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POSITIVE PERIODIC SOLUTIONS FOR A ϕ-LAPLACIAN GENERALIZED RAYLEIGH EQUATION WITH A SINGULARITY

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Abstract. This paper explores the existence of positive periodic solutions to a ϕ-Laplacian generalized Rayleigh equation with a singularity as $(\phi(v'(t)))' + f(t,v'(t)) + g(v(t)) = e(t)$, where the function g has a repulsive singularity at $v = 0$. According to the Manásevich-Mawhin continuation theorem, we prove the existence of positive periodic solutions to this equation. This result is feasible for the cases of a strong or weak singularity.

Keywords. ϕ-Laplacian; Generalized Rayleigh equation; Positive periodic solution; Strong singularity; Weak singularity.

1. INTRODUCTION

The study of singular differential equations can be traced back to the paper of Lazer and Solimini [1]. They explored a second-order differential equation with a singularity:

$$u'' - \frac{1}{u^\alpha} = h(t), \tag{1.1}$$

where $h(t)$ is a continuous and ω-periodic function. They proved the existence of a positive ω-periodic solution to this equation if all $\alpha > 0$ and $h(t)$ has a positive mean value. The condition of $\alpha \geq 1$ in equation (1.1) is one of the common conditions. It is a so-called strong force condition that can guarantee the existence of positive periodic solutions; see, e.g., [2, 3, 4, 5, 6, 7, 8] and the references therein. Correspondingly, the condition of $0 < \alpha < 1$ in equation (1.1) is a so-called weak force condition that can guarantee the existence of positive periodic solutions of singular differential equations; see, e.g., [9, 10, 11, 12, 13].

At the same time, Rayleigh equations with a singularity were also explored by authors [14, 15, 16, 17, 18, 19, 20, 21]. For example, Lu et al. [18] discussed p-Laplacian Rayleigh equations with a singularity in 2016 as follows:

$$(|u'|^{p-2}u')' + f(u') - g_1(u) + g_2(u) = h(t)$$

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and

$$(|u'|^{p-2}u')' + f(u') + g_1(u) - g_2(u) = h(t),$$

where $p > 1$ is a constant, f is a continuous function, $g_1,\ g_2 \in C((0,+\infty),\mathbb{R})$, when $u \to 0^+$, g_1 is unbounded, and it has a strong singularity at $u = 0$, namely,

$$\lim_{u\to 0^+} \int_1^u g_1(s)ds = +\infty.$$

According to the Manásevich-Mawhin's continuation theorem, they proved the existence of positive periodic solutions to the p-Laplacian Rayleigh equations. After that, Xin and Yao [20] in 2020 investigated the p-Laplacian Rayleigh equation with a singularity as follows:

$$(|u'|^{p-2}u')' + f(t,u') + g(u) = h(t). \tag{1.2}$$

Based on the Manásevich-Mawhin's continuation theorem, they obtained that equation (1.2) has a positive periodic solution.

Inspired by [18, 20], this paper explores the ϕ-Laplacian Rayleigh equation with a singularity as follows:

$$(\phi(v'))' + f(t,v') + g(v) = e(t), \tag{1.3}$$

where f is continuous, and it is a ω-periodic function about t, $f(t,0) \equiv 0$, $e(t)$ is a ω-periodic function, $g \in C((0,+\infty),\mathbb{R})$ has a repulsive singularity at $v = 0$, that is, $\lim_{v\to 0^+} g(v) = -\infty$. By using the Manásevich-Mawhin continuation theorem , we prove that a new existence criterion of the positive periodic solution to equation (1.3) can be obtained by a weak singularity of repulsive type. In addition, we obtain the existence interval of periodic solutions of equation (1.3). Usually, g has a weak singularity at $v = 0$, which means that

$$\lim_{v\to 0^+} \int_1^v g(s)ds < +\infty,$$

where $\phi : (-\infty,+\infty) \to (-\infty,+\infty)$ of equation (1.3) is a continuous function, which satisfies condition $\phi(0) = 0$ and the following conditions:

