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THE SMOOTHNESS OF MULTIFRACTAL HEWITT-STROMBERG AND BOX DIMENSIONS

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Abstract. The objective of this paper is to examine the set-theoretic intricacy associated with multifractal Box-dimensions and multifractal Hewitt-Stromberg maps.

Keywords. Hewitt-Stromberg measures and dimensions, Box dimensions, Fractals, Set-theoretic complexity and measurability.

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1. INTRODUCTION

The investigation of dimensions is a crucial aspect when analyzing fractal and multifractal geometry. Numerous mathematicians and physicists proposed various definitions of dimension, including the Hausdorff dimension, packing dimension, and modified lower and upper box dimensions. While the Hausdorff and packing dimensions are defined in terms of measures, the modified lower and upper box dimension is not. Fractal measures were introduced by Hewitt and Stromberg in their classical textbook [15, Exercise 10.51]. These measures play a significant role in analyzing the local properties of fractals and fractal products. Many authors delved into the investigation of these measures; see, e.g., [3, 11, 13, 14, 16, 23, 25, 29] and the references therein. Notably, Edgar's textbook [11, pp. 32-36] provides an important and explicit introduction to these measures and highlights their significance in studying the local properties of fractals. While the measures of Hausdorff and packing dimensions were established using coverings and packings by families of sets with diameters smaller than a positive number r, the Hewitt-Stromberg measures were established through coverings and packings of balls with a fixed diameter r. Attia and Selmi, in [2, 3], developed a new multifractal formalism for the Hewitt-Stromberg measures. This formalism is analogous to Olsen's multifractal formalism

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introduced in [20]. Olsen's multifractal formalism is based on the Hausdorff and packing measures, while Attia and Selmi's formalism relies on the lower and upper Hewitt-Stromberg measures. Over the past five years, numerous investigations were conducted by scholars [4, 5, 7, 26] on these measures and dimensions, highlighting their crucial role in examining local properties of fractals and products of fractals. Moreover, the significance and applications ([10]) of these measures were further supported by additional studies carried out in [6, 8, 9, 25, 27, 28]. The main motivation of this paper stems from the results in Douzi et al. [9], Falconer and Mauldin [12], Matilla and Mauldin [19], and Olsen [20, 22].

This paper aims to investigate the descriptive set theoretic complexity associated with the multifractal box dimension and multifractal Hewitt-Stromberg dimensions

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\overline{C}_\nu^q(K), \tag{1.1}$$

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\underline{C}_\nu^q(K), \tag{1.2}$$

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\overline{L}_\nu^q(K), \tag{1.3}$$

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\underline{L}_\nu^q(K), \tag{1.4}$$

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto b_\nu^q(K), \tag{1.5}$$

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto B_\nu^q(K), \tag{1.6}$$

where $\mathscr{K}(\mathbb{R}^d)$ represents the collection of non-empty compact sets in $\mathbb{R}^d$ equipped with the Hausdorff metric, and $\mathscr{M}(\mathbb{R}^d)$ denotes the family of Borel probability measures on $\mathbb{R}^d$ equipped with the Lévy-Prokhorov metric.

In this study, we demonstrate the measurability of (1.1) and (1.4) with respect to the σ-algebra generated by the Borel sets and determine its Baire class. Specifically, we show that they belong to Baire class 2. Considering (1.2) and (1.3), the analysis is confined to a suitable subspace determined by the family of doubling measures, which are of Baire class 2. On the other hand, under the doubling condition for the Borel probability measure, (1.5) and (1.6) are measurable with respect to the σ-algebra generated by the analytic sets and are, in general, not Borel measurable.

2. Multifractal Hewitt-Stromberg Measures and Dimensions

Let us begin by revisiting the definitions of the Hewitt-Stromberg measures. We define $\mathscr{M}(\mathbb{R}^d)$ as the set of Borel probability measures. For a given measure $\nu\in\mathscr{M}(\mathbb{R}^d)$, we denote $supp\,\nu$ as the topological support of ν. Consider q and s as real numbers, ν as an element of $\mathscr{M}(\mathbb{R}^d)$, and E as a subset of $supp\,\nu$. The lower Hewitt-Stromberg pre-measure is then defined as follows:

$$\overline{\mathscr{U}}_{\nu,0}^{q,s}(E)=\liminf_{r\to0}N_{\nu,r}^q(E)(2r)^s,$$

where

$$N_{\nu,r}^q(E)=\inf\left\{\sum_i\nu(B(x_i,r))^q\;\middle|\;(B(x_i,r))_i\text{ is a family of closed balls with } x_i\in E\text{ and }E\subseteq\bigcup_i B(x_i,r)\right\}.$$

The function $\overline{\mathscr{U}}^{q,s}_{\nu,0}$ is neither increasing nor σ-additive, and it satisfies $\overline{\mathscr{U}}^{q,s}_{\nu,0}(\emptyset) = 0$. In order to obtain an outer measure, a standard modification is required. Therefore, we proceed to adjust the definition as follows

$$\overline{\mathscr{U}}^{q,s}_{\nu}(E) = \sup_{F \subseteq E} \overline{\mathscr{U}}^{q,s}_{\nu,0}(F).$$

We introduce the lower multifractal Hewitt-Stromberg measure by

$$\mathscr{U}^{q,s}_{\nu}(E) = \inf\left\{ \sum_i \overline{\mathscr{U}}^{q,s}_{\nu}(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \text{ and } E_i' \text{ s are bounded } \right\}.$$

Similarly, we denote the upper Hewitt-Stromberg pre-measure as follows:

$$\overline{\mathscr{V}}^{q,s}_{\nu}(E) = \limsup_{r \to 0} M^q_{\nu,r}(E)(2r)^s,$$

where

$$M^q_{\nu,r}(E) = \sup\left\{ \sum_i \nu(B(x_i,r))^q \;\middle|\; (B(x_i,r))_i \text{ is a family of closed balls with } x_i \in E \text{ and } B(x_i,r) \cap B(x_j,r) = \emptyset \text{ for } i \neq j \right\}.$$

$\overline{\mathscr{V}}^{q,s}_{\nu}$ is increasing but not σ-additive. For this, we introduce the upper Hewitt-Stromberg measure by

$$\mathscr{V}^{q,s}_{\nu}(E) = \inf\left\{ \sum_i \overline{\mathscr{V}}^{q,s}_{\nu}(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \text{ and } E_i' \text{ s are bounded } \right\}, \quad \mathscr{V}^{q,s}_{\nu}(\emptyset) = 0.$$

The set-functions $\mathscr{U}^{q,s}_{\nu}$ and $\mathscr{V}^{q,s}_{\nu}$ are outer measures, which implies that they are measures on the algebra generated by Carathéodory-measurable sets. However, while $\mathscr{U}^{q,s}_{\mu}$ is a metric measure, $\mathscr{V}^{q,s}_{\nu}$ does not possess this property. For more detailed information, we refer to [11, 16, 21, 23, 25].

Proposition 2.1. *Let $E \subseteq \mathbb{R}^d$, $q \in \mathbb{R}$ and $\nu \in \mathscr{M}(\mathbb{R}^d)$.*

(1) There exists a unique number $\Theta^q_{\nu}(E) \in [-\infty, +\infty]$ such that

$$\overline{\mathscr{U}}^{q,s}_{\nu}(E) = \begin{cases} +\infty & if \quad s < \Theta^q_{\nu}(E) \\ 0 & if \quad s > \Theta^q_{\nu}(E). \end{cases}$$

(2) There exists a unique number $b^q_{\nu}(E) \in [-\infty, +\infty]$ such that

$$\mathscr{U}^{q,s}_{\nu}(E) = \begin{cases} +\infty & if \quad s < b^q_{\nu}(E) \\ 0 & if \quad s > b^q_{\nu}(E). \end{cases}$$

(3) There exists a unique number $\Delta^q_{\nu}(E) \in [-\infty, +\infty]$ such that

$$\overline{\mathscr{V}}^{q,s}_{\nu}(E) = \begin{cases} +\infty & if \quad s < \Delta^q_{\nu}(E) \\ 0 & if \quad s > \Delta^q_{\nu}(E). \end{cases}$$

(4) There exists a unique number $B^q_{\nu}(E) \in [-\infty, +\infty]$ such that

$$\mathscr{V}^{q,s}_{\nu}(E) = \begin{cases} +\infty & if \quad s < B^q_{\nu}(E) \\ 0 & if \quad s > B^q_{\nu}(E). \end{cases}$$

