilon \in (0, \varepsilon_0]$ is a positive parameter with ε_0 being a fixed number.

According to the Khasminskii type averaging principle, we consider the following averaged FSDEs which correspond to the original standard form (3.1)

$$\begin{aligned} \hat{x}(t) = & \frac{t^{(\aleph-1)(1-\hbar)}}{\Gamma(\aleph(1-\hbar)+\hbar)}\phi(0) + \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{f}(\hat{x}(s-), \hat{x}(s-\delta(s)))dE_s \\ & + \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{g}(\hat{x}(s-), \hat{x}(s-\delta(s)))dB_{E_s} \\ & + \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{h}(\hat{x}(s-), \hat{x}(s-\delta(s)), z)d\tilde{N}(dE_s, dz), \end{aligned}$$

where the measurable functions $\bar{f}, \bar{g}, \bar{h}$ satisfy Assumption 2.7.

Theorem 3.1. *Suppose that Assumptions 2.6 and 2.7 hold. Then, for a given arbitrarily small number $\delta_1 > 0$, there exist $L > 0, \varepsilon_1 \in (0, \varepsilon_0]$, and $\beta \in (0, \alpha-1)$ such that, for any $\varepsilon \in (0, \varepsilon_1]$,*

$$\mathbb{E}\left(\sup_{[-\tau, L\varepsilon^{-\beta}]} |x^{\varepsilon}(t) - \hat{x}(t)|^2\right) \le \delta_1.$$

Proof. For any $t' \in [0,T]$, we have

$$\begin{aligned}
& x^{\varepsilon}(t') - \hat{x}(t') \\
= \;& \frac{1}{\Gamma(\aleph)} \int_0^{t'} (t' - E_{\frac{s'}{\varepsilon}})^{\aleph-1} [f(\frac{s'}{\varepsilon}, E_{\frac{s'}{\varepsilon}}, x^{\varepsilon}(s'-), x^{\varepsilon}(s' - \delta(s'))) \\
& - \bar{f}(\hat{x}(s'-), \hat{x}(s' - \delta(s')))] dE_{s'} \\
& + \frac{1}{\Gamma(\aleph)} \int_0^{t'} (t' - E_{\frac{s'}{\varepsilon}})^{\aleph-1} [g(\frac{s'}{\varepsilon}, E_{\frac{s'}{\varepsilon}}, x^{\varepsilon}(s'-), x^{\varepsilon}(s' - \delta(s'))) \\
& - \bar{g}(\hat{x}(s'-), \hat{x}(s' - \delta(s')))] dB_{E_{s'}} \\
& + \frac{1}{\Gamma(\aleph)} \int_0^{t'} \int_{|z|<c} (t' - E_{\frac{s'}{\varepsilon}})^{\aleph-1} [h(\frac{s'}{\varepsilon}, E_{\frac{s'}{\varepsilon}}, x^{\varepsilon}(s'-), x^{\varepsilon}(s' - \delta(s')), z) \\
& - \bar{h}(\hat{x}(s'-), \hat{x}(s' - \delta(s')), z)] \tilde{N} d(E_{s'}, dz).
\end{aligned} \tag{3.2}$$

Letting $s = \frac{s'}{\varepsilon}$ and $t = \frac{t'}{\varepsilon}$, we can rewrite (3.2) as

$$\begin{aligned}
& x^{\varepsilon}(\varepsilon t) - \hat{x}(\varepsilon t) \\
= \;& \frac{\varepsilon^{\alpha}}{\Gamma(\aleph)} \int_0^{t} (t - E_s)^{\aleph-1} [f(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon))) \\
& - \bar{f}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)))] dE_s \\
& + \frac{\varepsilon^{\frac{\alpha}{2}}}{\Gamma(\aleph)} \int_0^{t} (t - E_s)^{\aleph-1} [g(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon))) \\
& - \bar{g}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)))] dB_{E_s} \\
& + \frac{\varepsilon^{\frac{\alpha}{2}}}{\Gamma(\aleph)} \int_0^{t} \int_{|z|<c} (t - E_s)^{\aleph-1} [h(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon)), z) \\
& - \bar{h}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)), z)] \tilde{N} d(E_s, dz).
\end{aligned}$$

