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INTERPOLATED COEFFICIENT CHARACTERISTIC MIXED FINITE ELEMENT METHOD FOR SEMILINEAR CONVECTION-DIFFUSION OPTIMAL CONTROL PROBLEMS

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Abstract. This paper presents an interpolated coefficient characteristic mixed finite element method for semilinear convection-diffusion optimal control problems. The hyperbolic parts of the state or co-state equations are combined to form a material derivative, which is then discretized by backward difference. The diffusion terms are discretized by the lowest order Raviart-Thomas mixed finite elements, and the nonlinear terms are treated with interpolated coefficient technique. The numerical solution of the control variable is obtained by the variational discretization. Optimal a priori error estimates are derived for the control, state and co-state. Numerical examples are provided to confirm the theoretical results.
Keywords. Characteristic mixed finite element method; Interpolated coefficient; Semilinear convection-diffusion optimal control problems.
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1. Introduction

Optimal control problems (OCPs) play a significant role in physics, engineering, finance, and so on. Since analytical solutions of most OCPs are almost impossible to obtain, numerical methods, such as finite element method (FEM) [1, 2, 3, 4, 5], space-time FEM [6, 7], mixed FEM [8, 9], characteristic FEM [10], immersed FEM [11], finite volume element method [12], spectral method [13, 14], virtual element method [15, 16] were investigated. In [17], the authors considered a variational discretization combined with a fully discrete splitting positive definite mixed finite element method for parabolic optimal control problems. A systematic introduction can be found in [18, 19, 20].

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To improve the efficiency of the FEM for nonlinear partial differential equations. Zlámal and Larsson provided an interpolated coefficient FEM for nonlinear parabolic equations in [21, 22]. Recently, the interpolated coefficient technique was used to solve nonlinear elliptic and parabolic OCPs [23, 24]. The conventional FEM or finite difference discretization of convection-diffusion (CD) equations may produce nonphysical oscillating solutions. Douglas et al. proposed a characteristic FEM to improve the stability and accuracy in [25]. Rui et al. considered a second-order characteristic FEM for CD equations in [26] and investigated a characteristic mixed FEM for linear convection-diffusion optimal control problems (CDOCPs) in [27]. However, there are few results on nonlinear CDOCPs.

In this paper, we investigate an interpolated coefficient characteristic mixed finite element (ICCMFE) discretization of semilinear CDOCPs. The interpolated coefficient technique and the characteristic method are used to treat the semilinear term and the material derivative, respectively. The main advantage of the discretization scheme is that it can overcome the numerical solution oscillations caused by convective dominance and significantly reduce computational costs. We derive optimal a priori error estimates for all variables.

We are interested in the following semilinear CDOCP:

$$\begin{cases} J(u)=\min\limits_{u\in K}\left\{\dfrac{1}{2}\displaystyle\int_0^T\left(\|\boldsymbol{Y}-\boldsymbol{Y}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\}, \\ y_t+\boldsymbol{b}\cdot\nabla y+\operatorname{div}\boldsymbol{Y}+\phi(y)=u+f, \quad \forall t\in J, x\in\Omega, \\ \boldsymbol{Y}=-A\nabla y, \quad \forall t\in J, x\in\Omega, \\ y(t,x)=0, \quad \forall t\in J, x\in\partial\Omega, \\ y(0,x)=y_0, \quad \forall x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$ and $J=(0,T]$. Let $\boldsymbol{Y}_d\in U^2$, $y_d\in U$ with $U=L^2(J;H^1(\Omega))$, $f\in X$ with $X=L^2(J;L^2(\Omega))$, $y_0\in H_0^1(\Omega)$, $\boldsymbol{b}=(b_1(t,x),b_2(t,x))^T$ with $\nabla\cdot\boldsymbol{b}=0, \forall(t,x)\in J\times\Omega$, $A=(a_{ij}(x))_{2\times2}\in W^{1,\infty}(\bar{\Omega})^{2\times2}$ be a real-valued symmetric and positive matrix function. For any $R>0$, the nonlinear function $\phi(\cdot)\in W^{2,\infty}(-R,R)$, $\phi(y)\in L^2(\Omega)$ and $\phi'(\cdot)\geq 0$. The constraint set K is defined as follows:

$$K=\{u\in X: u(t,x)\geq 0, \text{ a.e. in} J\times\Omega\}.$$

We use the standard notation for Sobolev spaces $W^{m,q}(\Omega)$ with a semi-norm $|w|_{m,q}$ and a norm $\|w\|_{m,q}$ given by $|w|_{m,q}^q=\sum_{|\alpha|=m}\|D^\alpha w\|_{L^q(\Omega)}^q$ and $\|w\|_{m,q}^q=\sum_{|\alpha|\leq m}\|D^\alpha w\|_{L^q(\Omega)}^q$. For $q=2$, let $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=\{w\in H^m(\Omega): w|_{\partial\Omega}=0\}$, $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. $L^p(J;W^{m,q}(\Omega))$ denotes all L^p integrable function space from J into $W^{m,q}(\Omega)$ with norm $\|w\|_{L^p(J;W^{m,q}(\Omega))}=\left(\int_0^T\|w\|_{W^{m,q}(\Omega)}^p dt\right)^{1/p}$ for $p\in[1,\infty)$, and the standard modification for $p=\infty$. Additionally, c or C denote a general positive constant independent of the discrete parameters h and k.

The paper is structured as follows: Section 2 presents the ICCMFE approximation scheme of (1.1). Section 3 introduces some auxiliary variables and important error estimates. Section 4 derives optimal a priori error estimates for all variables. Numerical examples are presented in Section 5, the last section, to verify our theoretical results.

2. APPROXIMATION SCHEMES OF SEMILINEAR CDOCPS

This section presents two approximation schemes of semilinear CDOCPs (1.1). For convenience, let $Q = H^1(J;W)$ and $\boldsymbol{L} = L^2(J;\boldsymbol{V})$, where W and $\boldsymbol{V}$ are defined by:

$$W = L^2(\Omega), \boldsymbol{V} = \left\{ \boldsymbol{\upsilon} \in W^2, \mathrm{div}\boldsymbol{\upsilon} \in W \right\}.$$

The weak form of (1.1) is as follows: Find $(\boldsymbol{Y}, y, u) \in \boldsymbol{L} \times Q \times K$ such that

$$\begin{cases} J(u) = \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\|\boldsymbol{Y} - \boldsymbol{Y}_d\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt, \\ (y_t, \omega) + (\boldsymbol{b}\cdot\nabla y, \omega) + (\mathrm{div}\boldsymbol{Y}, \omega) + (\phi(y), \omega) = (u + f, \omega), \quad \forall \omega \in W,\ t \in J, \\ (A^{-1}\boldsymbol{Y}, \boldsymbol{\upsilon}) = (y, \mathrm{div}\boldsymbol{\upsilon}), \quad \forall \boldsymbol{\upsilon} \in \boldsymbol{V},\ t \in J, \\ y(0,x) = y_0, \quad \forall x \in \Omega. \end{cases} \tag{2.1}$$

