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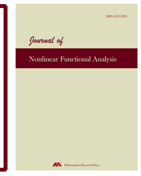

NONLINEAR IMPULSIVE (ρ_k, ψ_k)-HILFER FRACTIONAL PANTOGRAPH INTEGRO-DIFFERENTIAL EQUATIONS UNDER NONLOCAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper investigates the existence and uniqueness of solutions for a class of nonlinear impulsive fractional pantograph integro–differential equations with multi-point integral boundary conditions in the context of the (ρ_k, ψ_k)-Hilfer fractional operator. We transform our problem into an equivalent integral equation, and the uniqueness result is proved by applying Banach's fixed-point theorem. In addition, some types of Ulam's stability results are demonstrated and numerical examples are designed to illustrate the applicability of our theoretical results.

Keywords. (k, ψ)-Hilfer fractional derivative; Impulsive conditions; Nonlocal integral boundary conditions; Ulam's stability.

1. Introduction

A nonlinear system is one in which the change in output is not proportional to the change in input. In comparison to considerably simpler linear systems, which describe changes in variables over time, nonlinear dynamical systems may look chaotic, unexpected, or paradoxical. A nonlinear system of equations is a set of simultaneous equations in which the unknowns appear as variables of a polynomial of degree greater than one, in the argument of a function that is not a polynomial of degree one, or as unknown functions in the case of differential equations ($\mathbb{DE}$s), typically describes the behavior of a nonlinear system. $\mathbb{DE}$s have been used to describe occurrences in real-world problems. One famous class of $\mathbb{DE}$s involving proportional delay terms is

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called the pantograph equation ($\mathbb{PE}$), initially known in 1971 by Ockendon and Taylor [1]. It has been employed to explain the processes and phenomena that depend on previous states in various domains, such as economics, medicine, and engineering. Numerous studies have focused on the important characteristics of qualitative theory of solutions to $\mathbb{PE}$, we refer to [2, 3, 4] and the references cited therein. Besides, boundary value problems ($\mathbb{BVP}$s) involving integral boundary conditions commonly occur in engineering problems, such as population dynamics, chemical engineering, hydrodynamic, semiconductor, thermal conduction, underground water flow, blood flow, and so on. Note that impulsive differential equations play a significant role in describing physical phenomena, which are used to model some processes with discontinuous jumps and instantaneous moves. For results on $\mathbb{BVP}$s with impulsive conditions, we refer to [5, 6] and the references therein.

Over the past several decades, fractional calculus ($\mathbb{FC}$) has been immensely developed as the beginning of integral calculus. It is a branch of mathematical analysis concerned with derivative and integral of arbitrary-order (also known as fractional-order). Fractional dynamic equations were widely used in simulating and describing complex and chaotic systems in various scientific fields, such as physics, electrochemistry, bioengineering, financial, heat-transfer, and so on. This substantiality has frequently motivated researchers to discover the definitions of fractional derivative operator ($\mathbb{FDO}$) according to various kernels; see, e.g., [7, 8, 9]. It is known that the Riemann-Liouville ($\mathbb{RL}$) and Caputo senses are the oldest and most well-known. However, each operator is exceptional and has been continuously developed. For example, in 2012, the $\mathbb{FDO}$ of Hilfer's type was introduced by Furati *et. al.* [10] which is a generalization of $\mathbb{RL}$'s and Caputo's types. After that, the Caputo-$\mathbb{FDO}$ with respect to another function ψ was proposed by Almeida [11] in 2017, which is called ψ-Caputo-$\mathbb{FDO}$. Using the concepts in [10], the ψ-Hilfer-$\mathbb{FDO}$ and some important properties were discussed by [12] in 2018. Later, in 2021, the ψ-Hilfer-$\mathbb{FDO}$ and some properties were demonstrated in [13] with the aid of the definitions of the ρ-Gamma function [14], which was renamed the (ρ,ψ)-Hilfer-$\mathbb{FDO}$. In addition, by using Banach's fixed-point theorem, they studied the existence and uniqueness of solutions for a nonlinear fractional differential equations ($\mathbb{FDE}$s) in the context of the (ρ,ψ)-Hilfer-$\mathbb{FDO}$. As previously stated, the (ρ,ψ)-Hilfer-$\mathbb{FDO}$ can be generalized to a number of well-known $\mathbb{FDO}$s (see in Lemma 2.7). For more studies, we refer to [15, 16, 17] and the references cited therein.

Recently, the exclusive investigation of the significant qualitative theory for fractional differential equations was developed, such as existence theory and stability analysis. They are essential knowledge in studying the qualitative results for the differential equations with or/and without impulsive conditions because of some situations where finding the exact solution is quite a difficult task. Therefore, to guarantee the existence of solutions to the problem under consideration, Ulam's stability is an efficient choice that has been applied to verify the stability of functional problems. There are numerous kinds of Ulam's stability such as the Ulam-Hyers stability ($\mathbb{UH}$), which was initiated by Ulam and Hyers [18, 19] in 1941, and the Ulam-Hyers-Rassias ($\mathbb{UHR}$), which was provided by Rassias [20, 21] in 1978. This technique has inspired numerous researchers to examine the stability of a variety of mathematical problems; see, e.g., [22, 23, 24, 25, 26, 27] for more details.

This paper is inspired by the results in [1, 13, 16, 24]. Moreover, to the best of the authors' knowledge, no results have been published that take into account $\mathbb{PE}$ and integro-differential equations with impulsive conditions in the context of (ρ,ψ)-Hilfer-$\mathbb{FDO}$. Therefore, to make it novel and to promote more study in this area, we study the uniqueness result and Ulam's stability of the solutions for the following nonlinear impulsive (ρ_k, ψ_k)-Hilfer fractional pantograph integro-differential equation with multi-point integral boundary conditions (the impulsive (ρ_k, ψ_k)-$\mathbb{HFP}$-$\mathbb{IDE}$-$\mathbb{MIBC}$s):

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}u(t) = f(t,u(t),u(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+}u(t)),\ t \in \mathscr{J}_k, t \neq t_k,\ k = 0,1,\ldots,m, \\ {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k^*(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = 0, \quad u(T) = \sum_{i=0}^{m}\mu_i\, {}_{\rho_i}\mathscr{I}^{\kappa_i;\psi_i}_{t_i}u(\xi_i) + \mathscr{A}, \quad \xi_i \in (t_i, t_{i+1}], \end{cases} \tag{1.1}$$

where ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}$ denotes the (ρ_k, ψ_k)-Hilfer fractional derivative of order $\alpha_k \in (1,2]$ and type $\beta_k \in [0,1]$, $\rho_k > 0$, $\mathscr{J}_k := (t_k, t_{k+1}] \subset (a,b]$, for $k = 0,1,2,\ldots,m$, with $\mathscr{J}_0 := [a,t_1]$, $\mathscr{J} := [a,b]$, $0 \leq a = t_0 < t_1 \cdots < t_m < t_{m+1} = b \leq T$, ${}_{\rho_k}\mathscr{I}^{q;\psi_k}_{t_k^+}$ is the (ρ_k, ψ_k)-$\mathbb{RL}$ fractional integral with order $q \in \{\rho_k(2-\gamma_k), \rho_{k-1}(2-\gamma_{k-1}), \sigma_k, \kappa_i\}$ with $q > 0$, $k = 1,2,\ldots,m$, $i = 0,1,\ldots,m$, ${}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{p;\psi_k}_{t_k^+}$, is the (ρ_k, ψ_k)-$\mathbb{RL}$ fractional derivative with order $p \in \{\rho_k(\gamma_k-1), \rho_{k-1}(\gamma_{k-1}-1)\}$ with $p \in (1,2)$, $k = 1,2,\ldots,m$, ${}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+) = \lim_{t\to 0^+} {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k+h)$, ${}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+) = \lim_{h\to 0^+} {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k+h)$, ϕ_k, $\phi_k^* \in \mathscr{C}(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$, $f \in \mathscr{C}(\mathscr{J} \times \mathbb{R}^3, \mathbb{R})$, $\xi_i \in (t_i, t_{i+1}]$, $\mathscr{A}$, $\mu_i \in \mathbb{R}$ for $i = 0,1,\ldots,m$, and $\theta \in (0,1)$.

The highlight of this paper is to determine the existence and uniqueness of the solutions for (1.1), which will be described later in our first main result. The various Ulam's stability of the solutions is verified in our second main result. Finally, a few numerical examples of applications are illustrated to validate our theoretical results. The remaining frameworks of this study are structured as follows. Some essential lemmas which includes the process of transforming the considered problem into an integral equation are presented in Section 2. Our main results are proved by utilizing Banach's fixed-point theorem in Section 3. In addition, Section 4 provides some illustrative examples. Section 5, the final section, present a concluding remark.

2. Preliminaries

In this section, we provide some basic definitions and lemmas which are used throughout this paper.

Definition 2.1 ([28])**.** Assume that $f \in \mathscr{L}^1(\mathscr{J},\mathbb{R})$ and $\psi(t) : \mathscr{J} \to \mathbb{R}$ is an increasing function via $\psi'(t) \neq 0, t \in \mathscr{J}$. The (ρ, ψ)-$\mathbb{RL}$ fractional integral operator ($\mathbb{FIO}$) of a function f of order α is given by ${}_{\rho}\mathscr{I}^{\alpha;\psi}_{a^+}f(t)\frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t (\psi(t)-\psi(s))^{\frac{\alpha}{\rho}-1}\psi'(s)f(s)ds$, $\rho > 0$, $\alpha > 0$, where $\Gamma_\rho(\cdot)$ is the ρ-Gamma function, $\Gamma_\rho(z) = \int_0^\infty s^{z-1}e^{-\frac{s^\rho}{\rho}}ds$, $z \in \mathbb{C}$, $Re(z) > 0$, $\rho > 0$.

Some important properties of the ρ-Gamma function [29] are

$$\Gamma_\rho(z+\rho) = z\Gamma_\rho(z), \quad \Gamma_\rho(\rho) = 1, \quad \Gamma_\rho(z) = (\rho)^{\frac{z}{\rho}-1}\Gamma\left(\frac{z}{\rho}\right), \quad \Gamma(z) = \lim_{\rho\to 1}\Gamma_\rho(z).$$

Definition 2.2. *([13]).* Assume that $f \in \mathscr{C}^n(\mathscr{J},\mathbb{R})$ and $\psi(t) \in \mathscr{C}^n([a,b],\mathbb{R})$ with $\psi'(t) \neq 0$, $t \in \mathscr{J}$. Then, the (ρ,ψ)-$\mathbb{RL}$-$\mathbb{FDO}$ of a function f of order α, $\rho \in \mathbb{R}^+ := (0,\infty)$ is given by

$$ {}^{\mathbb{RL}}_{\rho}\mathfrak{D}^{\alpha;\psi}_{a^+} f(t) = \left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n {}_{\rho}\mathscr{I}^{\rho n-\alpha;\psi}_{a^+} f(t) = \delta^n_{\psi\rho}\mathscr{I}^{\rho n-\alpha;\psi}_{a^+} f(t), \quad \delta^n_{\psi} = \left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n. $$

Definition 2.3 ([13])**.** Assume that $f \in \mathscr{C}^n(\mathscr{J},\mathbb{R})$, $\psi \in \mathscr{C}^n(\mathscr{J},\mathbb{R})$, $\psi'(t) \neq 0$, $t \in \mathscr{J}$, α, $\rho \in \mathbb{R}^+$, and $\beta \in [0,1]$. The (ρ,ψ)-Hilfer $\mathbb{FDO}$ of a function f of order α and type β is given by

$$ {}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\psi}_{a^+} f(t) \;=\; {}_{\rho}\mathscr{I}^{\beta(\rho n-\alpha);\psi}_{a^+}\left({}^{\mathbb{RL}}_{\rho}\mathfrak{D}^{r_\rho;\psi}_{a^+} f\right)(t), \quad (1-\beta)(n\rho-\alpha) = n\rho - r_\rho, \tag{2.1} $$

where $\delta^n_{\psi} = \left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n$ and $n = \left\lceil \frac{\alpha}{\rho} \right\rceil$.