With the help of Jensen's inequality, for any $0 < u < T$, we have

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \le t\varepsilon \le u} |x^{\varepsilon}(\varepsilon t) - \hat{x}(\varepsilon t)|^2) \\
\le \;& \frac{3\varepsilon^{2\alpha}}{(\Gamma(\aleph))^2} \mathbb{E}\Bigg(\sup_{0 \le t\varepsilon \le u} |\int_0^{t} (t - E_s)^{\aleph-1} [f(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon))) \\
& - \bar{f}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)))] dE_s|^2 \Bigg) \\
& + \frac{3\varepsilon^{\alpha}}{(\Gamma(\aleph))^2} \mathbb{E}\Bigg(\sup_{0 \le t\varepsilon \le u} |\int_0^{t} (t - E_s)^{\aleph-1} [g(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon))) \\
& - \bar{g}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)))] dB_{E_s}|^2 \Bigg) \\
& + \frac{3\varepsilon^{\alpha}}{(\Gamma(\aleph))^2} \mathbb{E}\Bigg(\sup_{0 \le t\varepsilon \le u} |\int_0^{t} (t - E_s)^{\aleph-1} \int_{|z|<c} [h(s, E_s, x^{\varepsilon}(s\varepsilon-), x^{\varepsilon}(s\varepsilon - \delta(s\varepsilon)), z) \\
& - \bar{h}(\hat{x}(s\varepsilon-), \hat{x}(s\varepsilon - \delta(s\varepsilon)), z)] \tilde{N}(dE_s, dz)|^2 \Bigg) := I_1 + I_2 + I_3.
\end{aligned}$$

Now we show some necessary estimates for $I_i, i=1,2,3$. First, for the term I_1, we have

$$\begin{aligned} I_1 \le\ & \frac{6\varepsilon^{2\alpha}}{(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}[f(s,E_s,x^\varepsilon(s\varepsilon-),x^\varepsilon(s\varepsilon-\delta(s\varepsilon))) \\ & -f(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)))dE_s|^2\Big) \\ & +\frac{6\varepsilon^{2\alpha}}{(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}[f(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{f}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)))dE_s|^2\Big) := I_{11}+I_{12}. \end{aligned}$$

By Assumption 2.6, the Jensen's inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_{11} =\ & \frac{6\varepsilon^{2\alpha}}{(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}[f(s,E_s,x^\varepsilon(s\varepsilon-),x^\varepsilon(s\varepsilon-\delta(s\varepsilon))) \\ & -f(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)))dE_s|^2\Big) \\ \le\ & \frac{6\varepsilon^{2\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t \varphi(s)(|x^\varepsilon(s\varepsilon-)-\hat{x}(s\varepsilon-)|dE_s|^2 \\ & +|\int_0^t \varphi(s)(|x^\varepsilon(s\varepsilon-\delta(s\varepsilon))-\hat{x}(s\varepsilon-\delta(s\varepsilon))|dE_s|^2)\Big) \\ \le\ & \frac{12\varepsilon^{2\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}(|\int_0^t \varphi(s)|x^\varepsilon(s\varepsilon-)-\hat{x}(s\varepsilon-)|dE_s|^2 \\ & +|\int_0^t \varphi(s)|x^\varepsilon(s\varepsilon-\delta(s\varepsilon))-\hat{x}(s\varepsilon-\delta(s\varepsilon))|dE_s|^2)\Big) \\ \le\ & \frac{12\varepsilon^{2\alpha}Tk^2E_T}{(2\aleph-1)(\Gamma(\aleph))^2}\Big(\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\varepsilon\le s}|x^\varepsilon(r\varepsilon)-\hat{x}(r\varepsilon)|^2)dE_s \\ & +\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\varepsilon\le s}|x^\varepsilon(r\varepsilon-\delta(r\varepsilon))-\hat{x}(r\varepsilon-\delta(r\varepsilon))|^2)dE_s\Big). \end{aligned} \tag{3.3}$$

