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$(h-m)$-PREINVEX FUNCTIONS ON FRACTAL SETS AND LOCAL FRACTIONAL INTEGRAL INEQUALITIES WITH APPLICATIONS

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Abstract. Recently, there has been growing interest among researchers in extending and generalizing the concept of convexity to incorporate fractional theory and related inequalities. One notable extension which involves studying convexity on fractal sets is the $(h-m)$-convexity, which generalizes the h-convexity and m-convexity concepts. In this paper, we present a new notion of generalized $(h-m)$-preinvex functions, which is a generalization of convexity on fractal sets, along with various characteristics for the newly presented functions. Utilizing the new concept, we derive new Hermite-Hadamard-type inequalities and establish a new identity for generalized $(h-m)$-preinvex functions with parameters involving local fractional integrals. In addition, we derive some general local fractional integral inequalities for generalized $(h-m)$-preinvex functions. We utilize our findings in practical applications by selecting particular values for the variables to construct various generalizations of midpoint, trapezoidal, and Simpson type inequalities. The findings of this work provide essential extensions and generalizations of earlier studies conducted in the area.

Keywords. Convex functions; Fractal sets; Generalized Hermite-Hadamard inequality; Local fractional derivatives; Preinvex functions.

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1. INTRODUCTION

Several assumptions are linked with convex functions in the domain of mathematical inequalities. The most widely recognized inequality, with its extensive geometrical framework, as it offers both upper and lower bounds of the integral average value for convex functions, is the Hermite-Hadamard (HH) inequality, introduced by Hadamard in 1881. According to [15], this

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inequality is expressed thereby: If $\mathcal{H}: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, $r, s \in I$ and $r < s$, then

$$\mathcal{H}\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \mathcal{H}(y)dy \leq \frac{\mathcal{H}(r)+\mathcal{H}(s)}{2}. \tag{1.1}$$

For concave functions, the reverse of the aforementioned inequalities holds.

In the last decades, various generalizations for the classical properties of convexity were proposed. For a number of results which are extensions, generalizations, and enhancements of the basic Hermite–Hadamard inequality linked with different extensions of a convex function, we refer to [5, 8, 10, 12, 34] and the references therein. A novel family of generalized convex functions known as invex functions was introduced by Hanson in [16]. Let $\mathbb{B} \subseteq \mathbb{R}^n$ be a nonempty set, and let $\sigma(.,.): \mathbb{B} \times \mathbb{B} \to \mathbb{R}^n$ be a bi-function. The set $\mathbb{B}$ is called an *invex* set with respect to (w.r.t.) the bi-function $\sigma(.,.)$ if

$$u + \mu\sigma(v,u) \in \mathbb{B},$$

for every $u, v \in \mathbb{B}$ whenever $0 \leq \mu \leq 1$; see [11, 33]. If $\sigma(v,u) = v - u$ holds, the invex set is reduced to a convex set; nevertheless, some invex sets are not convex; examples of these can be seen in [4]. On the other hand, Ben-Isreal and Mond [11] defined preinvex functions, and demonstrated that differentiable preinvex functions are invex functions, but the converse may not be true. The characteristics of preinvex functions in equilibrium, mathematical programming, and variational inequalities were explored by Weir and Mond [33]. Let $\mathbb{B} \subseteq \mathbb{R}$ be an invex set. A function $\mathcal{H}: \mathbb{B} \to \mathbb{R}$ is *preinvex* on $\mathbb{B}$ by default. $\sigma(.,.)$ if

$$\mathcal{H}(u + \mu\sigma(v,u)) \leq \mu\mathcal{H}(v) + (1-\mu)\mathcal{H}(u), \tag{1.2}$$

for every $u, v \in \mathbb{B}$ and $0 \leq \mu \leq 1$. Recall that $\mathcal{H}$ is preconcave if $-\mathcal{H}$ is a preinvex function. Although the opposite is not true, it is evident that any convex function is a preinvex function.

Several researchers worked on and implemented the concept of preinvexity in various ways over the past few decades. In [21], Matoka introduced and explored the concept of h–preinvexity, which is given in the following definition:

Definition 1.1. Let $(0,1) \subseteq \mathbb{J} \subseteq \mathbb{R}$, $h: \mathbb{J} \to \mathbb{R}$, and $\mathbb{B}$ be an invex set w.r.t. $\sigma(\cdot,\cdot)$. A function $\mathcal{H}: \mathbb{B} \to \mathbb{R}$ is called an h-preinvex w.r.t. $\sigma(\cdot,\cdot)$ if

$$\mathcal{H}(u + \mu\sigma(v,u)) \leq h(\mu)\mathcal{H}(v) + h(1-\mu)\mathcal{H}(u), \tag{1.3}$$

for all $v, u \in \mathbb{B}$ and $\mu \in (0,1)$. If the above inequality is reversed, $-\mathcal{H}$ is called h-preconcave w.r.t. $\sigma(\cdot,\cdot)$.

It is evident that $\mathcal{H}$ becomes a preinvex function when $h(\mu) = \mu$ is used in (1.3). Moreover, if $\sigma(v,u) = v - u$, then $\mathcal{H}$ reduces to h-convex.