Lemma 2.4 ([13])**.** *Assume that $\alpha > 0$, $\rho > 0$ and $\mu \in \mathbb{R}$, where $\frac{\mu}{\rho} > -1$. Then,*

(i) ${}_{\rho}\mathscr{I}^{\alpha;\psi}_{a^+}\left[(\psi(t)-\psi(a))^{\frac{\mu}{\rho}}\right] = \frac{\Gamma_\rho(\mu+\rho)}{\Gamma_\rho(\mu+\rho+\alpha)}(\psi(t)-\psi(a))^{\frac{\mu+\alpha}{\rho}}$.

(ii) ${}^{H}_{\rho}\mathfrak{D}^{\alpha;\psi}_{a^+}\left[(\psi(t)-\psi(a))^{\frac{\mu}{\rho}}\right] = \frac{\Gamma_\rho(\mu+\rho)}{\Gamma_\rho(\mu+\rho-\alpha)}(\psi(t)-\psi(a))^{\frac{\mu-\alpha}{\rho}}$. *If $\alpha \in (1,2]$, then*

$$ {}^{H}_{\rho}\mathfrak{D}^{\alpha-1;\psi}_{a^+}\left[\Psi_{\psi}^{\frac{\alpha-1}{\rho}}(t,a)\right] = \Gamma_\rho(\alpha-1+\rho) \quad \textit{and} \quad {}^{H}_{\rho}\mathfrak{D}^{\alpha-1;\psi}_{a^+}\left[\Psi_{\psi}^{\frac{\alpha-2}{\rho}}(t,a)\right] = 0. $$

(iii) ${}_{\rho}\mathscr{I}^{\alpha;\phi}_{a^+}\,{}_{\rho}\mathscr{I}^{\mu;\psi}_{a^+} f(t) = {}_{\rho}\mathscr{I}^{\alpha+\mu;\psi}_{a^+} f(t) = {}_{\rho}\mathscr{I}^{\mu;\phi}_{a^+}\,{}_{\rho}\mathscr{I}^{\alpha;\psi}_{a^+} f(t)$.

Lemma 2.5 ([30])**.** *If $f \in \mathscr{C}^n(\mathscr{J},\mathbb{R})$, $\alpha \in \mathbb{R}^+$, $\beta \in [0,1]$, where $\rho > 0$, and $\gamma = \frac{1}{\rho}(\beta(\rho n-\alpha)+\alpha)$, then* $\left({}_{\rho}\mathscr{I}^{\alpha;\psi}_{a^+}\,{}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\psi}_{a^+} f\right)(t) = f(t) - \sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\gamma-i}}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))}\left[\delta^{n-i}_{\psi}\left({}_{\rho}\mathscr{I}^{\rho(n-\gamma);\psi}_{a^+} f(a)\right)\right]$, $n = \left\lceil \frac{\alpha}{\rho} \right\rceil$.

The following result is also needed for our main result.

Lemma 2.6. *Let $\nu \in (m-1,m)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, $\beta \in [0,1]$, and $\rho > 0$. If $h \in \mathscr{C}^n(\mathscr{J},\mathbb{R})$, then*

$$ {}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\psi}_{a^+}\left[{}_{\rho}\mathscr{I}^{\nu;\psi}_{a^+} h(t)\right] = {}_{\rho}\mathscr{I}^{\nu-\alpha;\psi}_{a^+} h(t). \tag{2.2} $$

Proof. Let $\gamma = \frac{1}{\rho}\left(\beta(\rho n-\alpha)+\alpha\right)$ with $\gamma \in (n-1,n]$. By applying Definition 2.2 and the (iii) of Lemma 2.4, we have

$$ {}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\psi}_{a^+}\left[{}_{\rho}\mathscr{I}^{\nu;\psi}_{a^+} h(t)\right] = {}_{\rho}\mathscr{I}^{\beta(\rho n-\alpha);\psi}_{a^+}\left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n \left[{}_{\rho}\mathscr{I}^{\rho(n-\gamma)+\nu;\psi}_{a^+} h(t)\right]. \tag{2.3} $$

By using Definition 2.1 for $n=1$, we obtain

$$ \begin{aligned} &\left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right) {}_{\rho}\mathscr{I}^{\rho(n-\gamma)+\nu;\psi}_{a^+} h(t) \\ =\;& \frac{1}{\rho\Gamma_\rho(\rho(n-\gamma)+\nu-\rho)}\int_a^t \Psi_{\psi}^{\frac{(n-\gamma)+\nu-\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds = {}_{\rho}\mathscr{I}^{\rho(n-\gamma)+\nu-\rho;\psi}_{a^+} h(t). \end{aligned} $$

In the same way, we have, for $n=2$,

$$ \begin{aligned} &\left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^2 {}_{\rho}\mathscr{I}^{\rho(n-\gamma)+\nu;\psi}_{a^+} h(t) \\ =\;& \frac{1}{\rho\Gamma_\rho(\rho(n-\gamma)+\nu-2\rho)}\int_a^t \Psi_{\psi}^{\frac{\rho(n-\gamma)+\nu-2\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds = {}_{\rho}\mathscr{I}^{\rho(n-\gamma)+\nu-2\rho;\psi}_{a^+} h(t). \end{aligned} $$

Repeating the above method, we obtain

$$\left(\frac{\rho}{\psi'(t)}\cdot\frac{d}{dt}\right)^{n}{}_{\rho}\mathscr{I}_{a^+}^{\rho(n-\gamma)+\nu;\psi}h(t)=\frac{1}{\rho\Gamma_{\rho}(\nu-\rho\gamma)}\int_a^t\Psi_{\psi}^{\frac{\nu-\rho\gamma}{\rho}-1}(t,s)\psi'(s)h(s)ds={}_{\rho}\mathscr{I}_{a^+}^{\nu-\rho\gamma;\psi}h(t).$$

From (2.3) with the property (iii) of Lemma 2.4, one sees that (2.2). The proof is completed. □

Remark 2.7. This investigation of (ρ,ψ)-Hilfer-$\mathbb{FDO}$ also yields the following conclusions:

($\mathbb{G}_1$) If we set $\psi(t)\in\mathscr{C}^n(\mathscr{J},\mathbb{R})$ with $\psi'(t)\neq 0$ as in Definition 2.3, we obtain the following special cases:
- (i) If $\beta=0$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the (ρ,ψ)-$\mathbb{RL}$-$\mathbb{FDO}$ defined in [13], while if $\beta=0$, $\rho=1$, then we obtain ψ-$\mathbb{RL}$-$\mathbb{FDO}$ defined as in [31, 32].
- (ii) If $\beta=1$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the (ρ,ψ)-Caputo-$\mathbb{FDO}$ defined as in [13], while if $\beta=\rho=1$, then we obtain ψ-Caputo-$\mathbb{FDO}$ defined in [31, 32].
- (iii) If $\rho=1$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ψ-Hilfer-$\mathbb{FDO}$ defined in [12].

($\mathbb{G}_2$) If we set $\psi(t)=t$ as in Definition 2.3, then we obtain the following special cases:
- (i) If $\beta=0$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-$\mathbb{RL}$-$\mathbb{FDO}$ defined as in [29], while if $\beta=0$, $\rho=1$, then we obtain $\mathbb{RL}$-$\mathbb{FDO}$ defined in [31, 32].
- (ii) If $\beta=1$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-Caputo-$\mathbb{FDO}$ defined as in [13], while if $\beta=\rho=1$, then we obtain the Caputo-$\mathbb{FDO}$ defined in [31, 32].
- (iii) If $\rho=1$ and $\beta\in[0,1]$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the Hilfer-$\mathbb{FDO}$ defined in [33].

($\mathbb{G}_3$) If we set $\psi(t)=t^{\mu}$ as in Definition 2.3, we obtain the following special cases:
- (i) If $\beta=0$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-Katugampola-$\mathbb{FDO}$ defined in [34].
- (ii) If $\beta=1$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-Caputo-Katugampola-$\mathbb{FDO}$ defined in [34].
- (iii) If $\rho=1$ and $\beta\in[0,1]$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to Hilfer-Katugampola-$\mathbb{FDO}$ defined in [13].

($\mathbb{G}_4$) If we set $\psi(t)=\log t$ as in Definition 2.3, we obtain the following special cases:
- (i) If $\beta=0$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-Hadamard-$\mathbb{FDO}$ defined in [13].
- (ii) If $\beta=1$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the ρ-Caputo-Hadamard-$\mathbb{FDO}$ defined in [13].
- (iii) If $\rho=1$ and $\beta\in[0,1]$, then (ρ,ψ)-Hilfer-$\mathbb{FDO}$ (2.1) reduces to the Hilfer-Hadamard-$\mathbb{FDO}$ defined in [13].

Now, we denote the weighted space

$$\mathscr{C}_{\psi}^{2-\gamma}(\mathscr{J},\mathbb{R})=\left\{u:(a,b]\to\mathbb{R}|u(a^+)\text{ exists and }\Psi_{\psi}^{2-\gamma}(t,a)u(t)\in\mathscr{C}(\mathscr{J},\mathbb{R})\right\},\ \gamma\in(1,2],$$

where $\mathscr{C}_{\psi}^{2-\gamma}=\mathscr{C}_{\psi}^{2-\gamma}(\mathscr{J},\mathbb{R})$ and the weighted space of piece-wise continuous functions is given by

$$\begin{aligned}&\mathscr{PC}_{\psi_k}^{2-\gamma_k}(\mathscr{J},\mathbb{R})\\&=\Big\{u:(a,b]\to\mathbb{R}\,|\,u\in\mathscr{C}_{\psi_k}^{2-\gamma_k},\ k=0,1,2,\ldots,m,\ {}_{\rho_k}\mathscr{I}_{t_k^+}^{\rho_k(2-\gamma_k);\psi_k}u(t_k^+),\ {}_{\rho_{k-1}}\mathscr{I}_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}u(t_k^-)\text{ exist and}\\&{}_{\rho_{k-1}}\mathscr{I}_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}u(t_k^-)={}_{\rho_{k-1}}\mathscr{I}_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}u(t_k),\ k=1,\ldots,m\Big\}.\end{aligned}$$

Note that $\mathscr{PC}=\mathscr{PC}_{\psi_k}^{2-\gamma_k}(\mathscr{J},\mathbb{R})$ is a Banach space equipped with $\|u\|=\sup_{t\in\mathscr{J}}|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)u(t)|$.