According to [18], CDOCP (2.1) has a solution $(\boldsymbol{Y}, y, u)$, and $(\boldsymbol{Y}, y, u)$ is the solution to (2.1). Then there exists a co-state $(\boldsymbol{Z}, z) \in \boldsymbol{L} \times Q$ such that $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ satisfies: For any $t \in J$,

$$(y_t, \omega) + (\boldsymbol{b}\cdot\nabla y, \omega) + (\mathrm{div}\boldsymbol{Y}, \omega) + (\phi(y), \omega) = (u + f, \omega), \quad \forall \omega \in W, \tag{2.2}$$

$$(A^{-1}\boldsymbol{Y}, \boldsymbol{\upsilon}) = (y, \mathrm{div}\boldsymbol{\upsilon}), \quad \forall \boldsymbol{\upsilon} \in \boldsymbol{V}, \tag{2.3}$$

$$y(0,x) = y_0, \quad \forall x \in \Omega, \tag{2.4}$$

$$-(z_t, \omega) - (\boldsymbol{b}\cdot\nabla z, \omega) + (\mathrm{div}\boldsymbol{Z}, \omega) + (\phi'(y)z, \omega) = (y - y_d, \omega), \quad \forall \omega \in W, \tag{2.5}$$

$$(A^{-1}\boldsymbol{Z}, \boldsymbol{\upsilon}) = (z, \mathrm{div}\boldsymbol{\upsilon}) - (\boldsymbol{Y} - \boldsymbol{Y}_d, \boldsymbol{\upsilon}), \quad \forall \boldsymbol{\upsilon} \in \boldsymbol{V}, \tag{2.6}$$

$$z(T,x) = 0, \quad \forall x \in \Omega, \tag{2.7}$$

$$\int_0^T (u + z, \tilde{u} - u)\,dt \geq 0, \quad \forall \tilde{u} \in K. \tag{2.8}$$

Recall that the pointwise projection [28] $P_K : X \to K$ satisfies: $P_K\psi = \max\{0, -\psi\}$. Then (2.8) can be rewritten as $u = P_K(z)$. Let $\mathscr{T}_h$ be regular triangulations of Ω and $h = \max\limits_{e\in\mathscr{T}_h}\{h_e\}$, where h_e is the diameter of element e. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ be the lowest order Raviart-Thomas space [29, 30] associated with the triangulations $\mathscr{T}_h$ of Ω. We define the $L^2(\Omega)$ projection [30] $P_h : W \to W_h$ such that

$$\begin{aligned} &(P_h\psi - \psi, \omega_h) = 0, \qquad \forall \omega_h \in W_h, \forall \psi \in W, \\ &\|\psi - P_h\psi\|_{-s,q} \leq Ch^{1+s}\|\psi\|_{1,q}, \quad 2 \leq q \leq \infty, s = 0,1, \forall \psi \in W^{1,q}(\Omega), \end{aligned} \tag{2.9}$$

and the Fortin projection [31] $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$ such that

$$\begin{aligned} &(\mathrm{div}\,(\Pi_h\boldsymbol{\upsilon} - \boldsymbol{\upsilon}), \omega_h) = 0, \qquad \forall \omega_h \in W_h, \forall \boldsymbol{\upsilon} \in \boldsymbol{V}, \\ &\|\boldsymbol{\upsilon} - \Pi_h\boldsymbol{\upsilon}\|_{0,q} \leq Ch\|\boldsymbol{\upsilon}\|_{1,q}, \quad 2 \leq q \leq \infty, \forall \boldsymbol{\upsilon} \in \left(W^{1,q}(\Omega)\right)^2. \end{aligned} \tag{2.10}$$

Let $0 = t_0 < t_1 < \cdots < t_N = T, N \in \mathbb{Z}^+, k = \max_{1\leq n\leq N}\{k_n\}$ with $k_n = t_n - t_{n-1}$. We set $\psi^n = \psi(t_n, x)$ and define the discrete norms

$$|||\psi|||_{l^p(J;W^{m,q}(\Omega))} := \left(\sum_{n=1-\iota}^{N-\iota} k_n \|\psi^n\|^p_{W^{m,q}(\Omega)}\right)^{1/p}, \quad 1 \leq p < \infty,$$

where $\iota = 1$ for the co-state and $\iota = 0$ for the control and state, with the standard modification for $p = \infty$. For simplicity, we denote $|||\cdot|||_{l^p(J;W^{m,q}(\Omega))}$ by $|||\cdot|||_{l^p(W^{m,q})}$ and let

$$l^p(J;W^{m,q}(\Omega)) := \left\{\psi : |||\psi|||_{l^p(W^{m,q})} < \infty\right\}, \qquad 1 \le p \le \infty.$$

Besides,

$$K' := \{v \in W : v(x) \ge 0, \text{ a.e. in } \Omega\}.$$

Then a fully discrete approximation scheme of (2.1) is: Find $(\boldsymbol{Y}_h^n, y_h^n, \boldsymbol{Z}_h^n, z_h^n, u_h^n) \in (\boldsymbol{V}_h \times W_h)^2 \times K, n = 1,2,\cdots,N$ such that

$$\begin{cases} J_{hk}(u_h) = \min_{u_h \in K} \dfrac{1}{2} \sum_{n=1}^{N} k_n \left(\|\boldsymbol{Y}_h^n - \boldsymbol{Y}_d^n\|^2 + \|y_h^n - y_d^n\|^2 + \|u_h^n\|^2\right), \\ \left(\dfrac{y_h^n - y_h^{n-1}}{k_n}, \omega_h\right) + (\boldsymbol{b} \cdot \nabla y_h, \omega_h) + (\mathrm{div}\boldsymbol{Y}_h^n, \omega_h) + (\phi(y_h^n), \omega_h) \\ = (u_h^n + f^n, \omega_h), \quad \forall \omega_h \in W_h, \\ (A^{-1}\boldsymbol{Y}_h^n, \boldsymbol{\upsilon}) = (y_h^n, \mathrm{div}\boldsymbol{\upsilon}), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \\ y_h^0 = P_h y_0. \end{cases} \tag{2.11}$$

2.1. Standard mixed finite elements approximation scheme.

A fully discrete standard mixed finite elements approximation of (2.2)-(2.8) is: Find $(\boldsymbol{Y}_h^n, y_h^n, \boldsymbol{Z}_h^n, z_h^n, u_h^n), n = 1,2,\cdots,N$, for any $\omega_h \in W_h, \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \tilde{u} \in K'$, such that

$$\left(\frac{y_h^n - y_h^{n-1}}{k_n}, \omega_h\right) + (\boldsymbol{b}^n \cdot \nabla y_h^n, \omega_h) + (\mathrm{div}\boldsymbol{Y}_h^n, \omega_h) + (\phi(y_h^n), \omega_h) = (f^n + u_h^n, \omega_h), \tag{2.12}$$

$$(A^{-1}\boldsymbol{Y}_h^n, \boldsymbol{\upsilon}_h) = (y_h^n, \mathrm{div}\boldsymbol{\upsilon}_h), \tag{2.13}$$

$$y_h^0 = P_h y_0, \tag{2.14}$$

$$\begin{aligned} &\left(\frac{z_h^{n-1} - z_h^n}{k_n}, \omega_h\right) - (\boldsymbol{b}^{n-1} \cdot \nabla z_h^{n-1}, \omega_h) + (\mathrm{div}\boldsymbol{Z}^{n-1}, \omega_h) + \left(\phi'(y_h^n) z_h^{n-1}, \omega_h\right) \\ &= (y_h^n - y_d, \omega_h), \end{aligned} \tag{2.15}$$

$$(A^{-1}\boldsymbol{Z}_h^{n-1}, \boldsymbol{\upsilon}_h) = (z_h^{n-1}, \mathrm{div}\boldsymbol{\upsilon}_h) - (\boldsymbol{Y}_h^n - \boldsymbol{Y}_d^n, \boldsymbol{\upsilon}_h), \tag{2.16}$$

$$z_h^N = 0, \tag{2.17}$$

$$(u_h^n + z_h^{n-1}, \tilde{u} - u_h^n) \ge 0. \tag{2.18}$$

We used variational discretization in (2.18). As in [32], (2.18) can be expressed as

$$u_h^n = P_K(z_h^{n-1}), \quad n = 1,2,\cdots,N, \tag{2.19}$$

u_h^n is not a finite element function corresponding to mesh $\mathscr{T}_h$.