The quantity $b^q_\nu(E)$ serves as a generalized analogue of the lower Hewitt-Stromberg dimension $\underline{dim}_{MB}(E)$ for any set E. Similarly, the value $B^q_\nu(E)$ serves as a generalized analogue of the upper Hewitt-Stromberg dimension $\overline{dim}_{MB}(E)$ for any set E. In addition, $\Theta^q_\nu(E)$ and $\Delta^q_\nu(E)$ naturally represent multifractal analogues of the lower box dimension $\underline{dim}_B(E)$ and upper box dimension $\overline{dim}_{MB}(E)$ of a set E. In particular,

$$b^0_\nu(E) = \underline{dim}_{MB}(E) \qquad B^0_\nu(E) = \overline{dim}_{MB}(E)$$

and

$$\Theta^0_\nu(E) = \underline{dim}_B(E) \qquad \Delta^0_\nu(E) = \overline{dim}_B(E).$$

Let us now revisit the definitions of multifractal box dimensions for a set $E \subset \mathbb{R}^d$, as introduced by Olsen in [20] (see also [1]). The multifractal lower box dimensions are defined as follows

$$\overline{L}^q_\nu(E) = \limsup_{r\to 0} \frac{\log N^q_{\nu,r}(E)}{-\log r} \text{ and } \underline{L}^q_\nu(E) = \liminf_{r\to 0} \frac{\log N^q_{\nu,r}(E)}{-\log r}.$$

Similarly, the upper multifractal box dimensions are defined as follows

$$\overline{C}^q_\nu(E) = \limsup_{r\to 0} \frac{\log M^q_{\nu,r}(E)}{-\log r} \text{ and } \underline{C}^q_\nu(E) = \liminf_{r\to 0} \frac{\log M^q_{\nu,r}(E)}{-\log r}.$$

It follows from [3, 25], for all $E \subseteq supp\,\nu$, that $\Theta^q_\nu(E) = \underline{L}^q_\nu(E)$ and $\Delta^q_\nu(E) = \overline{C}^q_\nu(E)$. Let ν be a Borel probability measure on $\mathbb{R}^d$ and $a > 1$. Note that

$$T_a(\nu) = \limsup_{r\to 0} \left(\sup_{x\in supp\,\nu} \frac{\nu(B(x,ar))}{\nu(B(x,r))} \right).$$

We say that a measure ν satisfies the doubling condition if there exists a constant $a > 1$ such that $T_a(\nu) < \infty$. The specific numerical value of the parameter a is not significant, meaning that

$$T_a(\nu) < \infty \text{ for some } a > 1 \text{ if and only if } T_a(\nu) < \infty, \text{ for all } a > 1.$$

Next, we denote

$$M_D(\mathbb{R}^d) = \Big\{ \nu \in \mathscr{M}(\mathbb{R}^d) \ \mid T_a(\nu) < \infty \text{ for some } a > 1 \Big\}$$

The family of Borel probability measures satisfies the doubling condition. For further details, please refer to [1, 20, 24].

3. Statement of Main Results

Throughout the paper, we write $\mathscr{K}(\mathbb{R}^d)$ the set of all compact subsets of $\mathbb{R}^d$, $\mathscr{M}(\mathbb{R}^d)$ the set of all compact supported Borel probability measures on $\mathbb{R}^d$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and

$$\Omega = \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}_D(\mathbb{R}^d) \times [0, +\infty[\cup \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times]-\infty, 0[.$$

The main results of this paper are presented as follows: The initial finding reveals that both the upper and lower multifractal box dimension maps belong to Baire class 2, and they are not classified under Baire class 1.

Theorem 3.1.

(1) The map $\mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \longrightarrow \overline{\mathbb{R}} : (K, \nu, q) \longmapsto \overline{C}^q_\nu(K)$ is of Baire class 2, but it is not of Baire class 1.

(2) *The map* $\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\underline{L}_\nu^q(K)$ *is of Baire class 2, but it is not of Baire class 1.*

(3) *The map* $\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\underline{C}_\nu^q(K)$ *is of Baire class 2, but it is not of Baire class 1.*

(4) *The map* $\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\overline{L}_\nu^q(K)$ *is of Baire class 2, but it is not of Baire class 1.*

Through the utilization of Theorem 3.1 and certain characterizations of the multifractal Hewitt-Stromberg dimensions, we establish that these dimension functions are measurable with respect to $\mathscr{B}(\mathscr{A}(\Omega))$, where $\mathscr{B}(\mathscr{A}(\Omega))$ represents the σ-algebra generated by the collection of analytic subsets of Ω.

Theorem 3.2.

(1) *The map* $\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto b_\nu^q(K)$ *is* $\mathscr{B}(\mathscr{A}(\Omega))$*-measurable.*

(2) *The map* $\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto B_\nu^q(K)$ *is* $\mathscr{B}(\mathscr{A}(\Omega))$*-measurable.*

Theorem 3.3. *The maps* b_ν^q, B_ν^q, $\mathscr{U}_\nu^{q,s}$, *and* $\mathscr{V}_\nu^{q,s}:\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}$ *are, in general, not Borel measurable.*

The difficulty that we encounter is in ascertaining the complexity of the multifractal Hewitt-Stromberg measures maps

$$\mathscr{K}\left(\mathbb{R}^d\right)\times\mathscr{M}\left(\mathbb{R}^d\right)\times\mathbb{R}\to\overline{\mathbb{R}}:(K,\mu,q)\to\mathscr{U}_\nu^{q,s}(K)$$

and

$$\mathscr{K}\left(\mathbb{R}^d\right)\times\mathscr{M}\left(\mathbb{R}^d\right)\times\mathbb{R}\to\overline{\mathbb{R}}:(K,\mu,q)\to\mathscr{V}_\nu^{q,s}(K).$$

Note that Mattila and Mauldin [19] established that, in the case where X is a Polish space and g is a dimension function that fulfills the doubling condition (where there exists a positive constant c such that $g(2t)\le cg(t)$ for all $t\ge 0$), the mapping

$$\mathscr{K}(X)\to\overline{\mathbb{R}}:K\to\mathscr{P}^g(K)$$

is $\mathscr{B}(\mathscr{A}(\mathscr{K}(X)))$-measurable where $\mathscr{B}(\mathscr{A}(\mathscr{K}(X)))$ denotes the σ-algebra generated by the family $\mathscr{A}(\mathscr{K}(X))$ of analytic subsets of $\mathscr{K}(X)$. Furthermore, Mattila and Mauldin presented an example illustrating that the packing measure map may not be Borel measurable. However, it is worth noting that the concepts discussed in [19] are not directly applicable to the multifractal scenario. To demonstrate the $\mathscr{B}(\mathscr{A}(\mathscr{K}(X)))$-measurability of this map, the authors in [19] employed the property that if the dimension function g satisfies the doubling condition, then the packing measure $\mathscr{P}^g$ possesses the "subset of positive and finite measure" property. This property asserts that, for any analytic subset A of X with $\mathscr{P}^g(A)=+\infty$, there exists a compact subset A' of A with $0<\mathscr{P}^g(A')<+\infty$. It is evident that the multifractal Hewitt-Stromberg measures $\mathscr{U}_\nu^{q,s}$ and $\mathscr{V}_\nu^{q,s}$ (given that these measures are derived using the conventional Method I Construction) lack the general property of having subsets with positive and finite measures. To explore this, we consider the case where the measure ν is the Lebesgue measure $\mathscr{L}^1$ on the unit interval I and select $q,s\in\mathbb{R}$ such that $q+s<0$. Now, given any $x\in I$ and for any $\delta>0$, we have

$$\mathscr{U}_\nu^{q,s}(\{x\})=\mathscr{V}_\nu^{q,s}(\{x\})=\lim_{\delta\to 0}(2\delta)^{q+s}=+\infty.$$

Hence, for any nonempty closed set $E\subseteq I$, every subset of E, including the empty set, has an infinite measure. This demonstrates that there exist measures $\mathscr{U}_\nu^{q,s}$ and $\mathscr{V}_\nu^{q,s}$ for which the "subset of finite and positive measure property" may not hold for all closed sets with infinite measures.

Consequently, the approach employed in [19] is not directly applicable to the multifractal scenario. Nevertheless, we hold the belief that the multifractal Hewitt-Stromberg measures maps exhibit analytic measurability. Based on this observation, we put forward the following conjectures:

(a) The maps

$$\mathscr{K}\left(\mathbb{R}^d\right) \times \mathscr{M}\left(\mathbb{R}^d\right) \times \mathbb{R} \to \overline{\mathbb{R}} : (K,\mu,q) \to \mathscr{U}_\nu^{q,s}(K)$$

and

$$\mathscr{K}\left(\mathbb{R}^d\right) \times \mathscr{M}\left(\mathbb{R}^d\right) \times \mathbb{R} \to \overline{\mathbb{R}} : (K,\mu,q) \to \mathscr{V}_\nu^{q,s}(K).$$

are measurable with respect to the σ-algebra generated by the analytic subsets of $\mathscr{K}\left(\mathbb{R}^d\right) \times \mathscr{M}\left(\mathbb{R}^d\right) \times \mathbb{R}$.