Lemma 2.8. *Let $\alpha_k\in(1,2)$, $\beta_k\in[0,1]$, $\rho_k>0$, $\mu_k>0$, $\nu_k>0$, $\gamma_k=(1/\rho_k)(\beta_k(2\rho_k-\alpha_k)+\alpha_k)$, $\psi_k\in\mathscr{C}(\mathscr{J},\mathbb{R})$ with $\psi_k'>0$, $k=0,1,2,\ldots,m$, $\mathscr{H}\in\mathscr{C}_{\psi_k}^{2-\gamma_k}$, and $\Omega\neq0$. Then the following linear impulsive (ρ_k,ψ_k)-$\mathbb{HFP}$-$\mathbb{IDE}$-$\mathbb{MIBC}$s*

$$\begin{cases}{}_{\rho_k}^{H}\mathfrak{D}_{t_k^+}^{\alpha_k,\beta_k;\psi_k}u(t)=\mathscr{H}(t),\ t\in\mathscr{J}_k,\ t\neq t_k,\quad k=0,1,\ldots,m,\\ {}_{\rho_k}^{\mathbb{RL}}\mathfrak{D}_{t_k^+}^{\rho_k(\gamma_k-1);\psi_k}u(t_k^+)-{}_{\rho_{k-1}}^{\mathbb{RL}}\mathfrak{D}_{t_{k-1}^+}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}u(t_k^-)=\phi_k(u(t_k)),\quad k=1,2,\ldots,m,\\ {}_{\rho_k}\mathscr{I}_{t_k^+}^{\rho_k(2-\gamma_k);\psi_k}u(t_k^+)-{}_{\rho_{k-1}}\mathscr{I}_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}u(t_k^-)=\phi_k^*(u(t_k)),\quad k=1,2,\ldots,m,\\ u(0)=0,\quad u(T)=\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}u(\xi_i)+\mathscr{A},\quad \xi_i\in(t_i,t_{i+1}],\end{cases}\tag{2.4}$$

satisfies the following integral equation, $u\in\mathscr{PC}$, as

$$\begin{aligned}u(t)\ =\ &\left\{\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k\gamma_k)}+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right.\\&\times\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\\&\times\left[\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\Big)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\&\times\left.\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\right]+\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i)+\mathscr{A}\\&-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{H}(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)\\&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\Big)\right.\\&+\left.\left.\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\right]\right\}\\&+{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\mathscr{H}(t)+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)\\&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\left[\sum_{j=0}^{k-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\Big)\right.\end{aligned}$$

$$+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\Bigg], \tag{2.5}$$

where

$$\begin{aligned}\Omega &:= \frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\ &\quad+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}.\end{aligned} \tag{2.6}$$

Proof. Let $u\in\mathscr{PC}$ be a solution to the impulsive (ρ_k,ψ_k)-Hilfer problem (2.4). For $t\in[t_0,t_1]$, we have

$$u(t)=\frac{\Psi_{\psi_0}^{\gamma_0-1}(t,t_0)}{\Gamma_{\rho_0}(\rho_0\gamma_0)}c_1+\frac{\Psi_{\psi_0}^{\gamma_0-2}(t,t_0)}{\Gamma_{\rho_0}(\rho_0(\gamma_0-1))}c_2+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0;\psi_0}\mathscr{H}(t),$$

where $c_1={}^{\mathbb{RL}}_{\rho_0}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t_0^+)$ and $c_2={}_{\rho_0}\mathscr{I}_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t_0^+)$. By using Lemma 2.4 and Lemma 2.6, we obtain

$${}_{\rho_0}\mathscr{I}_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t) = \frac{\Psi_{\psi_0}(t,t_0)}{\rho_0}c_1+c_2+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathscr{H}(t), \tag{2.7}$$

$${}^{\mathbb{RL}}_{\rho_0}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t) = c_1+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t). \tag{2.8}$$

Taking $t=t_1$ into (2.7)-(2.8), we have ${}^{\mathbb{RL}}_{\rho_0}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t_1)=c_1+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1)$ and

$${}_{\rho_0}\mathscr{I}_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t_1) = \frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}c_1+c_2+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathscr{H}(t_1).$$

For $t\in(t_1,t_2]$, we obtain

$$u(t)=\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}{}^{\mathbb{RL}}_{\rho_1}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t_1^+)+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}{}_{\rho_1}\mathscr{I}_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t_1^+)+{}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1;\psi_1}\mathscr{H}(t).$$

From the impulsive conditions, we see that

$$\begin{aligned}u(t) &= \left(\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}\cdot\frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}c_2\\ &\quad+\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}\left({}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1)+\phi_1(u(t_1))\right)\\ &\quad+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}\left({}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathscr{H}(t_1)+\phi_1^*(u(t_1))\right)+{}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1;\psi_1}\mathscr{H}(t).\end{aligned}$$

By applying Lemma 2.4 and Lemma 2.6, we arrive at

$$\begin{aligned}&{}_{\rho_1}\mathscr{I}_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t)\\ &=\left(\frac{\Psi_{\psi_1}(t,t_1)}{\rho_1}+\frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1+c_2+{}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathscr{H}(t_1)+\phi_1^*(u(t_1))\\ &\quad+\frac{\Psi_{\psi_1}(t,t_1)}{\rho_1}\left({}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1)+\phi_1(u(t_1))\right)+{}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1+\rho_1(2-\gamma_1);\psi_1}\mathscr{H}(t),\end{aligned}$$

and

$$ {}^{\mathbb{RL}}_{\rho_1}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t) = c_1 + {}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1) + \phi_1(u(t_1)) + {}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1-\rho_1(\gamma_1-1);\psi_1}\mathscr{H}(t). $$

In particular, for $t = t_2$, we have

$$ \begin{aligned} &{}_{\rho_1}\mathscr{I}_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t_2) \\ &= \left(\frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1} + \frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1 + c_2 + {}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathscr{H}(t_1) + \phi_1^*(u(t_1)) \\ &\quad + \frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1}\left({}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1) + \phi_1(u(t_1))\right) + {}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1+\rho_1(2-\gamma_1);\psi_1}\mathscr{H}(t_2), \end{aligned} $$

and

$$ {}^{\mathbb{RL}}_{\rho_1}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t_2) = c_1 + {}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1) + \phi_1(u(t_1)) + {}_{\rho_1}\mathscr{I}_{t_1}^{\alpha_1-\rho_1(\gamma_1-1);\psi_1}\mathscr{H}(t_2). $$

Under the impulsive conditions again, we obtain

$$ \begin{aligned} u(t) &= \left(\frac{\Psi_{\psi_2}^{\gamma_2-1}(t,t_2)}{\Gamma_{\rho_2}(\rho_2\gamma_2)} + \frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}\sum_{j=0}^{1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1 + \frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}c_2 \\ &\quad + \frac{\Psi_{\psi_2}^{\gamma_2-1}(t,t_2)}{\Gamma_{\rho_2}(\rho_2\gamma_2)}\sum_{j=0}^{1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\right) \\ &\quad + \frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}\left[\sum_{j=0}^{1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1}) + \phi_{j+1}^*(u(t_{j+1}))\right)\right. \\ &\quad \left. + \frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1}\left({}_{\rho_0}\mathscr{I}_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathscr{H}(t_1) + \phi_1(x(t_1))\right)\right] + {}_{\rho_2}\mathscr{I}_{t_2}^{\alpha_2;\psi_2}\mathscr{H}(t). \end{aligned} $$

Repeating the above process, for any $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$, one has

$$ \begin{aligned} u(t) &= \left(\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1 + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}c_2 \\ &\quad + {}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\mathscr{H}(t) + \frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\right) \\ &\quad + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\left[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1}) + \phi_{j+1}^*(u(t_{j+1}))\right)\right. \\ &\quad \left. + \sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\right)\right]. \end{aligned} \tag{2.9} $$

Thanks to Lemma 2.4, one sees that

$$
\begin{aligned}
&\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}u(\xi_i)=\\
&\left(\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1\\
&\quad+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}c_2+\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i)\\
&\quad+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\
&\quad+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi^{*}_{j+1}(u(t_{j+1}))\right)\right.\\
&\quad\left.+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right].
\end{aligned}
$$

By using the first boundary condition, $u(0)=0$, one obtains $c_2=0$. From the second boundary condition, $u(T)=\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}u(\xi_i)+\mathscr{A}$, one further obtains

$$
\begin{aligned}
c_1=\frac{1}{\Omega}&\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\right.\\
&+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi^{*}_{j+1}(u(t_{j+1}))\right)\right.\\
&\left.+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]+\sum_{i=0}^{m}\mu_i\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i)\\
&+\mathscr{A}-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{H}(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\
&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi^{*}_{j+1}(u(t_{j+1}))\right)\right.\\
&\left.\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]\right\},
\end{aligned}
$$

where Ω is given by (2.6). Taking the values c_1 and c_2 in (2.9), we achieve the solution (2.5).

On the other hand, one assumes that $u \in \mathscr{PC}$ satisfies equation (2.5). By taking the operator ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}$ into both sides of (2.5) with (ii) of Lemma 2.4 and Lemma 2.6, it follows that

$$
\begin{aligned}
{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}u(t) = &\left\{ \frac{{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}\left[\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)\right]}{\Omega\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}\left[\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)\right]}{\Omega\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\} \\
&\times\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\right. \\
&+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\right)\right. \\
&\left.+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right] \\
&+\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}^{\alpha_i+\kappa_i;\psi_i}_{t_i}\mathscr{H}(\xi_i)+\mathscr{A}-{}_{\rho_m}\mathscr{I}^{\alpha_m;\psi_m}_{t_m}\mathscr{H}(T) \\
&-\frac{\Psi^{\gamma_m-1}_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right) \\
&-\frac{\Psi^{\gamma_m-2}_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\right)\right. \\
&\left.\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]\right\} \\
&+{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}\left[{}_{\rho_k}\mathscr{I}^{\alpha_k;\psi_k}_{t_k}\mathscr{H}(t)\right]+\frac{{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}\left[\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)\right]}{\Gamma_{\rho_k}(\rho_k\gamma_k)} \\
&\times\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right) \\
&+\frac{{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}\left[\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)\right]}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\left[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\right)\right. \\
&\left.+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]=\mathscr{H}(t).
\end{aligned}
$$

Next, we show that u satisfies the multi-point integral boundary conditions. It is easy to see that $u(0)=0$ and

$$
\begin{aligned}
u(T) \quad = \quad &\left\{\frac{\Psi^{\gamma_m-1}_{\psi_m}(T,t_m)}{\Omega\Gamma_{\rho_m}(\rho_m\gamma_m)}+\frac{\Psi^{\gamma_m-2}_{\psi_m}(T,t_m)}{\Omega\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right. \\
&\times\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^{*}(u(t_{j+1}))\right)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\
&\left.\times\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]+\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i)\\
&+\mathscr{A}-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{H}(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\
&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^{*}(u(t_{j+1}))\right)\right.\\
&\left.\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]\right\}\\
&+{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{H}(T)+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\
&+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^{*}(u(t_{j+1}))\right)\right.\\
&\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right].
\end{aligned}
$$