Similar to [33], we can set $\{\boldsymbol{\psi}_i\}_{i=1}^M$ to be the edge-basis functions of $\boldsymbol{V}_h$ on edges of $\mathscr{T}_h$ and $\{\varphi_i\}_{i=1}^L$ to be the element-basis functions of W_h on elements of $\mathscr{T}_h$, where M and L be the number of edges and elements of grid $\mathscr{T}_h$, respectively. We set $\boldsymbol{Y}_h^n = \sum_{i=1}^{M} x_{\boldsymbol{Y},i}^n \boldsymbol{\psi}_i \in \boldsymbol{V}_h$ and

$y_h^n = \sum_{l=1}^{L} x_{y,M+l}^n \varphi_l \in W_h$. Selecting $\omega_h = \varphi_l, l = 1,2,\cdots,L$ in (2.12) and $\boldsymbol{\upsilon}_h = \boldsymbol{\psi}_i, i = 1,2,\cdots,M$ in (2.13), we obtain the following nonlinear system: For $i = 1,2,\cdots,M, l = 1,2,\cdots,L$,

$$\begin{aligned}
&\sum_{j=1}^{M}(A^{-1}x_{\boldsymbol{Y},j}^n\boldsymbol{\psi}_j,\boldsymbol{\psi}_i)+\sum_{j=1}^{M}(\mathrm{div}(x_{\boldsymbol{Y},j}^n\boldsymbol{\psi}_j),\varphi_l)+\left(\phi\left(\sum_{j=1}^{L}x_{y,M+j}^n\varphi_j\right),\varphi_l\right)\\
&+\sum_{j=1}^{L}x_{y,M+j}^n\left[\frac{1}{k_n}(\varphi_j,\varphi_l)+(\boldsymbol{b}^n\cdot\nabla\varphi_j,\varphi_l)\right]-\sum_{j=1}^{L}x_{y,M+j}^n(\varphi_j,\mathrm{div}\boldsymbol{\psi}_i)\\
&=(f^n,\varphi_l)+\frac{1}{k_n}\sum_{j=1}^{L}x_{y,M+j}^{n-1}(\varphi_j,\varphi_l)+\sum_{j=1}^{L}P_K\left(x_{z,M+j}^{n-1}\right)(\varphi_j,\varphi_l).
\end{aligned}\tag{2.20}$$

It is often solved by the Newton-like method. Its main concern is to compute the Jacobi matrix. The Jacobi matrix of (2.20) is as follows: For $i = 1,2,\cdots,M, l = 1,2,\cdots,L$,

$$\begin{aligned}
&\sum_{j=1}^{M}(A^{-1}\boldsymbol{\psi}_j,\boldsymbol{\psi}_i)+\sum_{j=1}^{M}(\mathrm{div}\boldsymbol{\psi}_j,\varphi_l)+\left(\phi'\left(\sum_{j=1}^{L}x_{y,M+j}^n\varphi_j\right)\varphi_j,\varphi_l\right)\\
&+\sum_{j=1}^{L}\left[\frac{1}{k_n}(\varphi_j,\varphi_l)+(\boldsymbol{b}^n\cdot\nabla\varphi_j,\varphi_l)\right]-\sum_{j=1}^{L}(\varphi_j,\mathrm{div}\boldsymbol{\psi}_i)=0.
\end{aligned}\tag{2.21}$$

The computation needs to be repeatedly performed as the iterations proceed, depending on the choice of y_h^n. The Jacobi matrix (2.21) can be time-consuming and expensive to compute.

2.2. **ICCMFE approximation scheme.** First, we take account of the hyperbolic part of (2.2) and (2.5) as a material derivative, namely $\boldsymbol{s}$ directional derivative as follows:

$$\begin{aligned}
&\lambda=\sqrt{|\boldsymbol{b}|+1}=\sqrt{b_1^2+b_2^2+1},\\
&y_t+\boldsymbol{b}\cdot\nabla y:=\lambda y_{\boldsymbol{s}},\ z_t+\boldsymbol{b}\cdot\nabla z:=\lambda z_{\boldsymbol{s}},
\end{aligned}$$

where $\boldsymbol{s}=\frac{1}{\lambda}(b_1,b_2,1)$. Then the optimality conditions (2.2)-(2.8) can be rewritten as: For any $t\in J$,

$$(\lambda y_{\boldsymbol{s}},\omega)+(\mathrm{div}\boldsymbol{Y},\omega)+(\phi(y),\omega)=(u+f,\omega),\quad\forall\omega\in W,\tag{2.22}$$

$$(A^{-1}\boldsymbol{Y},\boldsymbol{\upsilon})=(y,\mathrm{div}\boldsymbol{\upsilon}),\quad\forall\boldsymbol{\upsilon}\in\boldsymbol{V},\tag{2.23}$$

$$y(0,x)=y_0,\quad\forall x\in\Omega,\tag{2.24}$$

$$-(\lambda z_{\boldsymbol{s}},\omega)+(\mathrm{div}\boldsymbol{q},\omega)+(\phi'(y)z,\omega)=(y-y_d,\omega),\quad\forall\omega\in W,\tag{2.25}$$

$$(A^{-1}\boldsymbol{Z},\boldsymbol{\upsilon})=(z,\mathrm{div}\boldsymbol{\upsilon})-(\boldsymbol{Y}-\boldsymbol{Y}_d,\boldsymbol{\upsilon}),\quad\forall\boldsymbol{\upsilon}\in\boldsymbol{V},\tag{2.26}$$

$$z(T,x)=0,\quad\forall x\in\Omega,\tag{2.27}$$

$$\int_0^T(u+z,\tilde{u}-u)\geq 0,\quad\forall\tilde{u}\in K.\tag{2.28}$$

Similar to [25], we approximate $y_{\boldsymbol{s}}(t_n,x)$ by

$$y_{\boldsymbol{s}}^n\simeq\frac{y(t_n,x)-y(t_{n-1},\bar{x})}{k_n\sqrt{1+|\boldsymbol{b}(t_n,x)|^2}}:=\frac{y^n-\bar{y}^{n-1}}{k_n\lambda^n},\ n=1,2,\cdots,N,$$

where $\bar{x}=x-\boldsymbol{b}(t_n,x)k_n$.

Second, we introduce an interpolation operator [25] $I_h : W \rightarrow W_h$, which satisfies, for any $\psi \in W$, $I_h\psi = \sum_{i=1}^{L} x_{\psi,i}\varphi_i$, where $\{\varphi_i\}_{i=1}^{L}$ is the element-basis functions of W_h on elements of $\mathscr{T}_h$ and L is the number of elements of $\mathscr{T}_h$. For $0 \le r \le m+1$ and $1 \le p \le \infty$, the interpolation error estimate [22] is as follows:

$$\|\psi - I_h\psi\|_{r,p} \le Ch^{m+1-r}\|\psi\|_{m+1,p}, \qquad \forall \psi \in W^{m+1,p}(e) \cap C(\bar{\Omega}), \forall e \in \mathscr{T}_h. \tag{2.29}$$