By Assumption 2.7 and Jensen's inequality, we obtain

$$\begin{aligned} I_{12} =\ & \frac{6\varepsilon^{2\alpha}}{(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}[f(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{f}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)))dE_s|^2\Big) \\ \le\ & \frac{6\varepsilon^{2\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\sup_{0\le t\varepsilon\le u}\Big\{t^2\lambda_1^2(t)\mathbb{E}((\sup_{0\le s\le t}|\hat{x}(s\varepsilon)|+\sup_{0\le s\le t}|\hat{x}(s\varepsilon-\delta(s\varepsilon)))|)^2)\Big\} \\ \le\ & \frac{12\varepsilon^{2\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\sup_{0\le t\varepsilon\le u}\Big\{t^2\lambda_1^2(t)\mathbb{E}(\sup_{0\le s\le t}|\hat{x}(s\varepsilon)|^2+\sup_{0\le s\le t}|\hat{x}(s\varepsilon-\delta(s\varepsilon))|^2)\Big\} \\ \le\ & \frac{12\varepsilon^{2\alpha-2}Tu^2C_1^2}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big\{(\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon)|^2+\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon-\delta(s\varepsilon))|^2)\Big\}. \end{aligned} \tag{3.4}$$

Second, for the term I_2, we have

$$\begin{aligned} I_2 = & \frac{3\varepsilon^\alpha}{(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}[(g(s,E_s,x^\varepsilon(s\varepsilon-),x^\varepsilon(s\varepsilon-\delta(s\varepsilon)) \\ & -g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))+(g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{g}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))]dB_{E_s}|^2\Big) \\ \le & \frac{6\varepsilon^\alpha T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (g(s,E_s,x^\varepsilon(s\varepsilon-),x^\varepsilon(s\varepsilon-\delta(s\varepsilon))) \\ & -g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))dB_{E_s}|^2\Big) \\ & +\frac{6\varepsilon^\alpha T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{g}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))dB_{E_s}|^2\Big) \\ := & I_{21}+I_{22}. \end{aligned}$$

By Assumption 2.6 and the Burkholder-Davis-Gundy inequality (2.1), we have

$$\begin{aligned} I_{21} = & \frac{6\varepsilon^\alpha T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (g(s,E_s,x^\varepsilon(s\varepsilon-),x^\varepsilon(s\varepsilon-\delta(s\varepsilon))) \\ & -g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))dB_{E_s}|^2\Big) \\ \le & \frac{6\varepsilon^\alpha Tk^2b_2}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\int_0^{\frac{u}{\varepsilon}} (|x^\varepsilon(t\varepsilon-)-\hat{x}(t\varepsilon-) \\ & +|x^\varepsilon(t\varepsilon-\delta(t\varepsilon))-\bar{x}(t\varepsilon-\delta(t\varepsilon))|)^2 dE_t\Big) \\ \le & \frac{12\varepsilon^\alpha Tk^2b_2}{(2\aleph-1)(\Gamma(\aleph))^2}\Big(\int_0^{\frac{u}{\varepsilon}} (\mathbb{E}(\sup_{0\le r\le s}|x^\varepsilon(r\varepsilon)-\hat{x}(r\varepsilon)|^2)dE_s \\ & +\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\le s}|x^\varepsilon(r\varepsilon-\delta(t\varepsilon))-\bar{x}(t\varepsilon-\delta(t\varepsilon))|^2))dE_s\Big), \end{aligned} \tag{3.5}$$

where b_2 is a positive constant. According to Assumption 2.7 and the Burkgolder-Davis-Gundy inequality, we have

$$\begin{aligned} I_{22} = & \frac{6\varepsilon^\alpha T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le t\varepsilon\le u}|\int_0^t (g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{g}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))))dB_{E_s}|^2\Big) \\ \le & \frac{6\varepsilon^\alpha Tb_2}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\int_0^{\frac{u}{\varepsilon}} |g(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon))) \\ & -\bar{g}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)))|^2 dE_s\Big) \\ \le & \frac{6\varepsilon^{\alpha-1}Tb_2C_2}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Big(\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon)|^2+\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon-\delta(s\varepsilon))|^2\Big). \end{aligned} \tag{3.6}$$