Taking ${}_{\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}$ into both sides of (2.5) with (i) and (iii) of Lemma 2.4, we have

$$
\begin{aligned}
&\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}u(\xi_i)\\
=\;&\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\rho_i(\gamma_i-1)+\kappa_i}(\xi_i,t_i)}{\Omega\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\rho_i+\kappa_i)}+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\rho_i(\gamma_i-2)+\kappa_i}(\xi_i,t_i)}{\Omega\Gamma_{\rho_i}(\rho_i(\gamma_i-2)+\rho_i+\kappa_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\\
\times\;&\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\right.\\
+\;&\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^{*}(u(t_{j+1}))\right)\right.\\
+\;&\left.\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]+\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i)\\
+\;&\mathscr{A}-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{H}(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\
-\;&\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}^{*}(u(t_{j+1}))\right)\right.\\
&\left.\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\rho_i(\gamma_i-1)+\kappa_i}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\rho_i+\kappa_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{H}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)\\
&+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\rho_i(\gamma_i-2)+\kappa_i}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-2)+\rho_i+\kappa_i)}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{H}(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\Big)\\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{H}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\Bigg]+\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{H}(\xi_i),
\end{aligned}
$$

where Ω is given by (2.6). Hence, it is easy to show that $u(T)=\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\kappa_i;\psi_i}u(\xi_i)+\mathscr{A}$. The proof is completed. □

3. Main Results

3.1. Uniqueness Result under Banach's Fixed Point Theorem.

By Lemma 2.8, we define an operator $\mathscr{Q}:\mathscr{PC}(\mathscr{J},\mathbb{R})\to\mathscr{PC}(\mathscr{J},\mathbb{R})$ by

$$
\begin{aligned}
&(\mathscr{Q}u)(t)=\\
&\left\{\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k\gamma_k)}+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\left\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right.\\
&\times\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{F}_u(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\\
&\times\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{F}_u(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\Big)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\
&\times\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{F}_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\Bigg]+\sum_{i=0}^{m}\mu_{i\,\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}\mathscr{F}_u(\xi_i)+\mathscr{A}\\
&-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{F}_u(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{F}_u(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)\\
&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{F}_u(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\Big)\\
&\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{F}_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\Bigg]\right\}\\
&+{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\mathscr{F}_u(t)+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{F}_u(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\Big)\\
&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{F}_u(t_{j+1})+\phi^*_{j+1}(u(t_{j+1}))\Big)\\
&+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{F}_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\Big)\Bigg],
\end{aligned}
$$

where $\mathscr{F}_u(t)=f(t,u(t),u(\theta t),{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}u(t))$ Observe that, the problem (1.1) has a solution if and only if $\mathscr{Q}$ has fixed points. Next, we provide the symbols of constants that will be applied throughout in this work. For $c \in \{m, j\}$,

$$
\begin{aligned}
\Delta_1^c &:= \sum_{j=0}^{c-1} \frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_J}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))} + \sum_{j=1}^{c-1} \frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j} \sum_{r=0}^{j-1} \frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}, \\
\Lambda_1 &:= \frac{1}{|\Omega|}\left(\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)} + \sum_{j=0}^{m-1} \frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j \Gamma_{\rho_m}(\rho_m(\gamma_m-1))} \right), \\
\Lambda_2 &:= \frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)} + \sum_{i=0}^{m} \frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i+\kappa_i)} + \frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)} \sum_{j=0}^{m-1} \frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))} \\
&\quad + \frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)\Delta_1^m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))} + \sum_{i=0}^{m} \frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)} \sum_{j=0}^{i-1} \frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))} \\
&\quad + \sum_{i=0}^{m} \frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)\Delta_1^i}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}, \\
\Lambda_3 &:= \frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)} + \frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)} \sum_{j=0}^{m-1} \frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))} \\
&\quad + \frac{\Delta_1^m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}, \\
\Lambda_4 &:= \frac{m\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)} + \frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))} \sum_{j=1}^{m-1} \frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j} + \sum_{i=0}^{m} \frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)} \\
&\quad + \sum_{i=0}^{m} \frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)} \sum_{j=1}^{i-1} \frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j},
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_5 &:= \frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)} + \sum_{j=1}^{m-1} \frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}, \\
\Lambda_6 &:= \frac{m\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))} + \sum_{i=0}^{m} \frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}.
\end{aligned}
$$

Lemma 3.1. (Banach's fixed point theorem). *Assume that $\mathscr{D} \subset \mathscr{E}$ is a non-empty closed subset, where $\mathscr{E}$ is a Banach space. Then any contraction mapping $\mathscr{Q}$ from $\mathscr{D}$ into itself has a unique fixed-point.*

Theorem 3.2. *Let $\psi_k \in \mathscr{C}^2(\mathscr{J})$, where $\psi_k'(t) > 0$, $k = 0,1,2,\ldots,m$, $t \in \mathscr{J}$ and $f \in \mathscr{C}(\mathscr{J} \times \mathbb{R}^3, \mathbb{R})$, ϕ_k, $\phi_k^* \in \mathscr{C}(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$, and the following conditions hold:*

$(\mathbb{A}_1)$ *there are real constants $\mathbb{L}_i > 0$ such that, for any $t \in \mathscr{J}$ and u_i, v_i, $w_i \in \mathbb{R}$, $i = 1,2$,*

$$|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)| \le \mathbb{L}_1 \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(|u_1-u_2|+|v_1-v_2|) + \mathbb{L}_2 \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|w_1-w_2|;$$

($\mathbb{A}_2$) *there are real constants* $\mathbb{I}_i > 0$, $i = 1,2$, *such that, for* $k = 1,2,\ldots,m$,

$$|\phi_k(u)-\phi_k(v)| \le \mathbb{I}_1 \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u-v|\,,\ \ |\phi_k^*(u)-\phi_k^*(v)| \le \mathbb{I}_2 \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u-v|\,,\ t\in\mathscr{J}\ u,v\in\mathbb{R}.$$

Then, problem (1.1) *has a unique solution on an interval* $\mathscr{J}$ *provided that*

$$\Phi_1+\Phi_2<1 \tag{3.1}$$

where $\Phi_1 := (\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)$, $\Phi_2 := (\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+m\Psi_*^{\gamma_m})\mathbb{I}_2$, *and*

$$\Psi_*^{\sigma_m} := \frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}, \qquad \Psi_*^{\gamma_m} := \frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}.$$

Proof. Clearly, problem (1.1) corresponds to a fixed-point problem, that is, $u=\mathscr{Q}u$. Then, we show that $\mathscr{Q}$ has a fixed-point by the Banach's fixed-point theorem. Now, we suppose $\mathbb{M}_1 := \sup_{t\in\mathscr{J}}|f(t,0,0,0)|$, $\mathbb{M}_2 := \max\{\phi_k(0) : k=1,2,\ldots,m\}$ and $\mathbb{M}_3 := \max\{\phi_k^*(0) : k=1,2,\ldots,m\}$, and $\mathscr{B}_{r_1}$ is a bounded, closed, and convex subset of $\mathscr{E}$, where $\mathscr{B}_{r_1} := \{u\in\mathscr{E} : \|u\|\le r_1\}$ and

$$r_1 \ge \frac{(\Lambda_1\Lambda_2+\Lambda_3)\mathbb{M}_1+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{M}_2+(\Lambda_1\Lambda_6+m\Psi_*^{\gamma_m})\mathbb{M}_3+\Lambda_1|\mathscr{A}|}{1-(\Phi_1+\Phi_2)}.$$

Firstly, we show that $\mathscr{Q}\mathscr{B}_{r_1}\subset\mathscr{B}_{r_1}$. For any $u\in\mathscr{B}_{r_1}$, we obtain

$$\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}u)(t)\right| \\
\le\ & \left\{\frac{\Psi_{\psi_k}(t,t_k)}{|\Omega|\Gamma_{\rho_k}(\rho_k\gamma_k)}+\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\right\}\left\{\sum_{i=0}^{m}\frac{|\mu_i|\,\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right. \\
&\times\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))\right|\right)+\sum_{i=0}^{m}\frac{|\mu_i|\,\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)} \\
&\times\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}^*(u(t_{j+1}))\right|\right)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right. \\
&\left.\times\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|\mathscr{F}_u(t_{r+1})|+|\phi_{r+1}(u(t_{r+1}))|\right)\right]+\sum_{i=0}^{m}|\mu_i|\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}|\mathscr{F}_u(\xi_i)| \\
&+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))\right|\right) \\
&+|\mathscr{A}|+{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}|\mathscr{F}_u(T)|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}^*(u(t_{j+1}))\right|\right)\right.\\
&\left.\left.+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\left|\mathscr{F}_u(t_{r+1})\right|+\left|\phi_{r+1}(u(t_{r+1}))\right|\right)\right]\right\}\\
&+\frac{\Psi_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))\right|\right)\\
&+\frac{1}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\left[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})\right|+\left|\phi_{j+1}^*(u(t_{j+1}))\right|\right)\right.\\
&\left.+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\left|\mathscr{F}_u(t_{r+1})\right|+\left|\phi_{r+1}(u(t_{r+1}))\right|\right)\right]\\
&+\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\left|\mathscr{F}_u(t)\right|. \qquad (3.2)
\end{aligned}
$$

It follows from Lemma 2.4 (i) that

$$
\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left|{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}u(t)\right|\leq{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}(1)(t)\|u\|\leq\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}\|u\|_{\mathscr{PC}}. \qquad (3.3)
$$

Thanks to $(\mathbb{A}_1)$-$(\mathbb{A}_2)$ and (3.3), we obtain that

$$
\begin{aligned}
|\mathscr{F}_u(t)| &\leq \left|f(t,u(t),u(\theta t),{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}u(t))-f(t,0,0,0)\right|+|f(t,0,0,0)|\\
&\leq \mathbb{L}_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\big(|u(t)|+|u(\theta t)|\big)+\mathbb{L}_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left|{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}u(t)\right|+\mathbb{M}_1\\
&\leq \left(2\mathbb{L}_1+\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}\mathbb{L}_2\right)\|u\|+\mathbb{M}_1\\
&= (2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1,
\end{aligned} \qquad (3.4)
$$

$$
\begin{aligned}
|\phi_k(u(t_k))| &\leq |\phi_k(u(t_k))-\phi_k(0)|+|\phi_k(0)|\\
&\leq \mathbb{I}_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)|+\mathbb{M}_2\\
&\leq \mathbb{I}_1\|u\|+\mathbb{M}_2,
\end{aligned} \qquad (3.5)
$$

and

$$
\begin{aligned}
|\phi_k^*(u(t_k))| &\leq |\phi_k^*(u(t_k))-\phi_k^*(0)|+|\phi_k^*(0)|\\
&\leq \mathbb{I}_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)|+\mathbb{M}_3\\
&\leq \mathbb{I}_2\|u\|+\mathbb{M}_3.
\end{aligned} \qquad (3.6)
$$