Then an ICCMFE approximation of (2.22)-(2.28) is: Find $(\boldsymbol{Y}_h, y_h, \boldsymbol{Z}_h, z_h, u_h)$, for $n = 1, 2, \cdots, N$ and any $\omega_h \in W_h, \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \tilde{u} \in K'$, such that

$$\left(\frac{y_h^n - \bar{y}_h^{n-1}}{k_n}, \omega_h\right) + (\mathrm{div}\boldsymbol{Y}_h^n, \omega_h) + (I_h\phi(y_h^n), \omega_h) = (u_h^n + f^n, \omega_h), \tag{2.30}$$

$$\left(A^{-1}\boldsymbol{Y}_h^n, \boldsymbol{\upsilon}_h\right) = \left(y_h^n, \mathrm{div}\boldsymbol{\upsilon}_h\right), \tag{2.31}$$

$$y_h^0 = P_h y_0, \tag{2.32}$$

$$\left(\frac{z_h^{n-1} - \bar{\bar{z}}_h^n \cdot G^n}{k_n}, \omega_h\right) + (\mathrm{div}\boldsymbol{Z}_h^{n-1}, \omega_h) + (I_h\phi'(y_h^n)z_h^{n-1}, \omega_h) = (y_h^n - y_d^n, \omega_h), \tag{2.33}$$

$$(A^{-1}\boldsymbol{Z}_h^{n-1}, \boldsymbol{\upsilon}_h) = (z_h^{n-1}, \mathrm{div}\boldsymbol{v}_h) - (\boldsymbol{Y}_h^n - \boldsymbol{Y}_d^n, \boldsymbol{\upsilon}_h), \tag{2.34}$$

$$z_h^N = 0, \tag{2.35}$$

$$\left(u_h^n + z_h^{n-1}, \tilde{u} - u_h^n\right) \ge 0, \tag{2.36}$$

where $\bar{\bar{z}}_h^n = z_h^n(\bar{\bar{x}})$ and $x = \bar{\bar{x}} - \boldsymbol{b}(t_n, \bar{\bar{x}})k_n$. Note that $\nabla \cdot \boldsymbol{b} = 0, \forall (t,x) \in J \times \Omega$. Like in [34], we have $G^n = 1 + O(k_n^2)$. Similar to (2.19), (2.36) can be rewritten as $u_h^n = P_K(z_h^{n-1})$, $n = 1, 2, \cdots, N$. As in (2.20), by choosing $\omega_h = \varphi_l, l = 1, 2, \cdots, L$ in (2.30) and $\boldsymbol{\upsilon}_h = \boldsymbol{\psi}_i, i = 1, 2, \cdots, M$ in (2.31), we obtain the following nonlinear system: For $i = 1, 2, \cdots, M, l = 1, 2, \cdots, L$,

$$\begin{aligned}
&\sum_{j=1}^{M}(A^{-1}x_{\boldsymbol{Y},j}^n\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) + \sum_{j=1}^{M}(\mathrm{div}(x_{\boldsymbol{Y},j}^n\boldsymbol{\psi}_j), \varphi_l) + \sum_{j=1}^{L}\phi(x_{y,M+j}^n)(\varphi_j, \varphi_l)\\
&+ \sum_{j=1}^{L}x_{y,M+j}^n\frac{1}{k_n}(\varphi_j, \varphi_l) - \sum_{j=1}^{L}x_{y,M+j}^n(\varphi_j, \mathrm{div}\boldsymbol{\psi}_i)\\
&= (f^n, \varphi_l) + \frac{1}{k_n}\sum_{j=1}^{L}\bar{x}_{y,M+j}^{n-1}(\varphi_j, \varphi_l) + \sum_{j=1}^{L}P_K\left(x_{z,M+j}^{n-1}\right)(\varphi_j, \varphi_l).
\end{aligned} \tag{2.37}$$

Its Jacobi matrix of (2.37) is as follows: For $i = 1, 2, \cdots, M, l = 1, 2, \cdots, L$,

$$\begin{aligned}
&\sum_{j=1}^{M}(A^{-1}\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) + \sum_{j=1}^{M}(\mathrm{div}\boldsymbol{\psi}_j, \varphi_l) + \sum_{j=1}^{L}\phi'(x_{y,M+j}^n)(\varphi_j, \varphi_l)\\
&+ \sum_{j=1}^{L}\frac{1}{k_n}(\varphi_j, \varphi_l) - \sum_{j=1}^{L}(\varphi_j, \mathrm{div}\boldsymbol{\psi}_i) = 0.
\end{aligned} \tag{2.38}$$

Hence, Jacobi matrix (2.38) is reduced greatly compared with (2.21).

Remark 2.1. By introducing the characteristic method and interpolation coefficient technique, approximation scheme (2.30)-(2.36) can not only eliminate numerical oscillations, but also greatly save the computational cost and improve the solution efficiency.

3. Some Auxiliary Variables and Error Estimates

We introduce some auxiliary variables and important error estimates in this section. For any $\tilde{u} \in K$, $\omega_h \in W_h$, $\boldsymbol{\upsilon}_h \in \boldsymbol{V}_h$, let $(\boldsymbol{Y}_h^n(\tilde{u}), y_h^n(\tilde{u}), \boldsymbol{Z}_h^{n-1}(\tilde{u}), z_h^{n-1}(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2, n = 1, 2, \cdots, N$ satisfying the following discrete system:

$$\left(\frac{y_h^n(\tilde{u}) - \bar{y}_h^{n-1}(\tilde{u})}{k_n}, \omega_h\right) + (\mathrm{div}\boldsymbol{Y}_h^n(\tilde{u}), \omega_h) + (I_h\phi(y_h^n(\tilde{u})), \omega_h) = (\tilde{u}^n + f^n, \omega_h), \tag{3.1}$$

$$\left(A^{-1}\boldsymbol{Y}_h^n(\tilde{u}), \boldsymbol{\upsilon}_h\right) = \left(y_h^n(\tilde{u}), \mathrm{div}\boldsymbol{\upsilon}_h\right), \tag{3.2}$$

$$y_h^0(\tilde{u}) = P_h y_0, \tag{3.3}$$

$$\begin{aligned}&\left(\frac{z_h^{n-1}(\tilde{u}) - \bar{\bar{z}}_h^n(\tilde{u}) \cdot G^n}{k_n}, \omega_h\right) + (\mathrm{div}\boldsymbol{Z}_h^{n-1}(\tilde{u}), \omega_h) + (I_h\phi'(y_h^n(\tilde{u}))z_h^{n-1}(\tilde{u}), \omega_h)\\ &= (y_h^n(\tilde{u}) - y_d^n, w_h),\end{aligned} \tag{3.4}$$

$$(A^{-1}\boldsymbol{Z}_h^{n-1}(\tilde{u}), \boldsymbol{\upsilon}_h) = (z_h^{n-1}(\tilde{u}), \mathrm{div}\boldsymbol{\upsilon}_h) - (\boldsymbol{Y}_h^n(\tilde{u}) - \boldsymbol{Y}_d^n, \boldsymbol{\upsilon}_h), \tag{3.5}$$

$$z_h^N(\tilde{u}) = 0. \tag{3.6}$$

For simplicity, we set

$$\begin{aligned}&\eta = y_h - y_h(u), \boldsymbol{\theta} = \boldsymbol{Y}_h - \boldsymbol{Y}_h(u), \zeta = z_h - z_h(u), \boldsymbol{\xi} = \boldsymbol{Z}_h - \boldsymbol{Z}_h(u),\\ &\rho = y - y_h(u), \boldsymbol{\varsigma} = \boldsymbol{Y} - \boldsymbol{Y}_h(u), \mu = z - z_h(u), \boldsymbol{\nu} = \boldsymbol{Z} - \boldsymbol{Z}_h(u),\\ &\delta = P_h y - y_h(u), \boldsymbol{\vartheta} = \Pi_h \boldsymbol{Y} - \boldsymbol{Y}_h(u), \xi = P_h z - z_h(u), \boldsymbol{\sigma} = \Pi_h \boldsymbol{Z} - \boldsymbol{Z}_h(u).\end{aligned}$$

The following results are very important in a priori error estimates of characteristic mixed FEM approximation for CDOCPs.