Finally, for the term I_3, by using Doob's martingale inequality and It$\hat{o}$ isometry, we have

$$\begin{aligned}
I_3 = &\ \frac{3\varepsilon^{\alpha}}{(\Gamma(\aleph))^2}\mathbb{E}\Bigg(\sup_{0\le t\varepsilon\le u}|\int_0^t (t-E_s)^{\aleph-1}\int_{|z|<c}[h(s,E_s,x^{\varepsilon}(s\varepsilon-),x^{\varepsilon}(s\varepsilon-\delta(s\varepsilon)),z)\\
&-\bar{h}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)),z)]\tilde{N}(dE_s,dz)|^2\Bigg)\\
\le &\ \frac{12\varepsilon^{\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Bigg|\int_0^{\frac{u}{\varepsilon}}\int_{|z|<c}[h(s,E_s,x^{\varepsilon}(s\varepsilon-),x^{\varepsilon}(s\varepsilon-\delta(s\varepsilon)),z)\\
&-\bar{h}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)),z)]\tilde{N}(dE_s,dz)\Bigg|^2\\
\le &\ \frac{24\varepsilon^{\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\int_0^{\frac{u}{\varepsilon}}\int_{|z|<c}|h(s,E_s,x^{\varepsilon}(s\varepsilon-),x^{\varepsilon}(s\varepsilon-\delta(s\varepsilon)),z)\\
&-h(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s-\delta(s)),z)|^2 v(dz)dE_s\\
&+\frac{24\varepsilon^{\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\int_0^{\frac{u}{\varepsilon}}\int_{|z|<c}|h(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s-\delta(s)),z)\\
&-\bar{h}(\hat{x}(s\varepsilon-),\hat{x}(s-\delta(s)),z)|^2 v(dz)dE_s\\
:= &\ I_{31}+I_{32}.
\end{aligned}$$

By Assumption 2.6 and Jensen's inequality, we have

$$\begin{aligned}
I_{31}\le &\ \frac{24\varepsilon^{\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Bigg(\int_0^{\frac{u}{\varepsilon}}L_1(|x^{\varepsilon}(s\varepsilon-)-\hat{x}(s\varepsilon-)|^2\\
&+|x^{\varepsilon}(s\varepsilon-\delta(s\varepsilon))-\hat{x}(s-\delta(s))|^2)dE_s\Bigg)\\
\le &\ \frac{24\varepsilon^{\alpha}Tk}{(2\aleph-1)(\Gamma(\aleph))^2}\Bigg(\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\le s}|x^{\varepsilon}(r\varepsilon)-\hat{x}(r\varepsilon)|^2)dE_s\\
&+\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\le s}|x^{\varepsilon}(r\varepsilon-\delta(r\varepsilon))-\hat{x}(r-\delta(r))|^2)dE_s\Bigg).
\end{aligned}\tag{3.7}$$

By Assumption 2.7 and Jensen's inequality, we have

$$\begin{aligned}
I_{32}= &\ \frac{24\varepsilon^{\alpha}T}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Bigg(\int_0^{\frac{u}{\varepsilon}}\int_{|z|<c}|h(s,E_s,\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)),z)\\
&-\bar{h}(\hat{x}(s\varepsilon-),\hat{x}(s\varepsilon-\delta(s\varepsilon)),z)|^2 v(dz)dE_s\Bigg)\\
\le &\ \frac{24\varepsilon^{\alpha-1}TuC_3}{(2\aleph-1)(\Gamma(\aleph))^2}\mathbb{E}\Bigg(\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon)|^2+\sup_{0\le s\le\frac{u}{\varepsilon}}|\hat{x}(s\varepsilon-\delta(s\varepsilon))|^2\Bigg).
\end{aligned}\tag{3.8}$$