By substituting (3.4)-(3.6) into (3.2), we obtain

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}u)(t)\right| \\
&\leq \left\{\frac{\Psi_{\psi_k}(t,t_k)}{|\Omega|\Gamma_{\rho_k}(\rho_k\gamma_k)}+\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\right\}\left\{\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right. \\
&\quad\times\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_1\|u\|+\mathbb{M}_2\right) \\
&\quad+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left[\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_2\|u\|\right.\right. \\
&\quad\left.+\mathbb{M}_3\right)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_1\|u\|\right. \\
&\quad\left.\left.+\mathbb{M}_2\right)\right]+\sum_{i=0}^{m}|\mu_i|\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}(1)(\xi_i)\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+|\mathscr{A}|+{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}(1)(T)\Big[(2\mathbb{L}_1 \\
&\quad+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\Big]+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]\right. \\
&\quad\left.+\mathbb{I}_1\|u\|+\mathbb{M}_2\right)+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]\right.\right. \\
&\quad\left.+\mathbb{I}_2\|u\|+\mathbb{M}_3\right)+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]\right. \\
&\quad\left.\left.\left.+\mathbb{I}_1\|u\|+\mathbb{M}_2\right)\right]\right\}+\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}(1)(t)\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right] \\
&\quad+\frac{\Psi_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_1\|u\|+\mathbb{M}_2\right) \\
&\quad+\frac{1}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\left[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_2\|u\|_{\mathscr{PC}}+\mathbb{M}_3\right)\right. \\
&\quad\left.+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\left[(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right]+\mathbb{I}_1\|u\|+\mathbb{M}_2\right)\right].
\end{aligned}
$$

From Lemma 2.4 (i), it follows that

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}u)(t)\right| \\
\leq\ &\left\{(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right\}\left\{\left[\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right]\right. \\
&\times\left[\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i+\kappa_i)}+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left(\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_J}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\
& \left.\times\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}\right)+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}\\
& +\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\left(\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\
& \left.\left.\times\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}\right)\right]+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}\\
& +\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\left(\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1,t_j})}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\
& \left.\left.\times\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}\right)\right\}+\left\{\left[\frac{m\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\right.\\
& \left.+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\left\{\mathbb{I}_1\|u\|+\mathbb{M}_2\right\}\\
& +\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg]\left[\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right]\\
& +\left\{\left[\frac{m\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\right]\left[\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right]\right.\\
& \left.+\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\left\{\mathbb{I}_2\|u\|+\mathbb{M}_3\right\}+\frac{|\mathscr{A}|\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{|\mathscr{A}|\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\\
= \;& \left\{(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u\|+\mathbb{M}_1\right\}\left\{\Lambda_1\Lambda_2+\Lambda_3\right\}+\left\{\mathbb{I}_1\|u\|+\mathbb{M}_2\right\}\left\{\Lambda_1\Lambda_4+\Lambda_5\right\}\\
& +\left\{\mathbb{I}_2\|u\|+\mathbb{M}_3\right\}\left\{\Lambda_1\Lambda_6+m\Psi_*^{\gamma_m}\right\}+\Lambda_1|\mathscr{A}|\\
\leq \;& \left\{(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+m\Psi_*^{\gamma_m})\mathbb{I}_2\right\}r_1\\
& +(\Lambda_1\Lambda_2+\Lambda_3)\mathbb{M}_1+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{M}_2+(\Lambda_1\Lambda_6+m\Psi_*^{\gamma_m})\mathbb{M}_3+\Lambda_1|\mathscr{A}|\leq r_1.
\end{aligned}
$$

Then $\|\mathscr{Q}u\|\leq r_1$, which yields that $\mathscr{Q}\mathscr{B}_{r_1}\subset\mathscr{B}_{r_1}$.

Next, we demonstrate that $\mathscr{Q}$ is a contraction. If u, $v\in\mathscr{B}_{r_1}$ and $t\in\mathscr{J}$, then

$$
\begin{aligned}
& \left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}u)(t)-\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}v)(t)\right|\\
\leq \;& \left\{\frac{\Psi_{\psi_k}(t,t_k)}{|\Omega|\Gamma_{\rho_k}(\rho_k\gamma_k)}+\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\right\}\left\{\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))\right|\right)\\
&+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\Bigg[\sum_{j=0}^{i-1}\Bigg({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|\\
&+|\phi_{j+1}^*(u(t_{j+1}))-\phi_{j+1}^*(v(t_{j+1}))|\Bigg)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\
&\times\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|\mathscr{F}_u(t_{r+1})-\mathscr{F}_v(t_{r+1})|+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\right)\Bigg]\\
&+\sum_{i=0}^{m}|\mu_i|\,{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}|\mathscr{F}_u(\xi_i)-\mathscr{F}_v(\xi_i)|+{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}|\mathscr{F}_u(T)-\mathscr{F}_v(T)|\\
&+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))\right|\right)\\
&+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Bigg({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|\\
&+\left|\phi_{j+1}^*(u(t_{j+1}))-\phi_{j+1}^*(v(t_{j+1}))\right|\Bigg)+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|\mathscr{F}_u(t_{r+1})-\mathscr{F}_v(t_{r+1})|\\
&+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\Bigg)\Bigg]\Bigg\}+\frac{\Psi_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Bigg({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|\\
&+|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))|\Bigg)+\frac{1}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Bigg({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\left|\mathscr{F}_u(t_{j+1})-\mathscr{F}_v(t_{j+1})\right|\\
&+|\phi_{j+1}^*(u(t_{j+1}))-\phi_{j+1}^*(v(t_{j+1}))|\Bigg)+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|\mathscr{F}_u(t_{r+1})-\mathscr{F}_v(t_{r+1})|\\
&+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\Bigg)\Bigg]+\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}|\mathscr{F}_u(t)-\mathscr{F}_v(t)|. \qquad (3.7)
\end{aligned}
$$

It follows from Lemma 2.4 (i) that

$$
\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}|u(t)-v(t)|\leq\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}\|u-v\|. \qquad (3.8)
$$

From conditions $(\mathbb{A}_1)$-$(\mathbb{A}_2)$ with (3.8) again, one has

$$
\begin{aligned}
|\mathscr{F}_u(t)-\mathscr{F}_v(t)|&\leq\left|f(t,u(t),u(\theta t),{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}u(t))-f(t,v(t),,v(\theta t)\,{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}v(t))\right|\\
&\leq\mathbb{L}_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(|u(t)-v(t)|+|u(\theta t)-v(\theta t)|)\\
&\quad+\mathbb{L}_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,{}_{\rho_k}\mathscr{I}_{t_k^+}^{\sigma_k;\psi_k}|u(t)-v(t)|\\
&\leq(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|,
\end{aligned}\qquad (3.9)
$$

$$
|\phi_k(u(t_k))-\phi_k(v(t_k))|\leq\mathbb{I}_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)-v(t)|\leq\mathbb{I}_1\|u-v\|, \qquad (3.10)
$$

and

$$
|\phi_k^*(u(t_k))-\phi_k^*(v(t_k))|\leq\mathbb{I}_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)-v(t)|\leq\mathbb{I}_2\|u-v\|. \qquad (3.11)
$$

Substituting (3.9)-(3.11) into (3.7), we have

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}u)(t)-\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(\mathscr{Q}v)(t)\right| \\
\leq\ & \left\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\,\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\,|\Omega|\,\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\} \\
&\times\Bigg\{\sum_{i=0}^{m}\frac{|\mu_i|\,\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\sum_{j=0}^{i-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right) \\
&+\sum_{i=0}^{m}\frac{|\mu_i|\,\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\Bigg[\sum_{j=0}^{i-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_2\|u-v\|\right) \\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right)\Bigg] \\
&+\sum_{i=0}^{m}\frac{|\mu_i|\,\Psi_{\psi_i}^{\frac{\alpha_i+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i+\kappa_i)}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\| \\
&+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right) \\
&+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|_{\mathscr{PC}}+\mathbb{I}_2\|u-v\|_{\mathscr{PC}}\right) \\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right)\Bigg]\Bigg\} \\
&+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right) \\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_2\|u-v\|\right) \\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\|+\mathbb{I}_1\|u-v\|\right)\Bigg] \\
&+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)\|u-v\| \\
=\ & \left[(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+\Psi_*^{\gamma_m})\mathbb{I}_2\right]\|u-v\|.
\end{aligned}
$$

It follows that $\|\mathscr{Q}u-\mathscr{Q}v\|\leq[\Phi_1+\Phi_2]\|u-v\|$. Thus $\mathscr{Q}$ is a contraction. From Lemma 3.1, we see that (1.1) has a unique solution on $\mathscr{J}$. □

3.2. **Ulam's Stability Results.** In this section, we study some types of Ulam stability of problem (1.1). We first introduce the concepts of Ulam stability for problem (1.1). Suppose that

$\Theta \in \mathscr{C}(\mathscr{J},\mathbb{R}^+)$ is a non-decreasing function and $\varepsilon > 0$, $\chi \geq 0$, $z \in \mathscr{E}$, such that, for each $t \in \mathscr{J}_k$, $k = 1,2,\ldots,m$, the following key inequalities are fulfilled:

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) - f(t,z(t),z(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) \right| \leq \varepsilon, \\ \left| {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi_k(z(t_k)) \right| \leq \varepsilon, \\ \left| {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi^*_k(z(t_k)) \right| \leq \varepsilon, \end{cases} \tag{3.12}$$

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) - f(t,z(t),z(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) \right| \leq \Theta(t), \\ \left| {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi_k(z(t_k)) \right| \leq \chi, \\ \left| {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi^*_k(z(t_k)) \right| \leq \chi, \end{cases} \tag{3.13}$$

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) - f(t,z(t),z(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) \right| \leq \varepsilon\Theta(t), \\ \left| {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi_k(z(t_k)) \right| \leq \varepsilon\chi, \\ \left| {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \phi^*_k(z(t_k)) \right| \leq \varepsilon\chi. \end{cases} \tag{3.14}$$

Definition 3.3. Problem (1.1) is said to be $\mathbb{UH}$ stable if there exists a real constant $\mathfrak{C}_{\mathscr{F}} > 0$ such that, for any $\varepsilon > 0$ and $z \in \mathscr{E}$ of (3.12), there exists $u \in \mathscr{E}$ of (1.1) such that $|z(t) - u(t)| \leq \mathfrak{C}_{\mathscr{F}}\, \varepsilon$, $t \in \mathscr{J}$.

Definition 3.4. Problem (1.1) is said to be a $\mathbb{GUH}$ stable if there exists $\Theta \in \mathscr{C}(\mathbb{R}^+,\mathbb{R}^+)$ via $\Theta(0) = 0$ such that, for any $\varepsilon > 0$ and $z \in \mathscr{E}$ of (3.13), there exists $u \in \mathscr{E}$ of (1.1) that satisfies $|z(t) - u(t)| \leq \Theta(\varepsilon)$, $t \in \mathscr{J}$.

Definition 3.5. Problem (1.1) is said to be $\mathbb{UHR}$ stable with respect to (χ,Θ) if there exists a real constant $\mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}} > 0$ such that, for any $\varepsilon > 0$ and $z \in \mathscr{E}$ of (3.14), there exists $u \in \mathscr{E}$ of (1.1) that satisfies $|z(t) - u(t)| \leq \mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}}\, \varepsilon(\chi + \Theta(t))$, $t \in \mathscr{J}$.