Lemma 3.1. *(cf. Rui et al. [26]). If $\boldsymbol{b} \in (L^\infty(J; W^{1,\infty}(\Omega)))^2$, then, for $n = 0, 1, \cdots, N$ and $f \in L^2(\Omega)$, it holds that $\|\bar{f}^n\|^2 \leq (1 + Ck_n)\|f^n\|^2$.*

Lemma 3.2. *Let $\left(\boldsymbol{Y}_h^n, y_h^n, \boldsymbol{Z}_h^{n-1}, z_h^{n-1}, u_h^n\right)$ and $\left(\boldsymbol{Y}_h^n(u), y_h^n(u), \boldsymbol{Z}_h^{n-1}(u), z_h^{n-1}(u)\right)$ be the solutions of (2.30)-(2.36) and (3.1)-(3.6), respectively. Then*

$$|||\boldsymbol{Y}_h - \boldsymbol{Y}_h(u)|||_{l^2(L^2)} + |||y_h - y_h(u)|||_{l^\infty(L^2)} \leq C|||u_h - u|||_{l^2(L^2)}, \tag{3.7}$$

$$|||\boldsymbol{Z}_h - \boldsymbol{Z}_h(u)|||_{l^2(L^2)} + |||z_h - z_h(u)|||_{l^\infty(L^2)} \leq C|||u_h - u|||_{l^2(L^2)}. \tag{3.8}$$

Proof. According to (2.30)-(2.31), (3.1)-(3.2), there exits a constant $\theta \in (0,1)$ satisfying

$$\begin{aligned}&\left(\frac{\eta^n - \bar{\eta}^{n-1}}{k_n}, \omega_h\right) + (\mathrm{div}\boldsymbol{\theta}^n, \omega_h) + (I_h\phi'(y_h^n - \theta\eta^n)\eta^n, \omega_h)\\ &= (u_h^n - u^n, \omega_h), \quad \forall \omega_h \in W_h,\end{aligned} \tag{3.9}$$

$$(A^{-1}\boldsymbol{\theta}^n, \boldsymbol{\upsilon}_h) = (\eta^n, \mathrm{div}\boldsymbol{\upsilon}_h), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h. \tag{3.10}$$

From the inequality $\frac{a^2-b^2}{2} \le a(a-b)$, we have

$$\frac{\|\eta^n\|^2-\|\bar{\eta}^{n-1}\|^2}{2k_n} \le \left(\frac{\eta^n-\bar{\eta}^{n-1}}{k_n},\eta^n\right). \tag{3.11}$$

By using Lemma 3.1 and (3.11), we have

$$\frac{\|\eta^n\|^2-\|\eta^{n-1}\|^2}{2k_n} \le \left(\frac{\eta^n-\bar{\eta}^{n-1}}{k_n},\eta^n\right)+C\|\eta^{n-1}\|^2. \tag{3.12}$$

From the definition of I_h, we have $\eta^n = I_h\eta^n$. Note that $\phi(\cdot)\in W^{1,\infty}(\Omega)$ and $\phi'(\cdot)\ge 0$. Then

$$\left(I_h\phi'(y_h^n-\theta\eta^n)\eta^n,\eta^n\right) \ge (I_h\eta^n, I_h\eta^n) \ge 0. \tag{3.13}$$

It follows from ε-Cauchy inequality that

$$(u_h^n-u^n,\eta^n) \le C(\varepsilon)\|u_h^n-u^n\|^2+\varepsilon\|\eta^n\|^2. \tag{3.14}$$

By taking $\omega_h=\eta^n$ in (3.9) and $\boldsymbol{v}_h=\boldsymbol{\theta}^n$ in (3.10). From (3.9)-(3.14), we derive

$$\frac{\|\eta^n\|^2-\|\eta^{n-1}\|^2}{2k_n}+c\|\boldsymbol{\theta}^n\|^2 \le C(\varepsilon)\left(\|\eta^n\|^2+\|\eta^{n-1}\|^2\right)+C(\varepsilon)\|u_h^n-u^n\|^2. \tag{3.15}$$

Multiply both sides of (3.15) by $2k_n$ and then sum over n from 1 to $N^*(N^*\le N)$. In view of $\eta^0=0$, we obtain

$$\|\eta^{N^*}\|^2+2c\sum_{n=1}^{N^*}k_n\|\boldsymbol{\theta}^n\|^2 \le 2C\sum_{n=1}^{N^*}k_n\left(\|\eta^n\|^2+\|\eta^{n-1}\|^2\right)+2C\sum_{n=1}^{N^*}k_n\|u_h^n-u^n\|^2. \tag{3.16}$$

From the discrete Gronwall's inequality and (3.16), we arrive at (3.7). It follows from (2.33)-(2.34) and (3.4)-(3.5) that

$$\begin{aligned}&\left(\frac{\zeta^{n-1}-\bar{\bar{\zeta}}^n}{k_n},\omega_h\right)+\left(\mathrm{div}\boldsymbol{\xi}^{n-1},\omega_h\right)+\left(I_h\phi'(y_h^n)\zeta^{n-1},\omega_h\right)\\ &=(\eta^n,\omega_h)+\left(\frac{\bar{\bar{\zeta}}^n\cdot G^n-\bar{\bar{\zeta}}^n}{k_n},\omega_h\right)+\left(I_h\left(\phi'(y_h^n(u))-\phi'(y_h^n)\right)z_h^{n-1}(u),\omega_h\right),\end{aligned} \tag{3.17}$$

$$\left(A^{-1}\boldsymbol{\xi}^{n-1},\boldsymbol{v}_h\right)=\left(\zeta^{n-1},\mathrm{div}\boldsymbol{v}_h\right)-\left(\boldsymbol{\theta}^n,\boldsymbol{v}_h\right). \tag{3.18}$$

By choosing $\omega_h=\zeta^{n-1}$ in (3.17) and $\boldsymbol{v}_h=\boldsymbol{\xi}^{n-1}$ in (3.18), we can obtain (3.8) similarly. □

Lemma 3.3. *Let* $(\boldsymbol{Y},y,\boldsymbol{Z},z,u)$ *and* $\left(\boldsymbol{Y}_h^n(u),y_h^n(u),\boldsymbol{Z}_h^{n-1}(u),z_h^{n-1}(u)\right)$ *be the solutions of (2.22)-(2.28) and (3.1)-(3.6), respectively. Assume that* $y,z\in L^\infty(H_0^1)\cap L^\infty(H^2)\cap H^1(H^1)\cap H^2(L^2)$ *and* $\boldsymbol{Y},\boldsymbol{Z}\in L^2((H^1)^2)\cap H^1((L^2)^2)$. *Then*

$$|||\Pi_h\boldsymbol{Y}-\boldsymbol{Y}_h(u)|||_{l^2(L^2)}+|||P_hy-y_h(u)|||_{l^\infty(L^2)} \le C(h+k), \tag{3.19}$$

$$|||\Pi_h\boldsymbol{Z}-\boldsymbol{Z}_h(u)|||_{l^2(L^2)}+|||P_hz-z_h(u)|||_{l^\infty(L^2)} \le C(h+k). \tag{3.20}$$

Proof. It follows from (2.22)-(2.23) and (3.1)-(3.2) that

$$\begin{aligned}&\left(\frac{\rho^n-\bar{\rho}^{n-1}}{k_n},\omega_h\right)+(\mathrm{div}\boldsymbol{\varsigma}^n,\omega_h)+(\phi(y^n)-I_h\phi(y_h^n(u)),\omega_h)\\&=\left(\frac{y^n-\bar{y}^{n-1}}{k_n}-\lambda^n y_s^n,\omega_h\right),\quad\forall\omega_h\in W_h,\end{aligned}\tag{3.21}$$

$$(A^{-1}\boldsymbol{\varsigma}^n,\boldsymbol{\upsilon}_h)=(\rho^n,\mathrm{div}\boldsymbol{\upsilon}_h),\qquad\forall\boldsymbol{\upsilon}_h\in\boldsymbol{V}_h.\tag{3.22}$$