(b) The measures $\mathscr{U}_\nu^{q,s}$ and $\mathscr{V}_\nu^{q,s}$ have the "subset of finite and positive measure property" for some conditions.

4. Proof of the Main Results

We provide an overview of the tools, intermediate results, and notations that are utilized in the proof of our main results. Firstly, we revisit the fundamental concepts and key findings in the field of topological theory.

4.1. **Topological notations.**

4.1.1. *Polish spaces and analytic sets.* Let $(X,\mathscr{T})$ denote a topological space, which can be considered a Polish space if it satisfies both separability and topological completeness. This means that there exists a metric d on X that induces the topology $\mathscr{T}$. Specifically, if (X,d) is both separable and complete, it is classified as a Polish space. Now, let $(X,\mathscr{T})$ be a Polish space and consider a subset $E \subseteq X$. An analytic set refers to a set E for which there exists a Polish space Y and a continuous function $f : Y \longrightarrow X$ such that $f(Y) = E$. The complement of an analytic set is referred to as a co-analytic set. One method used to characterize analytic sets involves two Polish spaces, X and Y, and considers E as a subset of X. In this approach, E is considered an analytic set if it can be expressed as $E = \pi(F)$, where F is a Borel set in $X \times Y$ and π represents the projection onto the first factor. For further details on this topic, we refer to [17, Chapter 14].

4.1.2. *Hausdorff distance.* Consider a metric space (X,d), where $\mathscr{K}(X)$ represents the collection of all non-empty compact subsets of X. For any two non-empty compact sets K and L in X, the Hausdorff metric induced by d is defined as follows

$$D_H(K,L) = \inf\left\{ r > 0 \;\middle|\; K \subset L_r \text{ and } L \subset K_r \right\},$$

where $K_r = \left\{ x \in X \;\middle|\; d(x,K) < r \right\}$ and D_H is a distance on $\mathscr{K}(X)$. Moreover, we can equip $\mathscr{K}(X) \cup \{\emptyset\}$ with

$$D_{H_\emptyset}(K,L) = \begin{cases} 0 & if \quad K = \emptyset \text{ and } L = \emptyset, \\ 1 & if \quad \text{one of the sets equal to } \emptyset, \\ D_H(K,L) & if \quad K \neq \emptyset \text{ and } L \neq \emptyset. \end{cases}$$

Consequently, if (X,d) is a Polish space, then $(\mathscr{K}(X),D_H)$ is also a Polish space, as detailed in [17, Theorems 4.22 and 4.25].

4.1.3. *Lévy–Prokhorov metric.* Consider a metric space (X,d), and let μ and ν be two Borel probability measures on X. The Lévy-Prokhorov distance, denoted by P and induced by the metric d, between μ and ν, is defined as follows

$$D_{LP}(\mu,\nu)=\inf\Big\{\alpha>0 \mid \mu(E)\leqslant \nu(E_\alpha)+\alpha \text{ and } \nu(E)\leqslant \mu(E_\alpha)+\alpha,\ E \text{ a Borel set of } X\Big\},$$

where $E_\alpha=\{x \mid d(x,E)<\alpha\}$ and $d(x,E)=\inf\{d(x,y)|\ y\in E\}$. In [18], the metric D_{LP} is introduced as a metric on the set of Borel probability measures on X. If X is a separable metric space, the convergence in metric D_{LP} is equivalent to weak convergence in $\mathscr{M}(X)$. In other words, if X is a separable metric space and $(\nu_n)_n$ is a sequence of Borel probability measures converging to ν, we have the following relationship

$$D_{LP}(\nu_n,\nu)\to 0 \quad \text{if and only if} \quad \nu_n \overset{\text{weakly}}{\longrightarrow} \nu.$$

Furthermore, if (X,d) is a Polish space, then $(\mathscr{M}(X),D_{LP})$ is also a Polish space. It is worth recalling that a countable product of Polish spaces remains Polish, which implies that $\mathscr{K}(X)\times\mathscr{M}(X)\times X$ is a Polish space. Specifically, in the case of $\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}$, it is also a Polish space.

4.1.4. *Borel Hierarchy and Baire class.* Consider a metrizable space (X,d). The Borel hierarchy is a classification method used to measure the level of smoothness of functions defined on X. This hierarchy is defined recursively for ordinals θ satisfying $1\leqslant\theta<\omega_1$, where ω_1 denotes the first uncountable ordinal.

$$\Sigma_1^0(X)=\Big\{U\subset X \,\Big|\, U \text{ is open}\Big\}$$

and

$$\Pi_1^0(X)=\Big\{F\subset X \,\Big|\, F \text{ is closed}\Big\}.$$

Next,

$$\Sigma_\theta^0(X)=\left\{\bigcup_{i=1}^{+\infty}E_i \,\middle|\, E_i\in\bigcup_{\kappa<\theta}\Pi_\kappa^0(X)\right\}$$

and

$$\Pi_\theta^0(X)=\left\{\bigcap_{i=1}^{+\infty}E_i \,\middle|\, E_i\in\bigcup_{\kappa<\theta}\Sigma_\kappa^0(X)\right\}.$$

Also, we have

$$\bigcup_{\theta<\omega_1}\Sigma_\lambda^0(X)=\bigcup_{\theta<\omega_1}\Pi_\lambda^0(X)=\mathscr{B}(X),$$

where $\mathscr{B}(X)$ is a σ-algebra of X. Also, we note that $\mathscr{F}(X)$ is a set of all closed subsets of X and $\mathscr{G}(X)$ is a set of all open subsets of X. We have

$$\begin{array}{ll}\Sigma_1^0(X)=\mathscr{G}(X), & \Pi_1^0(X)=\mathscr{F}(X),\\ \Sigma_2^0(X)=\mathscr{F}_\delta(X), & \Pi_2^0(X)=\mathscr{G}_\sigma(X),\\ \Sigma_3^0(X)=\mathscr{G}_{\delta\sigma}(X), & \Pi_3^0(X)=\mathscr{F}_{\sigma\delta}(X),\\ \vdots & \vdots\end{array}$$

Let X and Y be two metric spaces, and let θ be a countable ordinal number. A function $f : X \to Y$ is said to be of Baire class θ if it can be expressed as the pointwise limit of a sequence of functions that are of Baire class less than θ. Additionally, f is of Baire class θ if it is $\Sigma^0_{\theta+1}(X)$-measurable. In the following, $\mathscr{B}(X)$ denotes the σ-algebra of Borel sets in X, and $\mathscr{A}(X)$ denotes the σ-algebra of analytic sets in X.

4.2. **Proof of Theorem 3.1.** In this section, we introduce the tools and intermediate results that are utilized in the proof of our main results.

Proposition 4.1. *Let $E \subset \mathbb{R}^d$, $q > 0$ and $\nu \in \mathscr{M}_D(\mathbb{R}^d)$.*

(1) If $q > 0$ and $\nu \in \mathscr{M}_D(\mathbb{R}^d)$, then there exist two positive constants $C_1, C_2 > 0$ such that
$$N^q_{\nu,r}(E) \leqslant C_1 M^q_{\nu,r}(E) \leqslant C_2 N^q_{\nu,\frac{r}{2}}(E).$$
(2) If $q \leqslant 0$, then there exist two positive constants $C_3, C_4 > 0$ such that
$$N^q_{\nu,r}(E) \leqslant C_3 M^q_{\nu,r}(E) \leqslant C_4 N^q_{\nu,\frac{r}{2}}(E).$$

Proof. Let ζ be an integer that appears in Besicovitch covering theorem [11]. Let $r > 0$ and $\mathscr{B} = \{B(x,r) \,\big|\, x \in E\}$. There exist ζ countable or finite families $\{B(x_{ij}, r)\}_j$ with $i \in \{1, \ldots, \zeta\}$ of $\mathscr{B}$ such that $\{B(x_{ij}, r)\}_{ij}$ is a cover of E and $\{B(x_{ij}, r)\}_j$ is a packing of E for all $i \in \{1, \ldots, \zeta\}$. Hence,
$$N^q_{\nu,r}(E) \leqslant \sum_i \sum_j \nu\Big(B(x_{ij}, r)\Big)^q \leqslant \sum_{i=1}^{\zeta} M^q_{\nu,r}(E) = \zeta M^q_{\nu,r}(E).$$
If $\nu \in \mathscr{M}_D(\mathbb{R}^d)$, then there exist $c > 0$ and $R > 0$ such that
$$c^{-1}\, \nu\Big(B(x,r)\Big) \leqslant \nu\Big(B(x,3r)\Big) \leqslant c\, \nu\Big(B(x,r)\Big), \text{ for all } x \in supp\, \nu \text{ and } 0 < r \leqslant R.$$
We can choose now $0 < r < R$, and we let $\{B(x_i, \frac{r}{2})\}$ be a centered covering of E and $\{B(y_i, r)\}$ be a packing of E. For all $i \in \mathbb{N}$, we choose i_k such that $y_i \in B(x_{i_k}, \frac{r}{2})$. It is easily seen that
$$B(x_{i_k}, \frac{r}{2}) \subset B(y_i, r) \subset B(x_{i_k}, \frac{3r}{2}).$$
We can see also if $i \neq j$ then $i_k \neq j_k$.