Definition 3.6. Problem (1.1) is said to be a $\mathbb{GUHR}$ stable with respect to (χ,Θ) if there exists a real constant $\mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}} > 0$ such that, for each $z \in \mathscr{E}$ of (3.13), there exists $u \in \mathscr{E}$ of (1.1) that satisfies $|z(t) - u(t)| \leq \mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}}(\chi + \Theta(t))$, $t \in \mathscr{J}$.

Remark 3.7. By Definition 3.3-3.6, we see that: (i) Definition 3.3 $\Rightarrow$ Definition 3.4; (ii) Definition 3.5 $\Rightarrow$ Definition 3.6; and (iii) Definition 3.5 $\Rightarrow$ Definition 3.3.

Remark 3.8. If $z \in \mathscr{E}$ is a solution to (3.12), then there exists $w \in \mathscr{E}$ with a sequence w_k for $k = 1,2,\ldots,m$, which depends on a function z such that

(i) $|w(t)| \leq \varepsilon$, $|w_k| \leq \varepsilon$, $t \in \mathscr{J}$;

(ii) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t,z(t),z(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) + w(t)$, $t \in \mathscr{J}$;

(iii) ${}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + w_k$, $t \in \mathscr{J}$;

(iv) ${}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi^*_k(z(t_k)) + w_k$, $t \in \mathscr{J}$.

Remark 3.9. If $z \in \mathscr{E}$ is a solution to (3.13), then there exists $w \in \mathscr{E}$ and w_k for $k = 1,2,\ldots,m$, which depends on a function z such that

(i) $|w(t)| \le \Theta(t)$, $|w_k| \le \chi$, $t \in \mathscr{J}$;

(ii) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), z(\theta t), {}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) + w(t)$, $t \in \mathscr{J}$;

(iii) ${}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + w_k$;

(iv) ${}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + w_k$, $t \in \mathscr{J}$.

Remark 3.10. If $z \in \mathscr{E}$ is a solution to (3.14), then there exits $w \in \mathscr{E}$ and w_k for $k = 1, 2, \dots, m$, which depends on a function z such that

(i) $|w(t)| \le \varepsilon\Theta(t)$, $|w_k| \le \varepsilon\chi$, $t \in \mathscr{J}$;

(ii) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), z(\theta t), {}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) + w(t)$, $t \in \mathscr{J}$;

(iii) ${}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + w_k$;

(iv) ${}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + w_k$, $t \in \mathscr{J}$.

3.2.1. $\mathbb{UH}$ *Stability Results.*

Theorem 3.11. *Let $\alpha_k \in (1,2]$, $\beta_k \in [0,1]$, $\rho_k \in \mathbb{R}^+$, $\gamma_k = (\beta_k(2\rho_k - \alpha_k) + \alpha_k)/\rho_k$, $\psi_k \in \mathscr{C}(\mathscr{J}, \mathbb{R})$, where $\psi_k' > 0$, $k = 1, 2, \dots, m$ and $f \in \mathscr{C}(\mathscr{J} \times \mathbb{R}^3, \mathbb{R})$. If assumptions $(\mathbb{A}_1)$-$(\mathbb{A}_2)$ and inequality (3.1) are fulfilled, then problem (1.1) is $\mathbb{UH}$ stable*

Proof. Let $z \in \mathscr{PC}$ be a solution to problem (3.12). Under the conditions (ii)-(iii) of Remark 3.8 and Lemma 2.8, we obtain

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), z(\theta t), {}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+} z(t)) + w(t), \quad t \neq t_k, \qquad k = 0, \dots, m, \\ {}^{\mathbb{RL}}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{\mathbb{RL}}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + w_k, k = 1, \dots, m, \\ {}_{\rho_k}\mathscr{I}^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathscr{I}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + w_k, k = 1, \dots, m, \\ z(0) = 0, \quad z(T) = \sum_{i=0}^{m} \mu_{i\,\rho_i}\mathscr{I}^{\kappa_i;\psi_i}_{t_i} z(\xi_i) + \mathscr{A}, \quad \xi_i \in (t_i, t_{i+1}]. \end{cases}$$

Observe that

$$\begin{aligned} z(t) \;=\; & \left\{ \frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k(\gamma_k-1))} \sum_{j=0}^{k-1} \frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j} \right\} \left\{ \sum_{i=0}^{m} \frac{\mu_i \Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)} \right. \\ & \times \sum_{j=0}^{i-1} \left({}_{\rho_j}\mathscr{I}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j} \mathscr{F}_z(t_{j+1}) + \phi_{j+1}(z(t_{j+1})) \right) + \sum_{i=0}^{m} \frac{\mu_i \Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)} \\ & \left[\sum_{j=0}^{i-1} \left({}_{\rho_j}\mathscr{I}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j} \mathscr{F}_z(t_{j+1}) + \phi^*_{j+1}(z(t_{j+1})) \right) + \sum_{j=1}^{i-1} \frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j} \right. \\ & \left. \sum_{r=0}^{j-1} \left({}_{\rho_r}\mathscr{I}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r} \mathscr{F}_z(t_{r+1}) + \phi_{r+1}(z(t_{r+1})) \right) \right] + \sum_{i=0}^{m} \mu_{i\,\rho_i}\mathscr{I}^{\alpha_i+\kappa_i;\psi_i}_{t_i} \mathscr{F}_z(\xi_i) + \mathscr{A} \end{aligned}$$

$$
\begin{aligned}
&-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}\mathscr{F}_z(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{F}_z(t_{j+1})+\phi_{j+1}(z(t_{j+1}))\right)\\
&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{F}_z(t_{j+1})+\phi_{j+1}^*(z(t_{j+1}))\right)\\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{F}_z(t_{r+1})+\phi_{r+1}(z(t_{r+1}))\right)\Bigg]\Bigg\}\\
&+{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\mathscr{F}_z(t)+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathscr{F}_z(t_{j+1})+\phi_{j+1}(z(t_{j+1}))\right)\\
&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathscr{F}_z(t_{j+1})+\phi_{j+1}^*(z(t_{j+1}))\right)\\
&+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathscr{F}_z(t_{r+1})+\phi_{r+1}(z(t_{r+1}))\right)\Bigg]\\
&+\left\{\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k\gamma_k)}+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Omega\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\\
&\times\sum_{j=0}^{i-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}w(t_{j+1})+w_{j+1}\right)+\sum_{i=0}^{m}\frac{\mu_i\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}w(t_{j+1})\\
&+w_{j+1}\Big)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}w(t_{r+1})+w_{r+1}\right)\Bigg]+\sum_{i=0}^{m}\mu_i{}_{\rho_i}\mathscr{I}_{t_i}^{\alpha_i+\kappa_i;\psi_i}w(\xi_i)\\
&-{}_{\rho_m}\mathscr{I}_{t_m}^{\alpha_m;\psi_m}w(T)-\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}w(t_{j+1})+w_{j+1}\right)\\
&-\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}w(t_{j+1})+w_{j+1}\right)+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\
&\times\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}w(t_{r+1})+w_{r+1}\right)\Bigg]\Bigg\}+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}w(t_{j+1})+w_{j+1}\right)\\
&+{}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}w(t)+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\left({}_{\rho_j}\mathscr{I}_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}w(t_{j+1})+w_{j+1}\right)\\
&+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}\mathscr{I}_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}w(t_{r+1})+w_{r+1}\right)\Bigg],
\end{aligned}
$$

for $t\in\mathscr{J}_k$, $k=0,1,\ldots,m$. Next, thanks to Remark 3.8 (i) and $(\mathbb{A}_1)$-$(\mathbb{A}_2)$, we have

$$
\begin{aligned}
\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\right| \quad\leq\quad & \left[(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+\Psi_*^{\gamma_m})\mathbb{I}_2\right]\|z-u\|\\
&+\varepsilon\left\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\Bigg\{\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\\
&\times\sum_{j=0}^{i-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\right)+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}
\end{aligned}
$$

$$\begin{aligned}
&\times\Bigg[\sum_{j=0}^{i-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+1\Bigg)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\Bigg)\Bigg]\\
&+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i+\kappa_i)}+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\frac{\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\Bigg)\\
&+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+1\Bigg)+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\
&\times\sum_{r=0}^{j-1}\Bigg(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\Bigg)\Bigg]\Bigg\}+\varepsilon\Bigg\{\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\\
&\times\sum_{j=0}^{m-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\Bigg)+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+1\Bigg)\\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\Bigg)\Bigg]\Bigg\}\\
=\;&\big[(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+\Psi_*^{\gamma_m})\mathbb{I}_2\big]\|z-u\|\\
&+\varepsilon\big[\Lambda_1(\Lambda_2+\Lambda_4+\Lambda_6)+\Lambda_3+\Lambda_5+m\Psi_*^{\gamma_m}\big]\\
=\;&[\Phi_1+\Phi_2]\|z-u\|+\varepsilon[\Lambda_1(\Lambda_2+\Lambda_4+\Lambda_6)+\Lambda_3+\Lambda_5+m\Psi_*^{\gamma_m}],
\end{aligned}$$

which implies that $\|z-u\|\le\mathfrak{C}_{\mathscr{F}}\varepsilon$, where $\mathfrak{C}_{\mathscr{F}}$ is given by

$$\mathfrak{C}_{\mathscr{F}}:=\frac{\Lambda_1(\Lambda_2+\Lambda_4+\Lambda_6)+\Lambda_3+\Lambda_5+m\Psi_*^{\gamma_m}}{1-(\Phi_1+\Phi_2)}. \tag{3.15}$$

Thus problem (1.1) is $\mathbb{UH}$ stable in $\mathscr{E}$. □

Corollary 3.12. *If $\Theta(\varepsilon)=\mathfrak{C}_{\mathscr{F}}\varepsilon$ and $\Theta(0)=0$ in Theorem 3.11, we obtain that problem (1.1) is $\mathbb{GUH}$ stable.*

3.3. $\mathbb{UHR}$ **stability results.** To ensure $\mathbb{UHR}$ and $\mathbb{GUHR}$ stability results, we give the following assumption:

($\mathbb{P}_1$) There exist a non-decreasing function $\Theta\in\mathscr{C}(\mathscr{J},\mathbb{R})$ and a positive real number $\mathfrak{C}_\Theta>0$ such that, for any $\varepsilon>0$, ${}_{\rho_k}\mathscr{I}_{t_k}^{\alpha_k;\psi_k}\Theta(t)\le\mathfrak{C}_\Theta\Theta(t)$,

and denote by $\Lambda_7:=\sum_{i=0}^m|\mu_i|+1$ and

$$\Lambda_8:=\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}.$$

Theorem 3.13. *Let $\alpha_k\in(1,2]$, $\beta_k\in[0,1]$, $\rho_k\in\mathbb{R}^+$, $\gamma_k=(\beta_k(2\rho_k-\alpha_k)+\alpha_k)/\rho_k$, $\psi_k\in\mathscr{C}(\mathscr{J},\mathbb{R})$ where $\psi_k'>0$, $k=1,2,\dots,m$ and $f\in\mathscr{C}(\mathscr{J}\times\mathbb{R}^3,\mathbb{R})$. If assumptions ($\mathbb{A}_1$)-($\mathbb{A}_2$) and inequality (3.1) are fulfilled, then (1.1) is $\mathbb{UHR}$ stable with respect to (χ,Θ) on $\mathscr{J}$.*