According to the definition of P_h and Π_h and (3.21)-(3.22), we have

$$\begin{aligned}&\left(\frac{\delta^n-\bar{\delta}^{n-1}}{k_n},\delta^n\right)+(A^{-1}\boldsymbol{\vartheta}^n,\boldsymbol{\vartheta}^n)\\&=\left(\frac{y^n-\bar{y}^{n-1}}{k_n}-\lambda^n y_{\boldsymbol{s}}^n,\delta^n\right)-\left(A^{-1}\left(\boldsymbol{p}^n-\Pi_h\boldsymbol{p}^n\right),\boldsymbol{\vartheta}^n\right)\\&\quad+(I_h\phi(P_hy^n)-\phi(y^n),\delta^n)-(I_h\phi(P_hy^n)-I_h\phi(y_h^n(u)),\delta^n).\end{aligned}\tag{3.23}$$

Similar to (3.11), we have

$$\frac{\|\delta^n\|^2-\|\bar{\delta}^{n-1}\|^2}{2k_n}\le\left(\frac{\delta^n-\delta^{n-1}}{k_n},\delta^n\right).\tag{3.24}$$

By using Lemma 3.1 and (3.24), we have

$$\frac{\|\delta^n\|^2-\|\delta^{n-1}\|^2}{2k_n}\le\left(\frac{\delta^n-\delta^{n-1}}{k_n},\delta^n\right)+C\|\delta^{n-1}\|^2.\tag{3.25}$$

From ε-Cauchy inequality and the finite difference error analysis [25], we obtain

$$\left(\frac{y^n-\bar{y}^{n-1}}{k_n}-\lambda^n y_{\boldsymbol{s}}^n,\delta^n\right)\le C(\varepsilon)k_n^2\left\|\frac{\partial^2 y}{\partial\boldsymbol{s}^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}+\varepsilon\|\delta^n\|^2.\tag{3.26}$$

According to ε-Cauchy inequality and (2.10), we have

$$\left(A^{-1}\left(\boldsymbol{p}^n-\Pi_h\boldsymbol{p}^n\right),\boldsymbol{\vartheta}^n\right)\le C(\varepsilon)h^2\|\boldsymbol{p}\|_1^2+\varepsilon\|\boldsymbol{\vartheta}^n\|^2.\tag{3.27}$$

By utilizing (2.9), (2.29) and ε-Cauchy inequality, we obtain

$$\begin{aligned}(I_h\phi(P_hy^n)-\phi(y^n),\delta^n)&=(I_h\phi(P_hy^n)-I_h\phi(y^n),\delta^n)+(I_h\phi(y^n)-\phi(y^n),\delta^n)\\&\le C(\varepsilon)h^2\|y^n\|_1^2+2\varepsilon\|\delta^n\|^2.\end{aligned}\tag{3.28}$$

Note that $\phi'(\cdot)\ge 0$. It is obvious that

$$(I_h\phi(P_hy^n)-I_h\phi(y_h^n(u)),\delta^n)\ge(I_h\delta^n,I_h\delta^n)\ge 0.\tag{3.29}$$

Combining (3.23)-(3.29), we derive

$$\begin{aligned}\frac{\|\delta^n\|^2-\|\delta^{n-1}\|^2}{2k_n}+c\|\boldsymbol{\vartheta}^n\|^2\le&C\left(\|\delta^n\|^2+\|\delta^{n-1}\|^2\right)+C(\varepsilon)k_n^2\left\|\frac{\partial^2 y}{\partial\boldsymbol{s}^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\\&+C(\varepsilon)h^2\left(\|y^n\|_1^2+\|\boldsymbol{p}^n\|_1^2\right).\end{aligned}\tag{3.30}$$

Multiplying both sides of (3.30) by $2k_n$ and then summing over n from 1 to $N_*(N_*\le N)$, we arrive at

$$\begin{aligned}\|\delta^{N_*}\|^2+2c\sum_{n=1}^{N_*}k_n\|\boldsymbol{\vartheta}^n\|^2 \le& 2C\sum_{n=1}^{N_*}k_n\left(\|\delta^n\|^2+\|\delta^{n-1}\|^2\right)+2Ck^3\left\|\frac{\partial^2 y}{\partial \boldsymbol{s}^2}\right\|^2_{L^2(J;L^2(\Omega))}\\ &+2Ch^2\sum_{n=1}^{N_*}k_n\left(\|y^n\|_1^2+\|\boldsymbol{p}^n\|_1^2\right).\end{aligned}\tag{3.31}$$

Then (3.19) follows from (3.31) and the discrete Gronwall's inequality. Similarly, from (2.25)-(2.27) and (3.4)-(3.6), we can derive (3.20) immediately. □

4. Optimal a Priori Error Estimates

We derive optimal a priori error estimates of (2.30)-(2.36). As in [35], we assume that objective functional $J_{hk}(\cdot)$ in (2.11) is uniformly convex near the solution u, namely

$$c|||u-u_h|||^2_{l^2(L^2)}\le\left(J'_{hk}(u)-J'_{hk}(u_h),u-u_h\right).\tag{4.1}$$

Theorem 4.1. *Let $(\boldsymbol{Y},y,\boldsymbol{Z},z,u)$ and $\left(\boldsymbol{P}_h^n,y_h^n,\boldsymbol{Z}_h^{n-1},z_h^{n-1},u_h^n\right),n=1,2,\cdots,N$ be the solutions of (2.22)-(2.28) and (2.30)-(2.36), respectively. The conditions in Lemma 3.2 and Lemma 3.3 are valid and $z_t\in l^2(L^2)$, $u+z\in L^\infty(H^1)$. Then*

$$|||u-u_h|||_{l^2(L^2)}\le C(h+k),\tag{4.2}$$

$$|||\boldsymbol{Y}-\boldsymbol{Y}_h|||_{l^2(L^2)}+|||y-y_h|||_{l^\infty(L^2)}\le C(h+k),\tag{4.3}$$

$$|||\boldsymbol{Z}-\boldsymbol{Z}_h|||_{l^2(L^2)}+|||z-z_h|||_{l^\infty(L^2)}\le C(h+k).\tag{4.4}$$

Proof. Taking $\widetilde{u}=u_h$ in (2.28) and $\widetilde{u}=P_hu^n$ in (2.36) and then substituting into (4.1), we arrive at

$$\begin{aligned}|||u-u_h|||^2_{l^2(L^2)}\le& k_n\sum_{n=1}^N\left(u^n+z_h^{n-1}(u),u^n-u_h^n\right)+k_n\sum_{n=1}^N\left(u_h^n+z_h^{n-1},u_h^n-u_n\right)\\ \le& k_n\sum_{n=1}^N\left(u_h^n+z_h^{n-1},P_hu^n-u^n\right)-k_n\sum_{n=1}^N\left(z_h^{n-1}(u)-z^n,u_h^n-u^n\right)\\ =& k_n\sum_{n=1}^N(u_h^n-u^n,P_hu^n-u^n)+k_n\sum_{n=1}^N(u^n+z^n,P_hu^n-u^n)\\ &+k_n\sum_{n=1}^N\left(z^{n-1}-z^n,P_hu^n-u^n\right)+k_n\sum_{n=1}^N\left(z_h^{n-1}(u)-z^{n-1},P_hu^n-u^n\right)\\ &+k_n\sum_{n=1}^N\left(z_h^{n-1}-z_h^{n-1}(u),P_hu^n-u^n\right)+k_n\sum_{n=1}^N\left(z^{n-1}-z^n,u^n-u_h^n\right)\\ &+k_n\sum_{n=1}^N\left(z_h^{n-1}(u)-z^{n-1},u^n-u_h^n\right):=\sum_{i=1}^7 I_i.\end{aligned}\tag{4.5}$$