(1) If $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ and $q \geqslant 0$, then
$$\begin{aligned}\sum_i \nu\Big(B(y_i, r)\Big)^q &= \sum_i \left(\frac{\nu\Big(B(y_i, r)\Big)}{\nu\Big(B(x_{i_k}, \frac{r}{2})\Big)}\right)^q \nu\Big(B(x_{i_k}, \frac{r}{2})\Big)^q \\ &\leqslant \sum_i \left(\frac{\nu\Big(B(x_{i_k}, \frac{3r}{2})\Big)}{\nu\Big(B(x_{i_k}, \frac{r}{2})\Big)}\right)^q \nu\Big(B(x_{i_k}, \frac{r}{2})\Big)^q \\ &\leqslant c^q \sum_i \nu\left(B\left(x_{i_k}, \frac{r}{2}\right)\right)^q \leqslant c^q \sum_i \nu\left(B\left(x_i, \frac{r}{2}\right)\right)^q,\end{aligned}$$
which implies that $M^q_{\nu,r}(E) \leqslant c^q N^q_{\nu,\frac{r}{2}}(E)$. Finally, we can conclude that
$$N^q_{\nu,r}(E) \leqslant C_1 M^q_{\nu,r}(E) \leqslant C_2 N^q_{\nu,\frac{r}{2}}(E),$$

where $C_1 = \zeta$ and $C_2 = \zeta c^q$.

(2) If $q \leqslant 0$, then $\sum_i \nu(B(y_i,r))^q \leqslant \sum_i \nu(B(x_{i_k},\frac{r}{2}))^q \leqslant \sum_i \nu(B(x,\frac{r}{2}))^q$. Consequently, $M^q_{\nu,r}(E) \leq N^q_{\nu,\frac{r}{2}}(E)$. We write $C_3 = C_4 = \zeta$.

□

As a consequence of Proposition 4.1, we have the following result.

Corollary 4.2. *Let $E \subset \mathbb{R}^d$.*

(1) $\overline{L}^q_\nu(E) = \overline{C}^q_\nu(E)$ for $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ and $q > 0$.
(2) $\overline{L}^q_\nu(E) = \overline{C}^q_\nu(E)$ for $q \leqslant 0$.
(3) $\underline{L}^q_\nu(E) = \underline{C}^q_\nu(E)$ for $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ and $q > 0$.
(4) $\underline{L}^q_\nu(E) = \underline{C}^q_\nu(E)$ for $q \leqslant 0$.

We now examine the semi-continuity of functions $N^q_{\nu,r}(.)$ and $M^q_{\nu,r}(.)$.

Proposition 4.3. *Let $r > 0$.*

(1) The map $\mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \longrightarrow \overline{\mathbb{R}} : (K,\nu,q) \longmapsto N^q_{\nu,r}(K)$ is upper semi-continuous.
(2) The map $\mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \longrightarrow \overline{\mathbb{R}} : (K,\nu,q) \longmapsto M^q_{\nu,r}(K)$ is lower semi-continuous.

Proof. (1) It is sufficient to prove that, for all $c \in \mathbb{R}$ and $r > 0$,

$$A = \left\{ (K,\nu,q) \in \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \;\middle|\; N^q_{\nu,r}(K) < c \right\},$$

is open. Let

$$B = \Bigg\{ (K,\nu,q) \in \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \;\Bigg|\; \text{there exist } m \in \mathbb{N},$$
$$x_1,\dots,x_m \in K \text{ and } c_1,\dots,c_m > 0 \;\text{ with } \sum_{i=1}^m c_i < c \text{ such that}$$
$$i) K \subseteq \bigcup_{i=1}^m B(x_i,r)$$
$$ii) \nu(B(x_i,r))^q < c_i, \forall 1 \leqslant i \leqslant m \Bigg\}.$$

Claim 4.4. We prove that $A = B$. It follows from $(K,\nu,q) \in B$ that

$$\sum_{i=1}^m \nu\Big(B(x_i,r)\Big)^q < \sum_{i=1}^m c_i < c,$$

which implies that $N^q_{\nu,r}(K) < c$. Thus $B \subseteq A$. Conversely, if $(K,\nu,q) \in A$, we consider a centered covering $(B(x_i,r))_i$ of K. As K is compact, there exists a natural number m such that $K \subseteq \bigcup_{i=1}^n B(x_i,r)$ and $\sum_{i=1}^m \nu(B(x_i,r))^q < c$.

Now, we can choose $\varepsilon > 0$ such that $\sum_{i=1}^m \nu\Big(B(x_i,r)\Big)^q + \varepsilon < c$, which implies that

$$N^q_{\nu,r}(K) < \sum_{i=1}^m \nu\Big(B(x_i,r)\Big)^q + \varepsilon = \sum_{i=1}^n \left(\nu\Big(B(x_i,r)\Big)^q + \frac{\varepsilon}{m} \right) < c.$$

Then , we can put $c_i = \nu(B(x_i,r))^q + \frac{\varepsilon}{m}$.

Let $F = \mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\backslash B$. Let (K_n,ν_n,q_n) be a sequence in K and $(K,\nu,q)\in \mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}$ such that $(K_n,\nu_n,q_n)_n \underset{n\to+\infty}{\longrightarrow} (K,\nu,q)$. We prove that $(K,\nu,q)\in F$. Fixing $m\in\mathbb{N}$, $x_1,\dots,x_m\in K$ and $c_1,\dots,c_m>0$ with $\sum_{i=1}^m c_i < c$, we will prove that

$$K \not\subseteq \bigcup_{i=1}^m B(x_i,r) \tag{4.1}$$

or

$$\nu\Big(B(x_i,r)\Big)^q \geqslant c_i, \text{ for some } 1\leqslant i\leqslant m. \tag{4.2}$$

It is clear that if (4.1) is satisfied, then we are done. If it is not verified, we assume that

$$K \subseteq \bigcup_{i=1}^m B(x_i,r), \tag{4.3}$$

and we prove (4.2). Since $D_H(K_n,K) \underset{n\to+\infty}{\longrightarrow} 0$, we see by (4.3) that there exists $N\in\mathbb{N}$ such that

$$\begin{cases} K_n \subseteq \bigcup\limits_{i=1}^m B(x_i,r), \text{ for all } n\geqslant N \text{ and} \\ K_n\cap B(x_i,\frac{r}{8}) \neq \emptyset, \text{ for all } i\in\{1,\dots,m\}. \end{cases} \tag{4.4}$$

From (4.4), we fix $n\geqslant N$ and choose $y_i^n\in K_n\cap B(x_i,\frac{r}{8})$. It is obvious that

$$B(x_i,r) \subset U\Big(y_i^n,\frac{5r}{4}\Big) \subset B\Big(x_i,\frac{3r}{2}\Big). \tag{4.5}$$

In particular, we have

$$\Big(B\Big(y_i^n,\frac{5r}{4}\Big)\Big)_{i=1}^m \text{ is a centered covering of } K_n.$$

It follows from $(K_n,\nu_n,q_n)\in F$ that

$$\nu_n\Big(B\Big(y_{i(n)}^n,\frac{5r}{4}\Big)\Big)^{q_n} \geqslant c_{i(n)}, \text{ for some } i(n)\in\{1,\dots,m\}. \tag{4.6}$$

Next, we select an index i from the set $1,\dots,m$ such that there exists a strictly increasing sequence of positive numbers $(n_k)_k$, where $i(n_k)=i$ for all k. Moreover, we have $D_{LP}(\nu_n,\nu) \underset{n\to+\infty}{\longrightarrow} 0$. Then, (4.5) implies that

$$\begin{aligned} \nu\Big(B\Big(x_i,\frac{3r}{2}\Big)\Big) &\geqslant \limsup_k \nu_{n_k}\Big(B\Big(x_i,\frac{3r}{2}\Big)\Big) \\ &\geqslant \limsup_k \nu_{n_k}\Big(B\Big(y_i^{n_k},\frac{5r}{4}\Big)\Big) \\ &= \limsup_k \nu_{n_k}\Big(B\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \nu\Big(B\Big(x_i,r\Big)\Big) &\leqslant \nu\Big(U\Big(y_i^n,\frac{5r}{4}\Big)\Big) \\ &\leqslant \liminf_k \nu_{n_k}\Big(U\Big(y_i^{n_k},\frac{5r}{4}\Big)\Big) \\ &= \liminf_k \nu_{n_k}\Big(U\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big) \\ &\leqslant \liminf_k \nu_{n_k}\Big(B\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big). \end{aligned} \tag{4.8}$$