(B_1) $(\phi(v_1) - \phi(v_2))(v_1 - v_2) > 0$ for $\forall\ v_1 \neq v_2,\ v_1,\ v_2 \in \mathbb{R}$;

(B_2) $\exists\ \kappa : [0,+\infty) \to [0,+\infty),\ \kappa(s) \to +\infty$ when $s \to +\infty$, s.t., $\phi(v)\cdot v \geq \kappa(|v|)|v|$ for $\forall\ v \in (-\infty,+\infty)$.

Obviously, ϕ represents many nonlinear operators, that is,

- $\phi_p(v) = |v|^{p-2}v : (-\infty,+\infty) \to (-\infty,+\infty)$, here the constant p satisfies the condition of $p > 1$;
- the nonlinear operator $\phi(v) = ve^{v^2} : (-\infty,+\infty) \to (-\infty,+\infty)$.

2. The Positive ω-Periodic Solution To Equation (1.3)

First, we introduce a parameter μ, which satisfies the condition of $\mu \in (0,1]$. Then, we embed equation (1.3) into the equation family as follows:

$$(\phi(v'(t)))' + \mu f(t,v'(t)) + \mu g(v(t)) = \mu e(t). \tag{2.1}$$

According to [22, Theorem 3.1], we can obtain the following result.

Lemma 2.1. *Let the function ϕ satisfy the condition of $\phi(0)=0$ and the conditions of (B_1) and (B_2). Let σ_1, σ_2, and σ_3 be positive constants, and $\sigma_1<\sigma_2$ such that the following conditions hold:*

(1) each possible periodic solution v to equation (2.1) satisfies $\sigma_1<v(t)<\sigma_2$ and $\|v'\|<\sigma_3$ for all $t\in[0,\omega]$, where $\|v'\|:=\max\limits_{t\in[0,\omega]}|v'(t)|$.

(2) σ_1 and σ_2 satisfy $\left(g(\sigma_1)-\frac{1}{\omega}\int_0^\omega e(t)dt\right)\left(g(\sigma_2)-\frac{1}{\omega}\int_0^\omega e(t)dt\right)<0$.

Then equation (1.3) has at least one ω-periodic solution.

We explore the existence of a positive periodic solution to equation (1.3) with strong or weak singularities. Here we introduce the following notations:

$$\|e\|:=\max_{t\in[0,\omega]}|e(t)|, e^*:=\max_{t\in[0,\omega]}e(t), e_*:=\min_{t\in[0,\omega]}e(t), g(+\infty):=\lim_{v\to+\infty}g(v).$$

By Lemma 2.1, we have the following main result.

Theorem 2.2. *Let the function ϕ satisfy the condition of $\phi(0)=0$ and the conditions of (B_1) and (B_2). Let the following conditions hold:*

(H_1) assume that α and m are constants, which satisfy $\alpha>0$ and $m>1$ such that $f(t,v)v\geq\alpha|v|^m$, for $(t,v)\in[0,\omega]\times(-\infty,+\infty)$;

(H_2) g is a strictly monotone-increasing function, $e^*<g(+\infty)$;

(H_3) assume that β and γ are constants, and $\beta>0$ and $\gamma>0$ such that $|f(t,v)|\leq\beta|v|^{m-1}+\gamma,(t,v)\in[0,\omega]\times R$.