Proof. Let $z \in \mathscr{E}$ be the solution to (3.14) and $u \in \mathscr{E}$ be the solution to problem (1.1). From Theorem 3.11, of Remark 3.10 (i), $(\mathbb{A}_1)$, $(\mathbb{A}_2)$, and $(\mathbb{P}_1)$, we have

$$
\begin{aligned}
& \left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\right| \\
\leq\ & \left[(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+\Psi_*^{\gamma_m})\mathbb{I}_2\right]\|z-u\| \\
& +\varepsilon\left\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Omega|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Omega|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\left\{\mathfrak{C}_\Theta\Theta(t)\left(\frac{m\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\right.\right. \\
& +\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)} \\
& \times\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\frac{m\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}+\left.\sum_{i=0}^{m}|\mu_i|+1\right) \\
& +\chi\left(\frac{m\Psi_{\psi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\frac{\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-1)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i+\kappa_i)}\right. \\
& \left.\left.+\sum_{i=0}^{m}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\frac{m\Psi_{\psi_m}^{\gamma_m-2}(T,t_m)}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\sum_{i=0}^{m}\frac{i|\mu_i|\Psi_{\psi_i}^{\frac{\rho_i(\gamma_i-2)+\kappa_i}{\rho_i}}(\xi_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1)+\kappa_i)}\right)\right\} \\
& +\varepsilon\left\{\mathfrak{C}_\Theta\Theta(t)\left(\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right)\right. \\
& \left.+\chi\left(\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right)\right\} \\
=\ & \left[(\Lambda_1\Lambda_2+\Lambda_3)(2\mathbb{L}_1+\Psi_*^{\sigma_m}\mathbb{L}_2)+(\Lambda_1\Lambda_4+\Lambda_5)\mathbb{I}_1+(\Lambda_1\Lambda_6+\Psi_*^{\gamma_m})\mathbb{I}_2\right]\|z-u\| \\
& +\varepsilon\left\{\Lambda_1\left[\mathfrak{C}_\Theta\Theta(t)(\Lambda_4+\Lambda_6+\Lambda_7)+\chi(\Lambda_4+\Lambda_6)\right]+\mathfrak{C}_\Theta\Theta(t)(m\Psi_*^{\gamma_m}+\Lambda_8)+\chi(m\Psi_*^{\gamma_m}+\Lambda_5)\right\} \\
\leq\ & [\Phi_1+\Phi_2]\|z-u\|+\left\{\Lambda_1[\mathfrak{C}_\Theta(\Lambda_4+\Lambda_6+\Lambda_7)+\Lambda_4+\Lambda_6]\right. \\
& \left.+\mathfrak{C}_\Theta(m\Psi_*^{\gamma_m}+\Lambda_8)+m\Psi_*^{\gamma_m}+\Lambda_5\right\}\varepsilon(\Theta(t)+\chi).
\end{aligned}
$$

It follows that, $\|z-u\| \leq \mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}}\ \varepsilon\ (\Theta+\chi)$, where

$$
\mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}} := \frac{\Lambda_1[\mathfrak{C}_\Theta(\Lambda_4+\Lambda_6+\Lambda_7)+\Lambda_4+\Lambda_6]+\mathfrak{C}_\Theta(m\Psi_*^{\gamma_m}+\Lambda_8)+m\Psi_*^{\gamma_m}+\Lambda_5}{1-(\Phi_1+\Phi_2)}. \tag{3.16}
$$

Hence, (1.1) is $\mathbb{UHR}$ stable with respect to (χ,Θ) in $\mathscr{E}$. □

Corollary 3.14. *By setting $\varepsilon = 1$ and $\Theta(0) = 0$ in Theorem 3.13, we achieve the designed problem (1.1) is $\mathbb{GUHR}$ stable.*

4. Numerical Examples

In this section, we provide two numerical examples to support our main results.

Example 4.1. Consider the nonlinear impulsive (ρ_k,ψ_k)-$\mathbb{HFP}$-$\mathbb{IDE}$-$\mathbb{MIBC}$s:

$$\begin{cases} {}^{H}_{\frac{k+21}{20}}\mathfrak{D}^{\frac{k+6}{5},\frac{k+3}{6};\psi_k}_{t_k^+}u(t)=f(t,u(t),u(0.3t),{}_{\frac{k+21}{20}}\mathscr{I}^{\frac{6-2k}{5-k};\psi_k}_{t_k^+}u(t)),\ t\neq t_k,\ k=0,1,2,\\ {}_{\frac{k+21}{20}}\mathscr{I}^{\frac{k+4}{k+5}(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+)-{}_{\frac{k+20}{20}}\mathscr{I}^{\frac{k+3}{k+4}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-)=\phi_k(u(t_k)),\quad k=1,2,\\ {}^{\mathbb{RL}}_{\frac{k+21}{20}}\mathfrak{D}^{\frac{k+4}{k+5}(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+)-{}^{\mathbb{RL}}_{\frac{k+20}{20}}\mathfrak{D}^{\frac{k+3}{k+4}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+}u(t_k^-)=\phi_k^*(u(t_k)),\quad k=1,2,\\ u(0)=0,\qquad u\left(\frac{6}{5}\right)=\sum_{i=0}^{2}\left(\frac{3-i}{4i+4}\right){}_{\frac{k+21}{20}}\mathscr{I}^{\frac{i+3}{5-i};\psi_i}_{t_i}u\left(\frac{2i+1}{5}\right)+2. \end{cases} \tag{4.1}$$

Let $\alpha_k=(k+6)/5$, $\beta_k=(k+3)/6$, $\rho_k=(k+21)/20$, $\psi_k=(k+2-(k+1)e^{-(k+2)t})/(2k+3)$, $\sigma_k=(6-2k)/(5-k)$, $t_k=2k/5$, for $k=0,1,2$, with $\theta=0.3$ and $T=6/5$. The parameters of the boundary conditions are provided by $\mu_i=(3-i)/(4i+4)$, $\kappa_i=(i+3)/(5-i)$, $\xi_i=(2i+1)/5$, $i=0,1,2$, and $\mathscr{A}=2$. By using all parameters, we can compute that $\Omega\approx 0.4481898$, $\Lambda_1\approx 0.5602552$, $\Lambda_2\approx 0.1642033$, $\Lambda_3\approx 0.0984347$, $\Lambda_4\approx 0.1525212$, $\Lambda_5\approx 0.0981496$, and $\Lambda_6\approx 2.9626091$. The nonlinear functions are constructed as follows:

$$\begin{aligned} f(t,u(t),v(t),w(t)) &= e^{2t+5}\sin(t)+\frac{6\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)}{(\cos(t)+8)e^{5\sin(t)}}\left(\frac{|u(t)|}{2|u(t)|+3}-\frac{|v(t)|}{5|u(t)|+4}\right)\\ &\quad+\frac{(3-2t)\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)}{(t+2)^2}\frac{|w(t)|}{3|w(t)|+2},\\ \phi_k(u(t_k)) &= \frac{5t_k+1}{5t_k+10}\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)u(t_k)+\sin(\pi t_k), \end{aligned}$$

and

$$\phi_k^*(u(t_k))=\frac{5t_k-1}{10t_k+16}\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)u(t_k)+\cos(\pi t_k).$$

For any u_i, v_i, $w_i\in\mathbb{R}$, $i=1,2$, and $t\in[0,6/5]$, we find that

$$\begin{aligned} &|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|=\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)\left(\frac{2}{7}(|u_1-u_2|+|v_1-v_2|)+\frac{3}{8}|w_1-w_2|\right),\\ &|\phi_k(u)-\phi_k(v)|=\frac{5}{12}\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)|u_1-u_2|,\ |\phi_k^*(u)-\phi_k(v)|=\frac{3}{20}\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)|u_1-u_2|. \end{aligned}$$

It is noticed that conditions $(\mathbb{A}_1)$-$(\mathbb{A}_2)$ are satisfied with $\mathbb{L}_1=2/7$, $\mathbb{L}_2=3/8$, $\mathbb{I}_1=5/12$, $\mathbb{I}_2=3/20$, $\Psi^{\sigma_m}_*\approx 0.0868959$, and $\Psi^{\gamma_m}_*\approx 0.9481016$. Then, we have $\Phi_1+\Phi_2\approx 0.7249261<1$. Since, all conditions of Theorem 3.2 hold that is $(\mathbb{A}_1)$-$(\mathbb{A}_2)$, and $\Phi_1+\Phi_2<1$. Thus problem (4.1) has a unique solution on $[0,6/5]$.

Next, we show that Ulam's stability results for problem (4.1). From (3.15) in Theorem 3.11, we calculate the constant $\mathfrak{C}_{\mathscr{F}}\approx 14.2872548$. Hence, problem (4.1) is $\mathbb{UH}$ stable on $[0,6/5]$. By applying Corollary 3.12 with $\Theta(\varepsilon)=\mathfrak{C}_{\mathscr{F}}\varepsilon$ and $\Theta(0)=0$, we conclude that problem (4.1) is $\mathbb{GUH}$ stable on $[0,6/5]$. Finally, by setting $\Theta(t)=\Psi^{\frac{k}{2\rho_k}}_{\psi_k}(t,t_k)$ in $(\mathbb{P}_1)$, we see that

$${}_{\rho_k}\mathscr{I}^{\alpha_k;\psi_k}_{t_k}\Theta(t)\leq\frac{\Gamma_{\rho_k}(\frac{k}{2}+\rho_k)\Psi^{\frac{\alpha_k}{\rho_k}}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\frac{k}{2}+\rho_k+\alpha_k)}\Psi^{\frac{k}{2\rho_k}}_{\psi_k}(t,t_k)\leq\max_{k=\{0,1,2\}}\left\{\frac{\Gamma_{\rho_k}(\frac{k}{2}+\rho_k)\Psi^{\frac{\alpha_k}{\rho_k}}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\frac{k}{2}+\rho_k+\alpha_k)}\right\}\Theta(t),$$

which yields that $\mathfrak{C}_\Theta \approx 0.1274595$, and $\Lambda_7 = 25/12$ and $\Lambda_8 \approx 0.746402792$. From (3.16) in Theorem 3.13, we see that $\mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}} \approx 16.1689904$. Thus problem (4.1) is $\mathbb{UHR}$ stable on $[0,6/5]$. Moreover, by using Corollary 3.14 under $\Theta(\varepsilon) = \mathfrak{C}_{\mathscr{F},\Theta_{\mathscr{F}}}\varepsilon$ with $\varepsilon = 1$ and $\Theta(0) = 0$, we see from by Corollary 3.14 that problem (4.1) is $\mathbb{GUHR}$ stable with respect to (χ,Θ) on $[0,6/5]$. In addition, we present the graphical representations of $\Phi_1+\Phi_2$ under $\alpha_k \in (1,2]$ and $\beta_k \in [0,1]$ for $k \in \{0,1,2\}$ in Figure 1. The relationship between values α_k, β_k, Ω, Λ_i, $i = 1,2,\ldots,6$, and $\Phi_1+\Phi_2$ as shown in Table 1.