For $k \in \mathbb{N}$, we write $\lambda_k = \nu_{n_k}\Big(B\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big)$. Moreover, it can be readily observed that

$$\lambda_k^{q-q_{n_k}} \underset{k\to+\infty}{\longrightarrow} 1.$$

• If $q < 0$, it follows from (4.6) and (4.8) that

$$\begin{aligned} \nu(B(x_i,r))^q &\geqslant \limsup_k \lambda_k^{q-q_{n_k}} \lambda_k^{q_{n_k}} \\ &\geqslant \limsup_k \nu_{n_k}\Big(B\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big)^{q_{n_k}} \\ &\geqslant \limsup_k c_{i(n_k)} = c_i. \end{aligned}$$

• If $q \geqslant 0$, by using (4.6) and (4.7), we have

$$\begin{aligned} \nu\Big(B\Big(x_i,\frac{3r}{2}\Big)\Big)^q &\geqslant \limsup_k \lambda_k^{q-q_{n_k}} \nu\Big(B\Big(x_i,\frac{3r}{2}\Big)\Big)^{q_{n_k}} \\ &\geqslant \limsup_k \nu_{n_k}\Big(B\Big(y_{i(n_k)}^{n_k},\frac{5r}{4}\Big)\Big)^{q_{n_k}} \\ &\geqslant \limsup_k c_{i(n_k)} = c_i. \end{aligned}$$

Therefore, we conclude that $(K,\nu,q) \in F$, which implies that B is an open set.

(2) We prove that

$$U = \Big\{(K,\nu,q) \in \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R} \mid M_{\nu,r}^q(K) > c\Big\}$$

is open. Let

$$V = \left\{ (K,\nu,q) \in \mathscr{K}(\mathbb{R}^d) \times \mathscr{N}(\mathbb{R}^d) \times \mathbb{R} \;\middle|\; \begin{array}{l} \text{There exist } m \in \mathbb{N}, \\ x_1,\dots,x_m \in K \text{ and } c_1,\dots,c_m > 0 \\ \text{with } \sum_{i=1}^m c_i > c \text{ such that} \\ i)(B(x_i,r))_i \text{ is a packing of } K \\ ii)\nu(B(x_i,r))^q > c_i, \text{ for all } 1 \leqslant i \leqslant m \end{array} \right\}.$$

Claim 4.5. We show now that $U = V$. For this, we put $(K, \nu, q) \in V$. Then

$$c < \sum_{i=1}^{m} c_i < \sum_{i=1}^{m} \nu(B(x_i, r))^q,$$

which implies that $M^q_{\nu,r}(K) > c$. On the other hand, letting $(K, \nu, q) \in U$, we see that there exists $(B(x_i, r))_{i=1}^m$ a packing of K such that $c < \sum_{i=1}^m \nu(B(x_i, r))^q$. We choose $\varepsilon > 0$ with $c < \sum_{i=1}^m \nu(B(x_i, r))^q - \varepsilon < M^q_{\nu,r}(K)$. We can write $c_i = \nu(B(x_i, r))^q - \frac{\varepsilon}{m}$.

Now, we let W be the complement of V, (K_n, ν_n, q_n) be a sequence of W ,and $(K, \nu, q) \in \mathscr{K}(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d) \times \mathbb{R}$ be such that

$$(K_n, \nu_n, q_n)_n \underset{n \to +\infty}{\longrightarrow} (K, \nu, q).$$

Now, we prove that $(K, \nu, q) \in W$. Fix $m \in \mathbb{N}$, $x_1, \dots, x_m \in K$ $c_1, \dots, c_m > 0$ with $\sum_{i=1}^m c_i > c$. Our aim is to demonstrate that

$$(B(x_i, r))_{i=1}^m \text{ is not a packing of } K \tag{4.9}$$

or

$$\nu(B(x_i, r))^q \leqslant c_i, \text{ for some } 1 \leqslant i \leqslant m. \tag{4.10}$$

If condition (4.9) is satisfied, then the task is considered successfully completed. However, we assume that this condition is not satisfied, and we now prove the validity of (4.10). Since $D_H(K_n, K) \underset{n \to +\infty}{\longrightarrow} 0$, then we can choose $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\begin{cases} \Big(B(x_i, r)\Big)_{i=1}^m \text{ is a packing of } K_n \text{ and} \\ K_n \cap B(x_i, \frac{r}{8}) \neq \emptyset, \text{ for all } i \in \{1, \dots, m\}. \end{cases} \tag{4.11}$$

Next, by using (4.11), we can choose $y_i^n \in K_n \cap B(x_i, \frac{r}{8})$, for all $1 \leqslant i \leqslant m$. It is clear that

$$B(x_i, \frac{r}{2}) \subseteq U(x_i, \frac{3r}{4}) \subseteq B(y_i^n, \frac{7r}{8}) \subseteq B(x_i, r). \tag{4.12}$$

In particular, we have

$$\Big(B(y_i^n, \frac{7r}{8})\Big)_{i=1}^m \text{ is a packing of } K_n.$$

It follows from $(K_n, \nu_n, q_n) \in W$ that

$$\nu\Big(B(y_{i(n)}^n, \frac{7r}{8})\Big)^q \leqslant c_{i(n)}, \text{ for some } 1 \leqslant i(n_k) \leqslant m. \tag{4.13}$$

Consider an index i from the set $1, \dots, m$ such that there exists a strictly increasing sequence of positive integers $(n_k)_k$ satisfying $i(n_k) = i$ for all k. Furthermore, it can be observed that

$$D_{LP}(\nu_n, \nu) \underset{n \to +\infty}{\longrightarrow} 0.$$

It follows from (4.12) that

$$\begin{aligned}\nu\Big(B(x_i,\frac{r}{2})\Big)\leqslant \nu\Big(U(x_i,\frac{3r}{4})\Big) &\leqslant \liminf_k \nu_{n_k}\Big(U(x_i,\frac{3r}{4})\Big)\\ &\leqslant \liminf_k \nu_{n_k}\Big(B(y_i^{n_k},\frac{7r}{8})\Big)\\ &= \liminf_k \nu_{n_k}\Big(B(y_{i(n_k)}^{n_k},\frac{7r}{8})\Big) \qquad (4.14)\end{aligned}$$

and

$$\begin{aligned}\nu(B(x_i,r)) &\geqslant \limsup_k \nu_{n_k}(B(x_i,r))\\ &\geqslant \limsup_k \nu_{n_k}(B(y_i^{n_k},\frac{7r}{8}))\\ &\geqslant \limsup_k \nu_{n_k}(B(y_{i(n_k)}^{n_k},\frac{7r}{8})). \qquad (4.15)\end{aligned}$$

Now, for $k\in\mathbb{N}$, we write $\lambda_k=\nu_{n_k}\left(B\left(y_{i(n_k)}^{n_k},\frac{7r}{8}\right)\right)$. Moreover, $\lambda_k^{q-q_{n_k}}\underset{n\to+\infty}{\longrightarrow}1$. Then, we have

• If $q<0$, it follows from (4.13) and (4.15) that

$$\begin{aligned}\nu(B(x_i,r))^q &\leqslant \liminf_k \lambda_k^{q-q_{n_k}}\lambda_k^{q_{n_k}}\\ &\leqslant \liminf_k \nu_{n_k}\left(B\left(y_{i(n_k)}^{n_k},\frac{7r}{8}\right)\right)^{q_{n_k}}\\ &\leqslant \liminf_k c_{i(n_k)}=c_i.\end{aligned}$$

• If $q\geqslant 0$, from (4.13) and (4.14), we have

$$\begin{aligned}\nu\left(B(x_i,\frac{r}{2}\right) &\leqslant \liminf_k \lambda_k^{q-q_{n_k}}\lambda_k^{q_n}\\ &\leqslant \liminf_k \nu_{n_k}\left(B\left(y_{i(n_k)}^{n_k},\frac{7r}{8}\right)\right)^{q_{n_k}}\\ &\leqslant \liminf_k c_{i(n_k)}=c_i.\end{aligned}$$

In conclusion, it becomes evident that $(K,\nu,q)\in W$, which leads us to the deduction that V is an open set. □

Proof of Theorem 3.1.

(1) It follows from Proposition 4.3 that

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\sup_n\frac{\log M^q_{\nu,2^{-n}}(K)}{n\log 2}$$

is lower semi-continuous. In particular, it is of Baire class 1. Since

$$\overline{C}^q_\nu(K)=\lim_{n\to+\infty}\left(\sup_{k\geqslant n}\frac{\log M^q_{\nu,2^{-k}}(K)}{k\log 2}\right),$$

then

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\overline{C}^q_\nu(K)$$

is of Baire class 2.