From (2.9) and the ε-Cauchy inequality, we derive

$$I_1=k_n\sum_{n=1}^N(u_h^n-u^n,P_hu^n-u^n)\le C(\varepsilon)h^2|||u|||_{l^2(H^1)}+\varepsilon|||u-u_h|||^2_{l^2(L^2)}.\tag{4.6}$$

By using $u+z\in L^\infty(H^1)$, (2.9) and Cauchy inequality, we obtain

$$I_2 = k_n\sum_{n=1}^{N}(u^n+z^n, P_hu^n-u^n)\le C(h+k)^2. \tag{4.7}$$

It follows from Lemma 3.3 and ε-Cauchy inequality that

$$I_3 = k_n\sum_{n=1}^{N}\left(z^{n-1}-z^n, u^n-u_h^n\right)\le C(\varepsilon)k^2|||z_t|||_{l^2(L^2)}^2+\varepsilon|||u-u_h|||_{l^2(L^2)}^2 \tag{4.8}$$

and

$$I_4 = k_n\sum_{n=1}^{N}\left(z_h^{n-1}(u)-z^{n-1}, u^n-u_h^n\right)\le C(\varepsilon)(h+k)^2+\varepsilon|||u-u_h|||_{l^2(L^2)}^2. \tag{4.9}$$

By using (2.9) and Lemma 3.2, we obtain

$$I_5 = k_n\sum_{n=1}^{N}\left(z_h^{n-1}-z_h^{n-1}(u), P_hu^n-u^n\right)\le C(\varepsilon)h^2|||u|||_{l^2(H^1)}+\varepsilon|||u-u_h|||_{l^2(L^2)}^2. \tag{4.10}$$

It follows from (2.9), Lemma 3.3, and Cauchy inequality that

$$I_6 = k_n\sum_{n=1}^{N}\left(z^{n-1}-z^n, P_hu^n-u^n\right)\le C\left(k^2|||z_t|||_{l^2(L^2)}^2+h^2\right) \tag{4.11}$$

and

$$I_7 = k_n\sum_{n=1}^{N}\left(z_h^{n-1}(u)-z^{n-1}, P_hu^n-u^n\right)\le C(h^2+k^2). \tag{4.12}$$

From (4.5)-(4.12), we obtain (4.2). From (2.9)-(2.10), (4.2), triangle inequality, Lemma 3.2 and Lemma 3.3, we arrive at (4.3) and (4.4). □

5. Numerical Experiments

In this section, we present two numerical examples to illustrate the correctness of our theoretical results. We use the projection gradient algorithm to solve the CDOCP (1.1) based on the fully discrete ICCMFE approximation scheme (2.32)-(2.37). Let

$$\begin{cases} b\left(u_{(i+\frac{1}{2})}, v\right) = b\left(u_{(i)}, v\right)-\rho_{(i)}\left(J'\left(u_{(i)}\right), v\right), & \forall v\in X,\\ u_{(i+1)} = P_K\left(u_{(i+\frac{1}{2})}\right), \end{cases} \tag{5.1}$$

where $b(\cdot,\cdot)=\int_0^T(\cdot,\cdot)$ is a symmetric and positive definite bilinear form.

For an acceptable error Tol, by using (5.1) and (2.32)-(2.37), we construct the following projection gradient algorithm based on ICCMFE approximation scheme.

Algorithm 1. ICCMFE projection gradient algorithm

Step 1. Initialize $i=1, y_{(i)}^0, z_{(i)}^N, u_{(i)}^n (n=1,2,\cdots,N)$;

Step 2. For $n=1,2,\cdots,N$, solve the following system:

$$\begin{cases} b\left(u^n_{(i+\frac{1}{2})},v\right)=b\left(u^n_{(i)},v\right)-\rho_{(i)}\left(u^n_{(i)}-z^{n-1}_{(i)},v\right), \quad \forall v\in X,\\ \left(\dfrac{y^n_{(i)}-\bar{y}^{n-1}_{(i)}}{k_n},w\right)+\left(\mathrm{div}\boldsymbol{Y}^n_{(i)},w\right)+\left(I_h\phi(y^n_{(i)}),w\right)=(f^n+u^n_{(i)},w), \quad \forall w\in W_h,\\ \left(A^{-1}\boldsymbol{Y}^n_{(i)},\boldsymbol{v}\right)=\left(y^n_{(i)},\mathrm{div}\boldsymbol{v}\right), \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h,\\ \left(\dfrac{z^{n-1}_{(i)}-\bar{\bar{z}}^n_{(i)}}{k_n},w\right)+\left(\mathrm{div}\boldsymbol{Z}^{n-1}_{(i)},w\right)+\left(I_h\phi'\left(y^n_{(i)}\right)z^{n-1}_{(i)},w\right)=\left(y^n_{(i)}-y^n_d,w\right), \quad \forall w\in W_h,\\ \left(A^{-1}\boldsymbol{Z}^{n-1}_{(i)},\boldsymbol{v}\right)=\left(z^{n-1}_{(i)},\mathrm{div}\boldsymbol{v}\right)-\left(\boldsymbol{Y}^n_{(i)}-\boldsymbol{Y}^n_d,\boldsymbol{v}\right), \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h,\\ u^n_{(i+1)}=P_K\left(u^n_{(i+\frac{1}{2})}\right), \end{cases}$$

where we have omitted the subscript h and selected the step size $\rho_{(i)}\in(0,1/4)$ in practice;

Step 3. Compute the iterative error: $E_i=\sqrt{\sum_{n=1}^{N}k_n\|u^n_{(i+1)}-u^n_{(i)}\|^2}$;

Step 4. If $E_i\le Tol$ or $i\ge 100$, stop; else $i=i+1$ go to Step 2;

Step 5. Compute the errors of $|||u-u_h|||_{l^2(L^2)}$, $|||y-y_h|||_{l^\infty(L^2)}$, $|||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^2(L^2)}$, $|||z-z_h|||_{l^\infty(L^2)}$, $|||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^2(L^2)}$.

Let E be 2×2 identity matrix, $\boldsymbol{b}=(1,1)$, $T=1$, and $\Omega=(0,1)\times(0,1)$. The following numerical examples were solved by AFEPack [36]. The discrete nonlinear systems are solved by simple iteration.

Example 1. The data are as follows:

$$\begin{aligned} &A=\varepsilon\cdot E,\phi(y)=e^y,\\ &\boldsymbol{Y}_d(t,x)=\boldsymbol{Y}(t,x)+\boldsymbol{Z}(t,x)+\nabla z(t,x),\\ &y_d(t,x)=y(t,x)+z_t(t,x)+\boldsymbol{b}\cdot\nabla z(t,x)-\mathrm{div}\boldsymbol{Z}(t,x)-\phi'(y(t,x))z(t,x),\\ &u(t,x)=\max\{0,-z(t,x)\},\\ &z(t,x)=\sin(\pi(1-t))x_1^2(1-2x_1)(1-x_1)\sin(2\pi x_2),\\ &y(t,x)=\sin(\pi t)\sin(2\pi x_1)x_2^2(1-2x_2)(1-x_2),\\ &\boldsymbol{Y}(t,x)=-\varepsilon\begin{pmatrix}2\pi\sin(\pi t)\cos(2\pi x_1)x_2^2(1-2x_2)(1-x_2)\\ \sin(\pi t)\sin(2\pi x_1)x_2(2-9x_2+8x_2^2)\end{pmatrix},\\ &\boldsymbol{Z}(t,x)=-\varepsilon\begin{pmatrix}\sin(\pi(1-t))x_1(2-9x_1+8x_1^2)\sin(2\pi x_2)\\ 2\pi\sin(\pi(1-t))x_1^2(1-2x_1)(1-x_1)\cos(2\pi x_2)\end{pmatrix},\\ &f(t,x)=y_t(t,x)+\boldsymbol{b}\cdot\nabla y(t,x)+\mathrm{div}\boldsymbol{Y}(t,x)+\phi(y(t,x))-u(t,x), \end{aligned}$$

where $\varepsilon=10^{-1},10^{-3}$.