If $\alpha>(\frac{\omega}{2g^{-1}(e_*)})^{m-1}\|e\|$, then equation (1.3) has a positive ω-periodic solution v with

$$v\in\left(g^{-1}(e_*)-\frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}},\ g^{-1}(e^*)+\frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}}\right).$$

Proof. In view of $\int_0^\omega v'(t)dt=0$, we see that there are two points $t_1,\ t_2\in(0,\omega)$ such that $v'(t_1)\geq0$ and $v'(t_2)\leq0$. It follows from (B_1) that

$$\phi(v'(t_1))\geq0 \text{ and } \phi(v'(t_2))\leq0.$$

Let $t_3,\ t_4\in(0,\omega)$ be the points where the maximum and minimum values of $\phi(v'(t))$ are obtained, respectively. Obviously, we have the following conditions:

$$(\phi(v'(t_3)))'=0\ ,\phi(v'(t_3))\geq0 \tag{2.2}$$

and $(\phi(v'(t_4)))'=0$, $\phi(v'(t_4))\leq0$. By (B_2), we obtain that $v'(t_3)\geq0$ and $v'(t_4)\leq0$. From (H_1), we have that $f(t_3,v'(t_3))\geq0$ and $f(t_4,v'(t_4))\leq0$. Substituting (2.2) into equation (2.1), we deduce $-\mu g(v(t_3))+\mu e(t_3)=\mu f(t_3,v'(t_3))$ and $-\mu g(v(t_4))+\mu e(t_4)=\mu f(t_4,v'(t_4))$. Since $f(t_3,v'(t_3))\geq0$ and $f(t_4,v'(t_4))\leq0$. It follows that

$$g(v(t_3))\leq e(t_3)\leq e^* \text{ and } g(v(t_4))\geq e(t_4)\geq e_*\ .$$

As g is a strictly monotone-increasing function, we obtain

$$v(t_3)\leq g^{-1}(e^*) \text{ and } v(t_4)\geq g^{-1}(e_*). \tag{2.3}$$

From (2.3) and the fact that g is a continuous function, we can see that exists a point $\tau\in(0,\omega)$ such that

$$g^{-1}(e_*)\leq v(\tau)\leq g^{-1}(e^*). \tag{2.4}$$

On the other hand, multiplying both sides of equation (2.1) by $v'(t)$, and then integrating both sides of equation (2.1) in $[0,\omega]$, one has

$$\begin{aligned}&\int_0^\omega(\phi(v'(t)))'v'(t)dt+\mu\int_0^\omega f(t,v'(t))v'(t)dt+\mu\int_0^\omega g(v(t))v'(t)dt\\&=\mu\int_0^\omega e(t)v'(t)dt.\end{aligned}\tag{2.5}$$

In view of

$$\int_0^\omega(\phi(v'(t)))'v'(t)dt=\int_0^\omega v'(t)d(\phi(v'(t)))=0$$

and

$$\int_0^\omega g(v(t))v'(t)dt=\int_0^\omega g(v(t))dv(t)=0,$$

we see from (2.5) that $\int_0^\omega f(t,v'(t))v'(t)dt=\int_0^\omega e(t)v'(t)dt$. Furthermore, we have

$$\left|\int_0^\omega f(t,v'(t))v'(t)dt\right|=\left|\int_0^\omega e(t)v'(t)dt\right|.$$

In view of $|\int_0^\omega f(t,v'(t))v'(t)dt|=\int_0^\omega |f(t,v'(t))v'(t)|dt$, we obtain from (H_1) that

$$\left|\int_0^\omega f(t,v'(t))v'(t)dt\right|\geq\alpha\int_0^\omega|v'(t)|^m dt.$$

By using the Hölder inequality, we can obtain

$$\alpha\int_0^\omega|v'(t)|^m dt\leq\int_0^\omega|e(t)||v'(t)|dt\leq\|e\|\omega^{\frac{m-1}{m}}\left(\int_0^\omega|v'(t)|^m dt\right)^{\frac{1}{m}}.$$