(2) By using Proposition 4.3, we have that

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\inf_n\frac{\log N^q_{\nu,2^{-n}}(K)}{n\log 2}$$

is upper semi-continuous. In particular, it is of Baire class 1. However,

$$\underline{L}^q_\nu(K)=\lim_{n\to+\infty}\left(\inf_{k\geqslant n}\frac{\log N^q_{\nu,2^{-k}}(K)}{k\log 2}\right),$$

which implies that

$$\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\longrightarrow\mathbb{R}:(K,\nu,q)\longmapsto\underline{L}^q_\nu(K)$$

is of Baire class 2.

(3) and (4): If $(K,\nu,q)\in\Omega$, then it follows from from Corollary 4.2 that the functions

$$\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\overline{L}^q_\nu(K)$$

and

$$\Omega\longrightarrow\overline{\mathbb{R}}:(K,\nu,q)\longmapsto\underline{C}^q_\nu(K)$$

are of Baire class 2. Now, let us consider the case that $q=0$. We examine the following sets:

$$X=\left\{E\in\mathscr{K}(\mathbb{R}^d)\;\middle|\;E\text{ is a finite }\right\}$$

and

$$Y=\left\{E\cup F\;\middle|\;E\in\mathscr{K}(\mathbb{R}^d)\text{ is finite and }F\text{ is a compact line segment in }\mathbb{R}\right.$$
$$\left.\text{of positive length}\right\}.$$

As both X and Y are dense in $\mathscr{K}(\mathbb{R}^d)$, we can make the following observations: If $K\in X$, then $\overline{\dim}_B(K)=0$. On the other hand, if $K\in Y$, there exists a finite set L and a compact line segment M in $\mathbb{R}$ with a positive length. Consequently,

$$1=\max\Big(\underline{\dim}_B(L),\underline{\dim}_B(M)\Big)\leqslant\underline{\dim}_B(K)\leqslant\overline{\dim}_B(K)$$
$$\leqslant\max\Big(\overline{\dim}_B(L),\overline{\dim}_B(M)\Big)=1.$$

This implies that

$$\underline{L}^0_\nu(K)=\overline{L}^0_\nu(K)=\underline{C}^0_\nu(K)=\overline{C}^0_\nu(K)=\begin{cases}0 & if\quad K\in X\\ 1 & if\quad K\in Y.\end{cases}$$

According to [17, Theorem 24.15], it is established that $\underline{L}$, $\overline{L}$, $\underline{C}$, and $\overline{C}$ exhibit discontinuity at every point. This demonstrates that they do not belong to Baire class 1.

4.3. **Proof of Theorem 3.2.** In order to prove Theorem 3.2, it is necessary to characterize b_ν^q and B_ν^q. To accomplish this, we require the following results.

Proposition 4.6. *Let E be a bounded set $\mathbb{R}^d$ and $s \in \mathbb{R}$.*

(1) (a) If $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ and $q \geqslant 0$, then there exists $C_1 > 0$ such that

$$\overline{\mathscr{U}}_\nu^{q,s}(E) \leqslant \overline{\mathscr{U}}_\nu^{q,s}(\overline{E}) \leqslant C_1 \overline{\mathscr{U}}_\nu^{q,s}(E).$$

(b) If $q < 0$, then there exists $C_2 > 0$ such that

$$\overline{\mathscr{U}}_\nu^{q,s}(E) \leqslant \overline{\mathscr{U}}_\nu^{q,s}(\overline{E}) \leqslant C_2 \overline{\mathscr{U}}_\nu^{q,s}(E).$$

(2) (a) If $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ and $q \geqslant 0$, then there exists $C_3 > 0$ such that

$$\overline{\mathscr{V}}_\nu^{q,s}(E) \leqslant \overline{\mathscr{V}}_\nu^{q,s}(\overline{E}) \leqslant C_3 \overline{\mathscr{V}}_\nu^{q,s}(E).$$

(b) If $q < 0$, then there exists $C_4 > 0$ such that

$$\overline{\mathscr{V}}_\nu^{q,s}(E) \leqslant \overline{\mathscr{V}}_\nu^{q,s}(\overline{E}) \leqslant C_4 \overline{\mathscr{V}}_\nu^{q,s}(E).$$

Proof. Since $\nu \in \mathscr{M}_D(\mathbb{R}^d)$, then there exist $c, R > 0$, such that

$$c^{-1}\nu\Big(B(z,\frac{r}{2})\Big) \leqslant \nu\Big(B(x,r)\Big) \leqslant c\nu\Big(B(z,\frac{r}{2})\Big), \text{ for } 0 < r < R \text{ and } x, z \in supp\ \nu.$$

(1) (a) Using the definition of $\overline{\mathscr{U}}_\nu^{q,s}$, we obtain the first part. Now, let $r > 0$ and $(B(x_i,r))_i$ be a covering of E and $F \subset \overline{E}$ and let

$$I = \Big\{ i \;\Big|\; B(x_i,r) \cap F \neq \emptyset \Big\}.$$

For each $i \in I$, we choose $z_i \in B(x_i, \frac{r}{2}) \cap E$. It is easily seen that $B(x_i,r) \subset B(z_i, 2r)$, which implies that $\Big(B(z_i,2r)\Big)_i$ is a covering of F. Then

$$N_{\nu,2r}^q(F)(4r)^s \leqslant \sum_{i\in I} \nu(B(z_i,2r))^q (4r)^s \leqslant 2^s c^q \sum_i \nu(B(x_i,r))^q (2r)^s.$$

Putting $C_1 = 2^s c^q$, one has that $N_{\nu,2r}^q(F)(4r)^s \leqslant C_1 N_{\nu,r}^q(E)(2r)^s$. Letting $r \to 0$, one obtains that $\overline{\mathscr{U}}_{\nu,0}^{q,s}(F) \leqslant C_1 \overline{\mathscr{U}}_{\nu,0}^{q,s}(E) \leqslant C_1 \overline{\mathscr{U}}_\nu^{q,s}(E)$. Since F is arbitrary, then $\overline{\mathscr{U}}_\nu^{q,s}(\overline{E}) \leqslant C_1 \overline{\mathscr{U}}_\nu^{q,s}(E)$.

(b) If $q < 0$, then $\nu(B(z_i,2r))^q(4r)^s \leqslant 2^s \nu(B(x_i,r))^q (2r)^s$ and $C_2 = 2^s$.

(2) (a) From the definition of $\overline{\mathscr{V}}_\nu^{q,s}$, we have $\overline{\mathscr{V}}_\nu^{q,s}(E) \leqslant \overline{\mathscr{V}}_\nu^{q,s}(\overline{E})$. We prove now that if $\nu \in \mathscr{M}_D(\mathbb{R}^d)$, then there exists $C > 0$ such that $\overline{\mathscr{V}}_\nu^{q,s}(\overline{E}) \leqslant C\overline{\mathscr{V}}_\nu^{q,s}(E)$. Let $r > 0$ and $(B(x_i,r))_i$ be a packing of $\overline{E}$. Next, we take $z_i \in B(x_i, \frac{r}{2}) \cap E$, for each i. Since $B(z_i, \frac{r}{2}) \subset B(x,r)$, then $\Big(B(z_i,\frac{r}{2})\Big)_i$ is a packing of E whence

$$\begin{aligned}\sum_i \nu\Big(B(x_i,r)\Big)^q (2r)^s &\leqslant c^q \sum_i \nu\Big(B(x,\frac{r}{2})\Big)^q (2r)^s \\ &\leqslant 2^s c^q \sum_i \nu\Big(B(x,\frac{r}{2})\Big)^q r^s \\ &\leqslant 2^s c^q M_{\nu,\frac{r}{2}}^q(E) r^s.\end{aligned}$$

Letting $r \to 0$ and putting $C_3 = 2^s c^q$, we have $\overline{\mathscr{V}}_\nu^{q,s}(\overline{E}) \leqslant C_3 \overline{\mathscr{V}}_\nu^{q,s}(E)$.

(b) If $q < 0$, then $\nu(B(x_i,r))^q(2r)^s \leqslant 2^s \nu(B(z_i,\frac{r}{2}))^q r^s$. Thus $C_4 = 2^s$.

□

Based on Proposition 4.6, we can derive the following corollary.

Corollary 4.7. *If E a bounded set $\mathbb{R}^d$, then*

(1) $\underline{L}^q_\nu(E) = \underline{L}^q_\nu(\overline{E})$ *if* $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ *and* $q \geqslant 0$*;*
(2) $\underline{L}^q_\nu(E) = \underline{L}^q_\nu(\overline{E})$ *if* $q < 0$*;*
(3) $\overline{C}^q_\nu(E) = \overline{C}^q_\nu(\overline{E})$ *if* $\nu \in \mathscr{M}_D(\mathbb{R}^d)$ *and* $q \geqslant 0$*;*
(4) $\overline{C}^q_\nu(E) = \overline{C}^q_\nu(\overline{E})$ *if* $q < 0$.