Some numerical results are list in Table 1, Table 2, and Figure 1. It is easy to see the error convergence rate $\mathscr{O}(h+k)$ of errors $|||u-u_h|||_{l^2(L^2)}$, $|||y-y_h|||_{l^\infty(L^2)}$, $|||\boldsymbol{Y}-\boldsymbol{Y}_h|||_{l^2(L^2)}$, $|||z-z_h|||_{l^\infty(L^2)}$, and $|||\boldsymbol{Z}-\boldsymbol{Z}_h|||_{l^2(L^2)}$.

TABLE 1. Errors with $\varepsilon = 10^{-1}$, Example 1.

$h=k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	5.86934e-3	2.94157e-3	1.47206e-3	7.36014e-4
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	3.24947e-3	1.62513e-3	8.12504e-4	4.06236e-4
$\|\|\|\boldsymbol{Y}-\boldsymbol{Y}_h\|\|\|_{l^2(L^2)}$	1.48462e-2	7.42351e-3	3.71205e-3	1.85048e-3
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	5.47438e-3	2.73715e-3	1.36834e-3	6.84170e-4
$\|\|\|\boldsymbol{Z}-\boldsymbol{Z}_h\|\|\|_{l^2(L^2)}$	3.26465e-2	1.63243e-2	8.16214e-3	4.08106e-3

TABLE 2. Errors with $\varepsilon = 10^{-3}$, Example 1.

$h=k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	5.97624e-2	2.98821e-2	1.49416e-2	7.50153e-3
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	5.68015e-2	2.84075e-2	1.42037e-2	7.10186e-3
$\|\|\|\boldsymbol{Y}-\boldsymbol{Y}_h\|\|\|_{l^2(L^2)}$	9.37998e-1	4.71271e-1	2.37218e-1	1.18808e-1
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	5.76573e-2	2.88306e-2	1.44153e-2	7.25761e-3
$\|\|\|\boldsymbol{Z}-\boldsymbol{Z}_h\|\|\|_{l^2(L^2)}$	9.48763e-1	4.74386e-1	2.37195e-1	1.13542e-1

Example 2. The data are as follows:

$$
\begin{aligned}
&A = \varepsilon \cdot E, \phi(y) = y^3,\\
&\boldsymbol{Y}_d(t,x) = \boldsymbol{Y}(t,x) + \boldsymbol{Z}(t,x) + \nabla z(t,x),\\
&y_d(t,x) = y(t,x) + z_t(t,x) + \boldsymbol{b}\cdot\nabla z(t,x) - \mathrm{div}\boldsymbol{Z}(t,x) - \phi'(y(t,x))z(t,x),\\
&u(t,x) = \max\{0, -z(t,x)\},\\
&z(t,x) = (1-t)^2\sin(2\pi x_1)\sin(2\pi x_2),\\
&y(t,x) = t^2\sin(2\pi x_1)\sin(2\pi x_2),\\
&\boldsymbol{Y}(t,x) = -\varepsilon\begin{pmatrix} 2\pi t^2\cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi t^2\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&\boldsymbol{Z}(t,x) = -\varepsilon\begin{pmatrix} 2\pi(1-t)^2\cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi(1-t)^2\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&f(t,x) = y_t(t,x) + \boldsymbol{b}\cdot\nabla y(t,x) + \mathrm{div}\boldsymbol{Y}(t,x) + \phi(y(t,x)) - u(t,x),
\end{aligned}
$$

where $\varepsilon = 10^{-1}, 10^{-3}$.

In Table 3, Table 4, and Figure 2, we also can see the same error convergence rate $\mathscr{O}(h+k)$, which confirms our theoretical analysis results in Section 4.

Funding

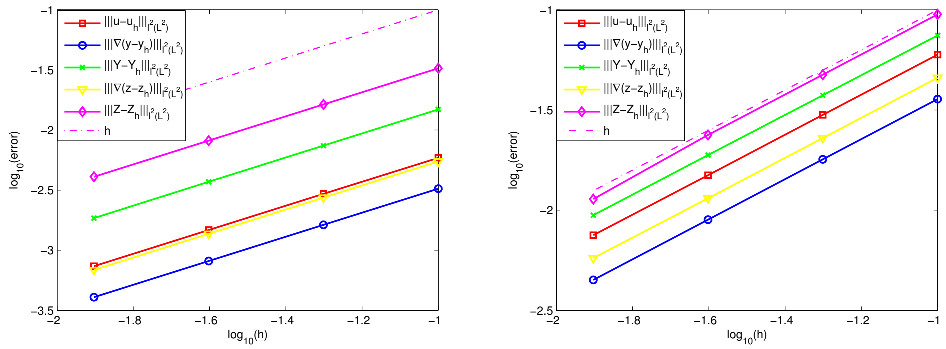

FIGURE 1. The error rate with $\varepsilon = 10^{-3}$ (left) and $\varepsilon = 10^{-1}$ (right), Example 1.

TABLE 3. Errors with $\varepsilon = 10^{-1}$, Example 2.

$h=k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	5.49283e-2	2.74642e-2	1.37321e-2	6.86604e-3
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	3.16752e-2	1.58376e-2	7.91882e-3	3.95941e-3
$\|\|\|\boldsymbol{Y}-\boldsymbol{Y}_h\|\|\|_{l^2(L^2)}$	1.35375e-1	6.76875e-2	3.38438e-2	1.69219e-2
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	5.27924e-2	2.63962e-2	1.31981e-2	6.59905e-3
$\|\|\|\boldsymbol{Z}-\boldsymbol{Z}_h\|\|\|_{l^2(L^2)}$	3.25824e-1	1.62912e-1	8.14560e-2	4.07281e-2

TABLE 4. Errors with $\varepsilon = 10^{-3}$, Example 2.

$h=k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	5.63578e-2	2.81789e-2	1.40895e-2	7.04473e-3
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	5.38725e-2	2.69363e-2	1.34681e-2	6.73406e-3
$\|\|\|\boldsymbol{Y}-\boldsymbol{Y}_h\|\|\|_{l^2(L^2)}$	8.62455e-1	4.31228e-1	2.15614e-1	1.07807e-1
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	5.46924e-2	2.73462e-2	1.36731e-2	6.83655e-3
$\|\|\|\boldsymbol{Z}-\boldsymbol{Z}_h\|\|\|_{l^2(L^2)}$	8.87254e-1	4.43627e-1	2.21814e-1	1.10907e-1

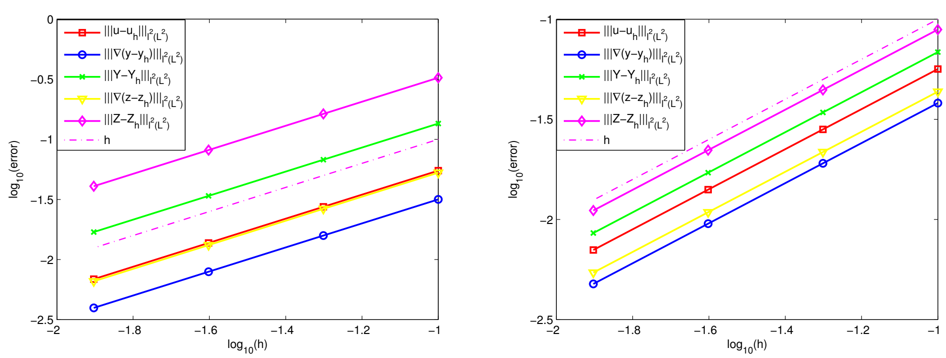

FIGURE 2. The error rate with $\varepsilon = 10^{-3}$ (left) and $\varepsilon = 10^{-1}$ (right), Example 2.

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