Since $\left(\int_0^\omega|v'(t)|^m dt\right)^{\frac{1}{m}}>0$, we deduce

$$\left(\int_0^\omega|v'(t)|^m dt\right)^{\frac{m-1}{m}}\leq\frac{\|e\|\omega^{\frac{m-1}{m}}}{\alpha}.$$

which together with (2.4) and the Hölder inequality yields that

$$\begin{aligned}v(t)=&\frac{1}{2}\left(v(\tau)+\int_\tau^t v'(\theta)d\theta+v(\tau)-\int_{t-\omega}^\tau v'(\theta)d\theta\right)\\\leq&v(\tau)+\frac{1}{2}\left(\int_\tau^t|v'(\theta)|d\theta+\int_{t-\omega}^\tau|v'(\theta)|d\theta\right)\\\leq&g^{-1}(e^*)+\frac{1}{2}\int_0^\omega|v'(\theta)|d\vartheta\\\leq&g^{-1}(e^*)+\frac{1}{2}\omega^{\frac{m-1}{m}}\left(\int_0^\omega|v'(\theta)|^m d\theta\right)^{\frac{1}{m}}\\\leq&g^{-1}(e^*)+\frac{1}{2}\omega^{\frac{m-1}{m}}\left(\frac{\|e\|\omega^{\frac{m-1}{m}}}{\alpha}\right)^{\frac{1}{m-1}}\\\leq&g^{-1}(e^*)+\frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}}:=M_1.\end{aligned}\tag{2.6}$$

Hence, it following from (2.4) and (2.6) that

$$\begin{aligned}v(t) \geq & g^{-1}(e_*) - \frac{1}{2}\left(\int_\tau^t |v'(\theta)|d\theta + \int_{t-\omega}^\tau |v'(\theta)|d\theta\right)\\ \geq & g^{-1}(e_*) - \frac{1}{2}\int_0^\omega |v'(\theta)|d\theta\\ \geq & g^{-1}(e_*) - \frac{1}{2}\omega^{\frac{m-1}{m}}\left(\int_0^\omega |v'(\theta)|^m d\theta\right)^{\frac{1}{m}}\\ \geq & g^{-1}(e^*) - \frac{1}{2}\omega^{\frac{m-1}{m}}\left(\frac{\|e\|\omega^{\frac{m-1}{m}}}{\alpha}\right)^{\frac{1}{m-1}}\\ \geq & g^{-1}(e_*) - \frac{\omega}{2}(\frac{\|e\|}{\alpha})^{\frac{1}{m-1}} := M_2,\end{aligned}$$

due to $\alpha > \|e\|(\frac{\omega}{2g^{-1}(e_*)})^{m-1}$.

Next, we explore a uniform bound of $v'(t)$. On account of $v(0) = v(\omega)$, we can obtain a point $t_5 \in [0,\omega]$ with $v'(t_5) = 0$. Furthermore, $\phi(v'(t_5)) = 0$. It follows from (H_3) that

$$\begin{aligned}\|\phi(v')\| = & \max_{t\in[t_5,t_5+\omega]}\left\{\left|\int_{t_5}^t (\phi(v'(\theta)))'d\theta\right|\right\}\\ \leq & \int_0^\omega |f(t,v'(t)|dt + \int_0^\omega |g(v(t))|dt + \int_0^\omega |e(t)|dt\\ \leq & \beta\int_0^\omega |v'(t))|^{m-1}dt + \gamma\omega + \int_0^\omega |g(v(t))|dt + \omega\|e\|\\ \leq & \beta\omega^{\frac{1}{m}}(\int_0^\omega |v'(t))|^m dt)^{\frac{m-1}{m}} + \gamma\omega + \int_0^\omega |g(v(t))|dt + \omega\|e\|\\ \leq & \beta\omega^{\frac{1}{m}}\frac{\|e\|\omega^{\frac{m-1}{m}}}{\alpha} + \gamma\omega + +\|g_{M_1}\|\omega + \omega\|e\|\\ \leq & \frac{\beta\|e\|\omega}{\alpha} + \gamma + \|g_{M_1}\|\omega + \omega\|e\| := M_3',\end{aligned}$$

where $\|g_{M_1}\| := \max_{M_2\leq v\leq M_1} |g(v)|$.