Lemma 4.8. *For* $E \subseteq \mathbb{R}^d$ *and* $\nu \in \mathscr{M}(\mathbb{R}^d)$*,*

(1) $b^q_\nu(E) = \inf\limits_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup\limits_i \underline{L}^q_\nu(E_i)$.
(2) $B^q_\nu(E) = \inf\limits_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup\limits_i \overline{C}^q_\nu(E_i)$.

Proof.

(1) Let $(E_i)_i$ be a sequence of bounded sets in $\mathbb{R}^d$ such that $E \subseteq \cup_i E_i$. Since b^q_ν is monotone and countable stable, then $b^q_\nu(E) \leqslant \sup_i b^q_\nu(E_i) \leqslant \sup_i \underline{L}^q_\nu(E_i)$. Hence,

$$b^q_\nu(E) \leqslant \inf_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup_i \underline{L}^q_\nu(E_i).$$

On the other hand, let ε and $t > b^q_\nu(E)$. Then $\mathscr{U}^{q,t}_\nu(E) = 0$. However, there is a sequence $(E_i)_i$ of bounded sets such that $\sum_i \overline{\mathscr{U}}^{q,t}_\nu(E_i) < \varepsilon$, which mean that,

$$\inf_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup_i \overline{\mathscr{U}}^{q,t}_\nu(E_i) < \infty.$$

That means

$$\inf_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup_i \underline{L}^q_\nu(E_i) < t, \text{ for all } t > b^q_\nu(E).$$

Finally, we can conclude that

$$b^q_\nu(E) = \inf_{\substack{E \subseteq \cup_i E_i \\ E_i \text{ is bounded}}} \sup_i \underline{L}^q_\nu(E_i).$$

(2) It is very similar to the proof of the first assertion.

□

By using Corollary 4.7 and Lemma 4.8 we have the following result.

Corollary 4.9. *For* $K \in \mathscr{K}(\mathbb{R}^d)$ *and* $\nu \in \mathscr{M}_D(\mathbb{R}^d)$*, we have*

(1) $b^q_\nu(K) = \inf\limits_{\substack{K \subseteq \cup_i K_i \\ K_i \text{ is compact}}} \sup\limits_i \underline{L}^q_\nu(K_i)$.
(2) $B^q_\nu(K) = \inf\limits_{\substack{K \subseteq \cup_i K_i \\ K_i \text{ is compact}}} \sup\limits_i \overline{C}^q_\nu(K_i)$.

Proposition 4.10.

(1) *For $q,s \in \mathbb{R}$, $K \in \mathscr{K}(\mathbb{R}^d)$ and $\nu \in \mathscr{M}(\mathbb{R}^d)$ with $\mathscr{U}_\nu^{q,s}(K) > 0$ (in holds, in particular, $b_\nu^q(K) \geqslant s$), then there exists $L \subseteq K$ such that*
(i) L is non-empty compact set.
(ii) If $O \subseteq \mathbb{R}^d$ is open with $O \cap L \neq \emptyset$, then $\underline{L}_\nu^q(L \cap \overline{O}) \geqslant s$.

(2) *For $q,s \in \mathbb{R}$, $K \in \mathscr{K}(\mathbb{R}^d)$ and $\nu \in \mathscr{M}(\mathbb{R}^d)$ with $\mathscr{V}_\nu^{q,s}(K) > 0$ (in holds, in particular, $B_\nu^q(K) \geqslant s$), then there exists $L \subseteq K$ such that*
(i) L is non-empty compact set.
(ii) If $O \subseteq \mathbb{R}^d$ is open with $O \cap L \neq \emptyset$, then $\overline{C}_\nu^q(L \cap \overline{O}) \geqslant s$.

Proof.

(1) Let μ the restriction of $\mathscr{U}_\nu^{q,s}(K)$ and put $L = supp\ \mu$. It is clear that $L \subseteq K$ and L is compact. Moreover, $\mu(L) = \mu(\mathbb{R}^d) = \mathscr{U}_\nu^{q,s}(K) > 0$. For this reason, we can conclude that $L \neq \emptyset$. Also, if $O \subseteq \mathbb{R}^d$ is open with $O \cap L \neq \emptyset$, then

$$\mu(O) = \mu(O \cap L) = \mathscr{U}_\nu^{q,s}(O \cap L) > 0.$$

Finally, $b_\nu^q(L \cap O) \geqslant s$ and L is the most important set of K satisfies the conditions (i) and (ii).

(2) The proof of the second assertion bears a strong resemblance to that of the first assertion.

□

Proposition 4.11.

(1) *Let $t \in \mathbb{R}$. Then*

$$\left\{ (K,\nu,q) \in \Omega \;\middle|\; b_\nu^q(K) \geqslant t \right\} = \left\{ (K,\nu,q) \in \Omega \;\middle|\; \begin{array}{l} \text{for all } s < t, \text{ there exists } L \subseteq K \text{ such that} \\ i) \ L \text{ is non-empty set and compact} \\ ii) \text{ If } O \subseteq \mathbb{R}^d \text{ is open set and } L \cap O \neq \emptyset \\ \text{then } \underline{L}_\nu^q(L \cap \overline{O}) \geqslant s \end{array} \right\}.$$

(2) *Let $t \in \mathbb{R}$. Then*

$$\left\{ (K,\nu,q) \in \Omega \;\middle|\; B_\nu^q(K) \geqslant t \right\} = \left\{ (K,\nu,q) \in \Omega \;\middle|\; \begin{array}{l} \text{for all } s < t, \text{ there exists } L \subseteq K \text{ such that} \\ i) \ L \text{ is non-empty set and compact} \\ ii) \text{ If } O \subseteq \mathbb{R}^d \text{ is open set and } L \cap O \neq \emptyset \\ \text{then } \overline{C}_\nu^q(L \cap \overline{O}) \geqslant s \end{array} \right\}.$$

Proof.

(1) " $\supseteq$ " It follows from Proposition 4.10-(1) that, for all $(K,\nu,q)\in\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}_D(\mathbb{R}^d)\times\mathbb{R}$ with $b_\nu^q(K)\geqslant t$, there exists $L\subseteq K$ a non-empty compact set. If $O\subseteq\mathbb{R}^d$ is open with $O\cap L\neq\emptyset$, then we have by using Corollary 4.9 that $t\leqslant b_\nu^q(L\cap O)\leqslant \underline{L}_\nu^q(L\cap\overline{O})$.
" $\subseteq$ " Let $(K,\nu,q)\in\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}_D(\mathbb{R}^d)\times\mathbb{R}$. If we assume that $s\leqslant t$, then there exists $L\subseteq K$ is non-empty compact set satisfying $\underline{L}_\nu^q(L\cap\overline{O})\geqslant s$ for all $O\subseteq\mathbb{R}^d$ open with $L\cap O\neq\emptyset$. We prove now that $b_\nu^q(K)\geqslant t$. We assume that $b_\nu^q(E)<t$. Next, we choose $s<t$. Thus there exists $L\subseteq K$ such that L is non-empty compact set and $\underline{L}_\nu^q(L\cap O)\geqslant s$ for all open set $O\subseteq\mathbb{R}^d$ with $L\cap O\neq\emptyset$. Since $b_\nu^q(K)<s$, we see by using Corollary 4.9-(1) that there exists a family $(K_i)_i$ of non-empty compact sets with

$$K\subseteq\bigcup_i K_i \text{ and } \underline{L}_\nu^q(K_i)<s. \tag{4.16}$$

We write now $I=\{i|\ L\cap K_i\neq\emptyset\}$. Since $L=\bigcup_{i\in I}\Big(L\cap K_i\Big)$, we see from the Baire's category theorem that there exists an open set O and $i_0\in I$ such that $L\cap O\neq\emptyset$ and $L\cap O\subseteq L\cap K_{i_0}$. We can choose an open U such that $L\cap U\neq\emptyset$ and $\overline{U}\subseteq O$. It follows from (4.16) that

$$s\leqslant \underline{L}_\nu^q(L\cap\overline{U})\leqslant \underline{L}_\nu^q(L\cap O)\leqslant \underline{L}_\nu^q(L\cap K_{i_0})\leqslant \underline{L}_\nu^q(K_{i_0})<s$$

which is a contradiction.
The proof of assertion (2) is very similar to the proof of assertion (1) and is therefore omitted.

□

Proof of Theorem 3.2.