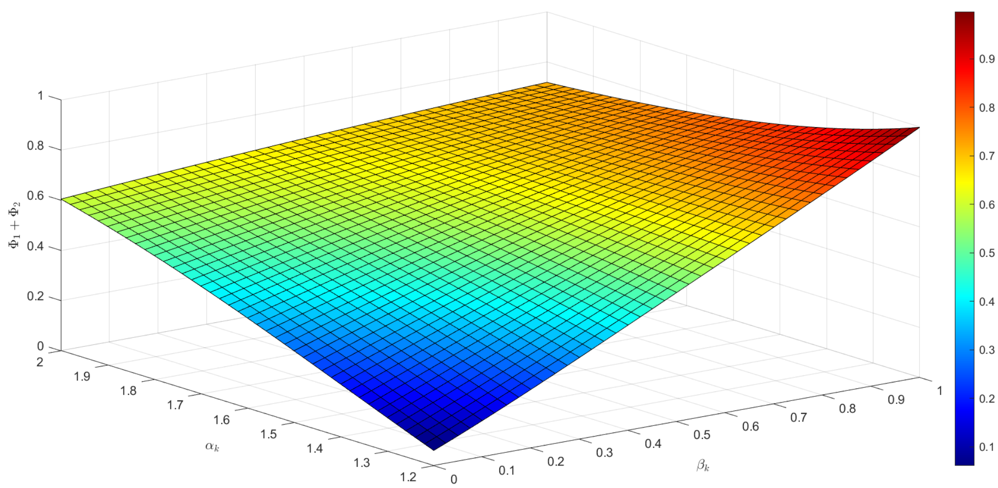

FIGURE 1. The condition $\Phi_1+\Phi_2$ of Problem (4.1).

TABLE 1. The values of Υ_1, Υ_2, Υ_3, and Υ_4.

α_k	β_k	Ω	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	$\Phi_1+\Phi_2$
1.20	0.0	1.82938	0.01475	0.33458	0.00534	1.93534	0.03227	6.32928	0.06081
1.28	0.1	2.46716	0.03103	0.44310	0.01420	1.47210	0.04835	15.06512	0.19990
1.36	0.2	2.09281	0.05820	0.41573	0.02482	1.02379	0.06235	14.02343	0.33044
1.44	0.3	1.57172	0.10194	0.34270	0.03577	0.69339	0.07376	10.91976	0.44353
1.52	0.4	1.13984	0.16767	0.26564	0.04562	0.47172	0.08257	8.06005	0.53548
1.60	0.5	0.82848	0.25941	0.19986	0.05318	0.32822	0.08907	5.91202	0.60594
1.68	0.6	0.61568	0.37776	0.14855	0.05769	0.23629	0.09367	4.41284	0.65652
1.76	0.7	0.47346	0.51800	0.11027	0.05888	0.17740	0.09678	3.39860	0.68991
1.84	0.8	0.37971	0.66903	0.08231	0.05694	0.13965	0.09881	2.72451	0.70927
1.92	0.9	0.31923	0.81403	0.06204	0.05241	0.11572	0.10006	2.28664	0.71774
2.00	1.0	0.28231	0.93328	0.04733	0.04602	0.10123	0.10078	2.01696	0.71819

Example 4.2. Consider the linear impulsive (ρ_k,ψ_k)-$\mathbb{HFP}$-$\mathbb{IDE}$-$\mathbb{MIBC}$s:

$$\begin{cases} {}^{H}_{\frac{k+21}{20}}\mathfrak{D}^{\alpha_k,\frac{k+3}{6};\psi_k}_{t_k^+}u(t)=5,\ t\neq t_k,\ k=0,1,2,\\ {}_{\frac{k+21}{20}}\mathscr{I}^{\frac{k+4}{k+5}(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+)-{}_{\frac{k+20}{20}}\mathscr{I}^{\frac{k+3}{k+4}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-)=\phi_k(u(t_k)),\quad k=1,2,\\ {}^{\mathbb{RL}}_{\frac{k+21}{20}}\mathfrak{D}^{\frac{k+4}{k+5}(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+)-{}^{\mathbb{RL}}_{\frac{k+20}{20}}\mathfrak{D}^{\frac{k+3}{k+4}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+}u(t_k^-)=\phi_k^*(u(t_k)),\quad k=1,2,\\ u(0)=0,\quad u\left(\frac{6}{5}\right)=\sum_{i=0}^{2}\left(\frac{3-i}{4i+4}\right){}_{\frac{k+21}{20}}\mathscr{I}^{\frac{i+3}{5-i};\psi_i}_{t_i}u\left(\frac{2i+1}{5}\right)+2. \end{cases} \tag{4.2}$$

Thanks to problem (4.2), we have all parameters that $\alpha_k\in\{(2k+12)/10,(2k+13)/10,(2k+14)/10,(2k+15)/10,(2k+16)/10\}$, $\beta_k=(k+3)/6$, $\rho_k=(k+21)/20$, $\psi_k=((5-k)t^{5-2k}+3-k)/((k+2)t^{5-2k}+6-2k)$, $t_k=(2k)/3$, $k=0,1,2$, $T=2$, $\phi_k(u(t_k))=((-1)^{k+1})(2k+1)/(2k)$, $\phi_k^*(u(t_k))=2$, $k=1,2$, $\mu_i=(3-i)/(4i+4)$, $\kappa_i=(i+3)/(5-i)$, $\xi_i=(i+1)/2$, $i=0,1,2$ and $\mathscr{A}=2$. By using Lemma 2.8 with $f(t,u(t),u(\theta t),{}_{\rho_k}\mathscr{I}^{\sigma_k;\psi_k}_{t_k^+}u(t))=5$, the solution of problem (4.2) with $\alpha_k\in\{(2k+12)/10,\ (2k+13)/10,\ (2k+14)/10,\ (2k+15)/10,\ (2k+16)/10\}$, and $\psi_k(t)=\frac{(5-k)t^{5-2k}+3-k}{(k+2)t^{5-2k}+6-2k}$ for $k=0,1,2$ is shown in Figure 2.

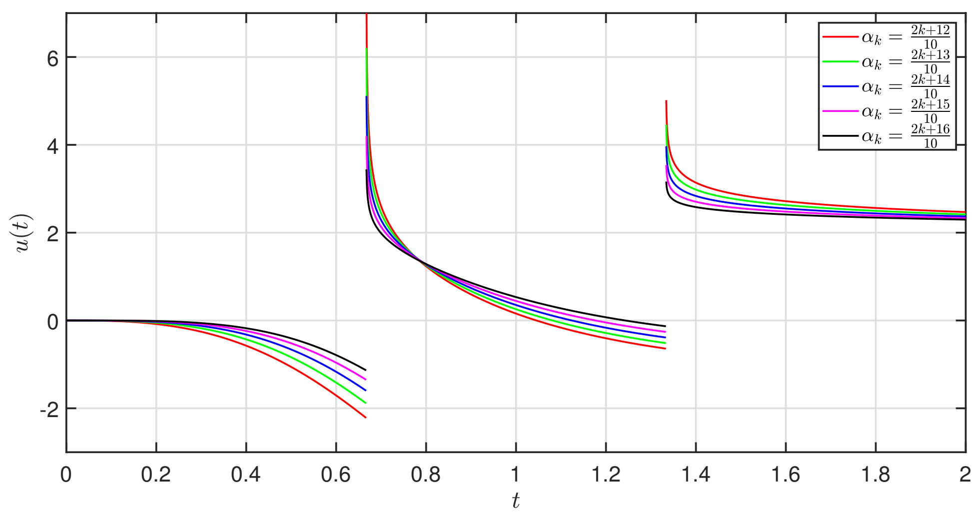

FIGURE 2. The solution of Example (4.2) via $\alpha_k\in\left\{\frac{2k+12}{10},\frac{2k+13}{10},\frac{2k+14}{10},\frac{2k+15}{10},\frac{2k+16}{10}\right\}$ and $\psi_k(t)=\frac{(5-k)t^{5-2k}+3-k}{(k+2)t^{5-2k}+6-2k}$ for $k=0,1,2$.

In addition, the solution $u(t)$ of problem (4.2) with $\alpha_0=1.4$, $\alpha_1=1.6$, $\alpha_2=1.8$, and $\psi_k(t)\in\left\{\frac{(5-k)t^{5-2k}+3-k}{(k+2)t^{5-2k}+6-2k},\ \frac{\ln((k+2)t+2k+2)}{\ln(t+k+3)},\ t^{t^2-\frac{k+1}{2}t+1},\ \tan\left(\frac{t}{k+1}+2k\right),\ \arcsin\left(\frac{t^2+kt-2k-1}{10}\right)\right\}$ for $k=0,1,2$ is shown in Figure 3.

5. CONCLUSION

In this paper, we studied a class of nonlinear impulsive fractional pantograph integro-differential equations in the context of the (ρ_k,ψ_k)-Hilfer fractional derivative under multi-point integral boundary conditions (problem (1.1) problem). The solution to problem (2.4) was obtained in

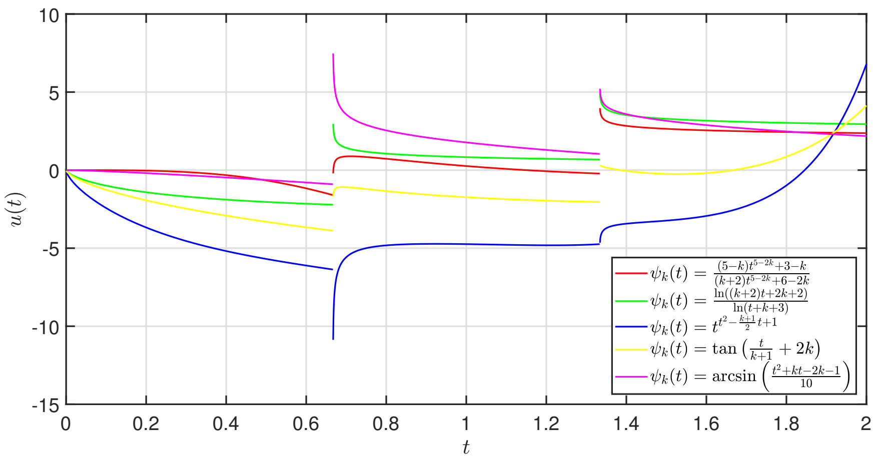

FIGURE 3. The solution of Example (4.2) via $\alpha_0 = 1.4$, $\alpha_1 = 1.6$, $\alpha_2 = 1.8$ and $\psi_k(t) \in \left\{\frac{(5-k)t^{5-2k}+3-k}{(k+2)t^{5-2k}+6-2k}, \frac{\ln((k+2)t+2k+2)}{\ln(t+k+3)}, t^{t^2-\frac{k+1}{2}t+1}, \tan\left(\frac{t}{k+1}+2k\right), \arcsin\left(\frac{t^2+kt-2k-1}{10}\right)\right\}$ for $k = 0,1,2$.

Volterra integral equation form. In the first main result, the existence and uniqueness result was investigated by applying Banach's fixed point theorem, while Ulam's stability, such as $\mathbb{UH}$, $\mathbb{GUH}$, $\mathbb{UHR}$, and $\mathbb{GUHR}$, was proved with nonlinear functional analysis in the second main result. Suitable examples were provided to support the validity of these theoretical results. It is of interest to further discuss the existence and uniqueness of solutions for the other types of nonlinear differential-integral equations in the context of the different fractional operators with various boundary conditions.

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