Combining (3.3)-(3.8), we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\varepsilon\le u}|x^{\varepsilon}(\varepsilon t)-\hat{x}(\varepsilon t)|^2\right)\\
\le\ &\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\
&\times\mathbb{E}\left(\sup_{0\le t\varepsilon\le u}|\hat{x}(\varepsilon t)|^2+\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\
&+\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha}k^2E_T+12\varepsilon^{2\alpha}k^2b_2+24\varepsilon^{\alpha}k)\\
&\times\left[\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\le s}|x^{\varepsilon}(r\varepsilon)-\hat{x}(r\varepsilon)|^2)dE_s\right.\\
&\left.+\int_0^{\frac{u}{\varepsilon}}\mathbb{E}(\sup_{0\le r\le s}|x^{\varepsilon}(r\varepsilon-\delta(r\varepsilon))-\hat{x}(r\varepsilon-\delta(r\varepsilon))|^2)dE_s\right].
\end{aligned}\tag{3.9}$$

Set

$$\Lambda(\frac{u}{\varepsilon}):=\mathbb{E}(\sup_{0\le t\le\frac{u}{\varepsilon}}|x^{\varepsilon}(\varepsilon t)-\hat{x}(\varepsilon t)|^2).$$

Thus inequality (3.9) can be written as follows:

$$\begin{aligned}
\Lambda(\frac{u}{\varepsilon})\le\ &\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\
&\times\mathbb{E}\left(\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon)|^2+sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\
&+\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha}k^2E_T+12\varepsilon^{2\alpha}k^2b_2+24\varepsilon^{\alpha}k)\\
&\times(\int_0^{\frac{u}{\varepsilon}}\Lambda(s)dE_s+\int_0^{\frac{u}{\varepsilon}}\Lambda(s-\delta(s))dE_s).
\end{aligned}$$

Letting $\Theta(u):=\sup_{\theta\in[-\tau,u]}\Lambda(\theta)$, for every $u\in[0,T]$, one has $\Lambda(s)\le\Theta(s)$ and $\Lambda(s-\delta(s))\le\Theta(s)$. Thus

$$\begin{aligned}
\Lambda(\frac{u}{\varepsilon})\le\ &\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\
&\times\mathbb{E}\left(\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon)|^2+\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\
&+\frac{2T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha}k^2E_T+12\varepsilon^{2\alpha}k^2b_2+24\varepsilon^{\alpha}k)\int_0^{\frac{u}{\varepsilon}}\Theta(s)dE_s.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Theta(\frac{u}{\varepsilon})=\ &\sup_{\theta\in[-\tau,\frac{u}{\varepsilon}]}\Lambda(\theta)\le\max\{\sup_{\theta\in[-\tau,0]}\Lambda(\theta),\sup_{\theta\in[0,\frac{u}{\varepsilon}]}\Lambda(\theta)\}\\
\le\ &\frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\
&\times\mathbb{E}\left(\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon)|^2+\sup_{0\le t\varepsilon\le u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\
&+\frac{2T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha}L_1^2E_T+12\varepsilon^{2\alpha}L_1^2b_2+24\varepsilon^{\alpha}L_2)\int_0^{\frac{u}{\varepsilon}}\Theta(s)dE_s.
\end{aligned}$$

With the help of time-changed Gronwall's inequality, we arrive at

$$\begin{aligned}\Theta(\frac{u}{\varepsilon}) \leq\ & \frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\ &\times\mathbb{E}\left(\sup_{0\leq t\varepsilon\leq u}|\hat{x}(t\varepsilon)|^2+\sup_{0\leq t\varepsilon\leq u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\ &\times e^{\frac{2T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha}k^2E_T+12\varepsilon^{2\alpha}k^2b_2+24\varepsilon^{\alpha}k)E_{\frac{u}{\varepsilon}}}.\end{aligned}$$

In addition, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\varepsilon\leq u}|x^{\varepsilon}(\varepsilon t)-\hat{x}(\varepsilon t)|^2\right)\\ \leq\ & \frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{2\alpha-2}u^2C_1^2+6\varepsilon^{\alpha-1}b_2uC_2+24\varepsilon^{\alpha-1}uC_3)\\ &\times\mathbb{E}\left(\sup_{0\leq t\varepsilon\leq u}|\hat{x}(t\varepsilon)|^2+\sup_{0\leq t\varepsilon\leq u}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\ &\times e^{\frac{2T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{\alpha}k^2E_T+12\varepsilon^{\alpha}k^2b_2+24\varepsilon^{\alpha}k)E_T}.\end{aligned}$$