We claim that there is a positive constant M_3 which satisfies the condition of $M_3 > M_3' + 1$ such that $\|v'(t)\| \leq M_3$. for all $t \in (-\infty,+\infty)$. In fact, if not, there exists a positive constant M_4 with $\kappa(|v'|) > M_4$ for some $v' \in (-\infty,+\infty)$. We obtain from (B_2) that

$$\kappa(|v'|)|v'| \leq |\phi(v')|v' \leq |\phi(v')||v'| \leq M_3'|v'|.$$

Thus $\kappa(|v'|) \leq M_3'$ for all $v' \in (-\infty,+\infty)$, which is a contradiction.

Let $\sigma_1 < M_2$, $\sigma_2 > M_1$, and $\sigma_3 > M_3$ be constants. We obtain a periodic solution v to equation (2.1), and we have

$$\sigma_1 < v(t) < \sigma_2,\ \|v'(t)\| < \sigma_3,$$

and the condition (1) of Lemma 2.1 is satisfied. Furthermore, let us explore the condition (2) of Lemma 2.1, Actually, because (H_2), we obtain

$$g(\sigma_1) - \frac{1}{\omega}\int_0^\omega e(t)dt < 0,$$

and

$$g(\sigma_2) - \frac{1}{\omega}\int_0^{\omega} e(t)dt > 0.$$

Hence, condition (2) is also satisfied. By using Lemma 2.1, we can obtain at least one positive periodic solution v of equation (1.3) which satisfies

$$v \in \left(g^{-1}(e_*) - \frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}}, \ g^{-1}(e^*) + \frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}} \right).$$

□

Nest, we present a numerical example that illustrates our results.

Example 2.3. We give the following ϕ-Laplacian Rayleigh equation, which has a repulsive and strong singularity.

$$(\phi(v'(t)))' + (6+\sin 8t)v'(t) + 4 - \frac{1}{v(t)} = e^{\cos^2 4t}, \tag{2.7}$$

where relativistic operator $\phi(v) = ve^{v^2}$. Obviously, $\omega = \frac{\pi}{4}$, $f(t,v) = (6+\sin 8t)v$, $g(v) = 4 - \frac{1}{v}$, $e(t) = e^{\cos^2 4t}$, and $e_* = 1,\ e^* = e$. Thus condition (H_2) holds. Since $f(t,v)\cdot v = (6+\sin 8t)\cdot v^2 \geq 5|v|^2, \alpha = 5, m = 2$, then condition (H_1) holds. Besides, $|f(t,v)| \leq 7v+1,\ \beta = 7, \gamma = 1$, condition (H_3) is satisfied.

Next, we consider the conditions (B_1) and (B_2)

$$(\phi(v))' = (ve^{v^2})' = e^{v^2}(1+2v^2) > 0,$$

and

$$\phi(v)\cdot v = v^2 e^{v^2} \geq (|v|e^{|v|^2})|v|.$$

It is easy to see that conditions (B_1) and (B_2) hold. Hence, $g^{-1}(v) = \frac{1}{4-v}$, and then we obtain

$$\left(\frac{\omega}{2g^{-1}(e_*)}\right)^{m-1} \|e\| = \frac{3}{8}\times\pi\times e \approx 3.2024 < \alpha = 5,$$

$$g^{-1}(e_*) - \frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}} = \frac{1}{3} - \frac{\pi\times e}{40} \approx 0.1198 > 0.119,$$

and

$$g^{-1}(e^*) + \frac{\omega}{2}\left(\frac{\|e\|}{\alpha}\right)^{\frac{1}{m-1}} = \frac{1}{4-e} + \frac{\pi\times e}{40} \approx 0.9937 < 0.994.$$

By using Theorem 2.2, we can obtain at least one positive and $\frac{\pi}{4}$-periodic solution v of equation (2.7), which satisfies $v \in (0.119, 0.994)$.

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