(1) It sufficient to prove that $\Big\{(K,\nu,q)\in\Omega\ \Big|\ b_\nu^q(K)\geqslant c\Big\}$ is an analytic set. Consider the projection function

$$\begin{aligned}\pi:\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\times\mathscr{K}(\mathbb{R}^d)&\longrightarrow\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\\ (K,\nu,q,L)&\longmapsto(K,\nu,q)\end{aligned}$$

and the following sets

$$C=\Big\{(K,\nu,q,L)\in\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\ \Big|\ L\subseteq K\Big\},$$

$$F_i(r)=\Big\{(K,\nu,q,L)\in\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\ \Big|\ U(x_i,r)\cap L\neq\emptyset\Big\}$$

and

$$D_{i,n}(r)=\Big\{(K,\nu,q,L)\in\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d)\times\mathbb{R}\ \Big|\ \underline{L}_\nu^q(B(x_i,r)\cap L)\geqslant c-\frac{1}{n}\Big\}.$$

It follows from Proposition 4.11 that

$$\Big\{(K,\nu,q)\in\Omega \;\Big|\; b_\nu^q(K)\geqslant c\Big\}$$

$$=\bigcap_{n\in\mathbb{N}}\Big\{(K,\nu,q)\in\Omega \;\Big|\; \exists L\subseteq K \text{ such that}$$

$i)$ L is non-empty compact set

$$ii) \text{ If } O\subseteq\mathbb{R}^d \text{ is open set with } L\cap O\neq\emptyset, \text{ then } \underline{L}_\nu^q(L\cap\overline{O})\geqslant c-\frac{1}{n}\Big\}$$

$$=\bigcap_{n\in\mathbb{N}}\pi\Bigg(\Big\{(K,\nu,q,L)\in\Omega\times\mathscr{K}(\mathbb{R}^d)\times \;\Big|\; L\subseteq K\Big\}$$

$$\cap\Big\{(K,\nu,q,L)\in\Omega\times\mathscr{K}(\mathbb{R}^d) \;\Big|\; \text{if } i\in\mathbb{N}, r\in\mathbb{Q}_+^* \text{ with } L\cap U(x_i,r)\neq\emptyset, \text{ then}$$

$$\underline{L}_\nu^q\Big(L\cap B(x_i,r)\Big)\geqslant c-\frac{1}{n}\Big\}\Bigg)$$

$$=\Omega\cap\bigcap_{n\in\mathbb{N}}\pi\bigg(C\cap\bigcap_{i\in\mathbb{N}}\bigcap_{r\in\mathbb{Q}_+^*}\Big(F_i\cup D_{i,n}(r)\Big)\bigg)=\Omega\cap S,$$

where

$$S=\bigcap_{n\in\mathbb{N}}\pi\bigg(C\cap\bigcap_{i\in\mathbb{N}}\bigcap_{r\in\mathbb{Q}_+^*}\Big(F_i\cup D_{i,n}(r)\Big)\bigg).$$

The closed sets C and F_i can be achieved. From [22, Proposition 4.6] and Theorem 3.1-(2), it is evident that $D_{i,n}(r)$ qualifies as a Borel set. Consequently, S is an analytic subset of $\mathscr{K}(\mathbb{R}^d)\times\mathscr{M}_D(\mathbb{R}^d)\times\mathbb{R}$. Thus it follows that $\Omega\cap S$ is also an analytic subset.

The proof of assertion (2) is very similar to the proof of assertion (1) and is therefore omitted. This achieves the proof of Theorem 3.2.

4.4. **Proof of Theorem 3.3.** To prove Theorem 3.3, we need to prove that

$$\Omega_1:=\Big\{(K,\nu,q)\in\mathscr{K}([0,1])\times\mathscr{M}([0,1])\times]-\infty,0] \;\Big|\; b_\nu^q(K)>0\Big\}$$

and

$$\Omega_2:=\Big\{(K,\nu,q)\in\mathscr{K}([0,1])\times\mathscr{M}([0,1])\times]-\infty,0] \;\Big|\; B_\nu^q(K)>0\Big\}$$

are analytic non-Borel sets. Every real number $x\in\mathbb{R}$ is known to have an exclusive continued fraction expansion represented as

$$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}.$$

Let $x \in \mathbb{R}$, and its continued fraction expansion be expressed as $x = [a_1, a_2, a_3, \cdots]$, where each $a_k \in \mathbb{N} := 1, 2, 3, \ldots$ represents the k-th partial quotient of x. This notation is commonly used to represent the continued fraction expansion of x. If the sequence $(a_k)_k$ is finite, then x belongs to the set of rational numbers $\mathbb{Q}$. On the other hand, if the sequence $(a_k)_k$ is infinite, then x is an irrational number. We denote the set of irrational numbers in the interval $[0,1]$ as $\mathbb{Q}^c$. For such x, we define the following

$$\Gamma(x) = \Big\{ [a_1, k_1, a_2, k_2, \ldots] \;\Big|\; k_i \in \mathbb{N} \quad \text{for } i \in \mathbb{N} \Big\}.$$

By employing an argument closely resembling that presented by in [9] and [19, Section 7], we can establish the existence of a constant $c \in (0, \frac{1}{2}]$. This constant holds the property that, for all $x \in [0,1] \cap \mathbb{Q}^c$, the following inequality is satisfied $c \leq b_\nu^0(\Gamma(x))$. It follows from [3, 25] that

$$c \leq b_\nu^q(\Gamma(x)) \leq B_\nu^q(\Gamma(x)) \quad \text{for all} \quad q \leq 0 \text{ and all } \nu \in \mathscr{M}([0,1]). \tag{4.17}$$

The demonstration of Theorem 3.2 reveals that Ω_1 and Ω_2 possess the property of being an analytic set. In order to establish that Ω_1 and Ω_2 are non-Borel sets, we employ the completeness method as outlined in [17]. We are going to demonstrate the analytic non-Borel nature of set Ω_1. As for set Ω_2, it shares a highly similar proof. For brevity, we omit its presentation. Let $Y := \mathscr{M}(\mathbb{R}^d) \times]-\infty, 0]$. To achieve this, it suffices to demonstrate that given an analytic subset A of the Polish space $X := \mathbb{R}^d$, we can find a Borel measurable function f that maps elements of $X \times Y$ into the space of continuous functions $\mathscr{K}([0,1]) \times \mathscr{M}([0,1]) \times]-\infty, 0]$ in such a way that $f^{-1}(\Omega_1) = A \times Y$. In other words, Ω_1 is classified as a Σ_1^1-complete set. To achieve this goal, we consider an analytic subset A belonging to the Polish space X, and let g be a continuous function from the space $\mathscr{Y} = \mathbb{N}^{\mathbb{N}}$ onto A. For $x = (a_1, a_2, a_3, \ldots) \in \mathscr{Y}$, we define the function

$$\phi(x) = (a_1, a_3, a_5, \ldots).$$

The continuity of ϕ as a function from $\mathscr{Y}$ to $\mathscr{Y}$ is evident. Now, let $\psi = g \circ \phi : \mathscr{Y} \to A$, which is also a continuous function. Furthermore, for every $t \in A$, represented as $g(a_1, a_2, a_3, \ldots) = t$, we have the following relationship $\Gamma(x) = \Gamma(a_1, a_2, a_3, \ldots) \subseteq \psi^{-1}\{t\}$. By using (4.17) we obtain

$$c \leq b_\nu^q(\psi^{-1}\{t\}) \leq B_\nu^q(\psi^{-1}\{t\}), \qquad \text{for all } t \in A.$$

Consider the closure K of the set $\{(\psi(\tau), \tau) \mid \tau \in \mathscr{Y}\}$, forming a closed subset of $X \times [0,1]$. Let $t \in A$. Then

$$\psi^{-1}\{t\} \subseteq \mathrm{K}_t := \Big\{ y \in [0,1] \;\Big|\; (t, y) \in K \Big\},$$

which implies that $c \leq b_\nu^q(\mathrm{K}_t) \leq B_\nu^q(\mathrm{K}_t)$. Now, for $t \in X$ and $y \in \mathrm{K}_t$, we can select a sequence $\tau_i \in \mathscr{Y}$ such that $(\psi(\tau_i), \tau_i) \to (t, y)$ as $i \to +\infty$. If $y \in \mathscr{Y}$ is an irrational number, then $t = \lim_{i \to +\infty} \psi(\tau_i) = \psi(y) \in A$ which implies that, for all $t \in X \backslash A$, $\mathrm{K}_t \subseteq \mathbb{Q}$ and $b_\nu^q(\mathrm{K}_t) = B_\nu^q(\mathrm{K}_t) = 0$. In conclusion, we have successfully demonstrated the existence of a function $f : X \times Y \to \mathscr{K}([0,1]) \times Y$, where $(t, \nu, q) \mapsto f(t, \nu, q) = (\mathrm{K}_t, \nu, q)$, satisfying the property that $f^{-1}(\Omega_1) = A \times Y$. This completes the proof of Theorem 3.3.

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