Select $\beta\in(0,\alpha-1)$ and $L>0$ such that, for any $t\in[0,L\varepsilon^{-\beta-1}]\subseteq[0,\frac{T}{\varepsilon}]$,

$$\mathbb{E}\left(\sup_{0\leq t\varepsilon\leq u}|x^{\varepsilon}(\varepsilon t)-\hat{x}(\varepsilon t)|^2\right)\leq\xi\varepsilon^{\alpha-\beta-1},$$

where we have the constant

$$\begin{aligned}\xi :=\ & \frac{T}{(2\aleph-1)(\Gamma(\aleph))^2}(12L^2\varepsilon^{\alpha-\beta-1}C_1^2+6b_2LC_2+24LC_3)\\ &\times\mathbb{E}\left(\sup_{0\leq t\varepsilon\leq L\varepsilon^{-\beta}}|\hat{x}(t\varepsilon)|^2+\sup_{0\leq t\varepsilon\leq L\varepsilon^{-\beta}}|\hat{x}(t\varepsilon-\delta(t\varepsilon))|^2\right)\\ &\times e^{\frac{2T}{(2\aleph-1)(\Gamma(\aleph))^2}(12\varepsilon^{\alpha}k^2E_T+12\varepsilon^{\alpha}k^2b_2+24\varepsilon^{\alpha}k)E_T}.\end{aligned}$$

Consequently, for any given $\delta_1>0$, there exists a $\varepsilon_1\in(0,\varepsilon_0]$ such that, for each $\varepsilon\in(0,\varepsilon_1]$ and$t\in[-\tau,L\varepsilon^{-\beta}]$,

$$\mathbb{E}\left(\sup_{-\tau\leq t\leq L\varepsilon^{-\beta}}|x^{\varepsilon}(t)-\hat{x}(t)|^2\right)\leq\delta_1.$$

This completes the proof. □

Remark 3.2. According to Remark 2.6, let $\hbar=0, 0<\aleph<1$, equation (1.1) is R-L type fractional stochastic equations driven by time-changed Lévy noise with variable delays:

$$\begin{cases} {}^{L}D_{0^+}^{\aleph}x(t)=f(t,E_t,x(t-),x(t-\delta(t))dE_t+g(t,E_t,x(t-),x(t-\delta(t)))dB_{E_t}\\ +\int_{|z|<c}h(t,E_t,x(t-),x(t-\delta(t)),z)\tilde{N}(dE_t,dz), \qquad t\in J,\\ x(t)=\phi(t), \qquad -\tau\leq t\leq 0,\\ I_{0^+}^{1-\aleph}x(0)=\phi(0), \end{cases} \tag{3.10}$$

with the initial value $I_{0^+}^{1-\aleph}x(0)=\phi(0)$, $\phi(\theta):-\tau\leq\theta\leq 0\in C([-\tau,0];\mathbb{R}^n)$ fulfilling $\phi(0)\in\mathbb{R}^n$, where the functions $f:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n, g:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}, h:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\})\to\mathbb{R}^n$ are measurable continuous functions, $\delta:[0,T]\to[0,\tau]$, and the constant $c>0$ is the maximum allowable jump size. The standard form of equation (3.10) is

$$\begin{aligned} x^{\varepsilon}(t) = \frac{t^{\aleph-1}}{\Gamma(\aleph)}\phi(0) + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} f(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dE_s \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} g(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dB_{E_s} \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} h(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)), z)\tilde{N}(dE_s, dz), \end{aligned} \tag{3.11}$$

with initial value $I_{0^+}^{1-\aleph}x^{\varepsilon}(0) = \phi(0)$, $\phi(\theta): -\tau \le \theta \le 0 \in C([-\tau,0];\mathbb{R}^n)$; the coefficients have the same definitions and conditions as in Equation (3.10) , and $\varepsilon \in (0,\varepsilon_0]$ is a positive parameter with ε_0 being a fixed number. According to Khasminskii type averaging principle, we consider the following averaged FSDEs which correspond to the original standard form (3.11)

$$\begin{aligned} \hat{x}(t) = \frac{t^{\aleph-1}}{\Gamma(\aleph)}\phi(0) + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{f}(\hat{x}(s-), \hat{x}(s-\delta(s)))dE_s \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{g}(\hat{x}(s-), \hat{x}(s-\delta(s)))dB_{E_s} \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{h}(\hat{x}(s-), \hat{x}(s-\delta(s)), z)d\tilde{N}(dE_s, dz), \end{aligned}$$

where the measurable functions $\bar{f}, \bar{g}, \bar{h}$ satisfy Assumption 2.7.

Corollary 3.3. *For equation* (3.10)*, suppose that Assumptions 2.6 and 2.7 hold. Then, for a given arbitrarily small number $\delta_1 > 0$, there exist $L > 0, \varepsilon_1 \in (0,\varepsilon_0]$ and $\beta \in (0,\alpha-1)$ such that, for any $\varepsilon \in (0,\varepsilon_1]$,*

$$\mathbb{E}\left(\sup_{[-\tau, L\varepsilon^{-\beta}]} |x^{\varepsilon}(t) - \hat{x}(t)|^2\right) \le \delta_1.$$

From Theorem 3.1, we can obtain this corollary immediately.

Remark 3.4. According to Remark 2.6, let $\hbar = 1, 0 < \aleph < 1$, equation (1.1) is Caputo type fractional stochastic equations driven by time-changed Lévy noise with variable delays:

$$\begin{aligned} {}^{C}D_{0^+}^{\aleph} x(t) = & f(t, E_t, x(t-), x(t-\delta(t))dE_t + g(t, E_t, x(t-), x(t-\delta(t)))dB_{E_t} \\ & + \int_{|z|<c} h(t, E_t, x(t-), x(t-\delta(t)), z)\tilde{N}(dE_t, dz), \qquad t \in [0,T], \end{aligned} \tag{3.12}$$

with the initial value $x(0) = \phi = \phi(\theta): -\tau \le \theta \le 0 \in C([-\tau,0];\mathbb{R}^n)$ fulfilling $\phi(0) \in \mathbb{R}^n$, where the functions $f: [0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n, g: [0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times m}, h: [0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\}) \to \mathbb{R}^n$ are measurable continuous functions, $\delta: [0,T] \to [0,\tau]$, and the constant $c>0$ is the maximum allowable jump size. The standard form of equation

(3.12) is

$$\begin{aligned} x^{\varepsilon}(t) = \phi(0) + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} f(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dE_s \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} g(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)))dB_{E_s} \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} h(\frac{s}{\varepsilon}, E_{\frac{s}{\varepsilon}}, x^{\varepsilon}(s-), x^{\varepsilon}(s-\delta(s)), z)\tilde{N}(dE_s, dz), \end{aligned} \tag{3.13}$$

with initial value $x^{\varepsilon}(0) = \phi = \phi(\theta) : -\tau \le \theta \le 0 \in C([-\tau, 0]; \mathbb{R}^n)$, and $\varepsilon \in (0, \varepsilon_0]$ is a positive parameter with ε_0 being a fixed number. According to (3.13),

$$\begin{aligned} \hat{x}(t) = \phi(0) + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{f}(\hat{x}(s-), \hat{x}(s-\delta(s)))dE_s \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{g}(\hat{x}(s-), \hat{x}(s-\delta(s)))dB_{E_s} \\ + & \frac{1}{\Gamma(\aleph)}\int_0^t \int_{|z|<c} (t-E_{\frac{s}{\varepsilon}})^{\aleph-1} \bar{h}(\hat{x}(s-), \hat{x}(s-\delta(s)), z)d\tilde{N}(dE_s, dz), \end{aligned}$$

where the measurable functions $\bar{f}, \bar{g}, \bar{h}$ satisfy Assumption 2.7.

Corollary 3.5. *For equation (3.12), suppose that Assumptions 2.6 and 2.7 hold. Then, for a given arbitrarily small number $\delta_1 > 0$, there exist $L > 0, \varepsilon_1 \in (0, \varepsilon_0]$ and $\beta \in (0, \alpha-1)$ such that, for any $\varepsilon \in (0, \varepsilon_1]$,*

$$\mathbb{E}\left(\sup_{[-\tau, L\varepsilon^{-\beta}]} |x^{\varepsilon}(t) - \hat{x}(t)|^2\right) \le \delta_1.$$

From Theorem 3.1, we can obtain this corollary immediately.

Remark 3.6. When $\aleph = 1$ and $\hbar = 0$, we obtain the same results presented in [13].

4. Example

We consider the fractional stochastic differential equations driven by time-changed Lévy noise with time-delays:

$$\begin{cases} D_{0^+}^{\aleph,\hbar} x_{\varepsilon}(t) = \varepsilon^{\alpha}\Big(x_{\varepsilon}\sin^2(E_t) - E_t x_{\varepsilon} cos(E_t - 1)\Big)dE_t + \varepsilon^{\frac{\alpha}{2}} 2\lambda dB_{E_t}, \\ +\varepsilon^{\frac{\alpha}{2}} \int_{|z|<c} 2\tilde{N}(dE_t, dz), t \in (0,1], \\ x_{\varepsilon}(t) = t, t \in [-1, 0], \\ I_{0^+}^{(1-\aleph)(1-\hbar)} x_{\varepsilon}(0) = 0.1, \end{cases} \tag{4.1}$$

where $v(z)dz = |z|^{-2}$, $\lambda \in \mathbb{R}$ and denote $\aleph = \frac{1}{2}, \hbar = \frac{2}{3}$,

$$f(t, E_t, x_{\varepsilon}(t), x_{\varepsilon}(t-\tau)) = x_{\varepsilon}\sin^2(E_t) - E_t x_{\varepsilon}\cos(E_t - 1),$$
$$g(t, E_t, x_{\varepsilon}(t), x_{\varepsilon}(t-\tau)) = 2\lambda, h(t, E_t, x_{\varepsilon}(t), x_{\varepsilon}(t-\tau)) = 2.$$

Let

$$\begin{aligned}\bar{f}(\hat{x}(s),\hat{x}(s-r)) = & \int_0^1 f(t,E_t,x_\varepsilon(t),x_\varepsilon(t-\tau))dE_t \\ = & \left[\frac{1}{2}E_1 - \frac{\sin 2E_1}{4} - E_1\sin(E_1-1) - \cos(E_1-1)\right]x_\varepsilon,\end{aligned}$$

and

$$\bar{g}(\hat{x}(s),\hat{x}(s-r)) = 2\lambda, \bar{h}(\hat{x}(s),\hat{x}(s-r),z) = 2.$$

We have the following corresponding averaged fractional stochastic differential equations driven by time-changed Lévy noise with variable delays:

$$\begin{cases} D_{0^+}^{\aleph,\hbar}x_\varepsilon(t) = \varepsilon^\alpha x_\varepsilon\left(\frac{1}{2}E_1 - \frac{\sin 2E_1}{4} - E_1\sin(E_1-1) - \cos(E_1-1)\right)dE_t, \\ +\varepsilon^{\frac{\alpha}{2}}2\lambda dB_{E_t} + \varepsilon^{\frac{\alpha}{2}}\int_{|z|<c}2\tilde{N}(dE_t,dz), t\in(0,1], \\ x_\varepsilon(t) = t, t\in[-1,0], \\ I_{0^+}^{(1-\aleph)(1-\hbar)}x_\varepsilon(0) = 0.1. \end{cases} \tag{4.2}$$

Define the error $E_{rr} = [|x_\varepsilon(t) - \bar{x}_\varepsilon(t)|^2]^{\frac{1}{2}}$. We carry out the numerical simulation to obtain the solutions of (4.1) and (4.2) under the conditions that $\alpha = 1.2, \varepsilon = 0.001, \lambda = 1$ and $\alpha = 1.2, \varepsilon = 0.001$, and $\lambda = -1$. One can see a good agreement between the solutions of the original equation and the averaged equation. This example illustrates the effectiveness of the proposed averaging principle for the fractional stochastic differential equations driven by time-changed Lévy noise with time-delays.

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