Definition 1.2. (Ref. [19]) Let $\mathbb{B} \subseteq [0, b^*]$, $b^* > 0$, be an invex set. For $m \in (0,1]$, a function $\mathcal{H}: \mathbb{B} \to \mathbb{R}$ is m-preinvex w.r.t. σ if

$$\mathcal{H}(u + \mu\sigma(v,u)) \leqslant (1-\mu)\mathcal{H}(u) + m\mu\mathcal{H}\left(\frac{v}{m}\right)$$

for all $v, u \in \mathbb{B}$, $\mu \in [0,1]$. Furthermore, $\mathcal{H}$ is m-preconcave whenever $-\mathcal{H}$ is m-preinvex.

Definition 1.3. (Ref. [31]) Let $[0, \mathrm{b}^*] \subseteq \mathbb{R}$, $\mathrm{b}^* > 0$, be a invex set w.r.t. σ and $h: J \to \mathbb{R}$, $(0,1) \subset J$, be a non-negative function. The function $\mathcal{H}$ is called (h,m) - preinvex, if

$$\mathcal{H}(u + \mu\sigma(v,u)) \leq h(1-\mu)\mathcal{H}(u) + mh(\mu)\mathcal{H}(v)$$

for all $v, u \in [0, b^*]$, $m \in [0,1]$ and $\mu \in (0,1)$.

The concept of preinvexity was further generalized by Sun in [26], who introduced the concept of generalized preinvex function. For $\ell \in (0,1]$, a function $\mathcal{H} : \mathbb{B} \to \mathbb{R}$ is said to be *generalized preinvex* on $\mathbb{B}$ if

$$\mathcal{H}(u+\mu\sigma(v,u)) \leq t^{\ell}\mathcal{H}(v) + (1-\mu)^{\ell}\mathcal{H}(u), \tag{1.4}$$

for every $u, v \in \mathbb{B}$ and $0 \leq \mu \leq 1$. In [27], Sun combined the idea of the h-preinvexity and the generalized convexity to establish the concept of generalized h-preinvex functions as follows.

Definition 1.4. Let $h : J \to \mathbb{R}$, $(0,1) \subseteq J$, be a nonnegative function and $h^{\ell} \not\equiv 0^{\ell}$. Let $\mathcal{H}: \mathbb{B} \to \mathbb{R}^{\ell}$ $(0 < \ell \leq 1)$, where $\mathbb{B}$ is an invex set w.r.t. $\sigma(\cdot,\cdot)$. Then $\mathcal{H}$is a generalized h-preinvex function w.r.t. $\sigma(\cdot,\cdot)$ if

$$\mathcal{H}(u+\mu\sigma(v,u)) \leq h^{\ell}(\mu)\mathcal{H}(v) + h^{\ell}(1-\mu)\mathcal{H}(u)$$

for all $u, v \in \mathbb{B}$ and $\mu \in (0,1)$. Moreover, $\mathcal{H}$ is called generalized h-preconcave w.r.t. $\sigma(\cdot,\cdot)$ if the above inequality is reversed.

We consider the following criteria proposed by Mohan and Samir for the bi-function σ; see [20].

Condition C.: Let $\sigma(.,.) : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^n$ be a bi-function, where $\mathbb{B} \subseteq \mathbb{R}^n$ is an invex set. Then

$$\sigma(\vartheta, \vartheta + \mu\sigma(\zeta,\vartheta)) = -t\sigma(\zeta,\vartheta),$$
$$\sigma(\zeta, \vartheta + \mu\sigma(\zeta,\vartheta)) = (1-\mu)\sigma(\zeta,\vartheta),$$

for $0 \leq \mu \leq 1$, and all $\vartheta, \zeta \in \mathbb{B}$. Furthermore, from Condition C, the subsequent equality holds true:

$$\sigma(\zeta + \nu_2\sigma(\vartheta,\zeta), \zeta + \nu_1\sigma(\vartheta,\zeta)) = (\nu_2 - \nu_1)\sigma(\vartheta,\zeta),$$

for every $\vartheta, \zeta \in \mathbb{B}$ and $\nu_1, \nu_2 \in [0,1]$.

The philosophy of local fractional calculus was initiated by Yang in [36, 37]. Aimed at modeling various non-differentiable phenomena that arise in many fields of science and engineering, local fractional calculus theory is widely applied in differential equations. Applications of this theory are also relevant to physics, control theory, random walk process, and communication engineering [9, 18, 35]. Many researchers recently concentrated on extending and generalizing the concept of preinvexity to encompass fractional theory and related inequalities; see, e.g., [2, 13, 17, 22, 25, 30, 32, 38]. In [27–29], Sun examined various extensions of preinvex functions relative to fractal theory. Al-Sa'di et al. [6, 7] established and investigated the notation of generalized γ-preinvexity and m-preinvexity functions on the Yang's fractal sets and established some generalized Hermite–Hadamard's inequality for such classes of functions. In [24], Peng and Du developed various inequalities of Hermite–Hadamard type in association with *differentiable multiplicative m-preinvexity and (s,m)-preinvexity.

Inspired by these results, we present a generalized $(h-m)$-preinvex function on fractal sets and discuss a few properties of this function. Utilizing the new concept, we derive HH-type inequalities for the latest function. To provide an integrated framework for developing a generalizations of Simpson and HH-like inequalities for functions where the derivative absolute values are generalized $(h-m)$-preinvex functions, we first establish a number of some comprehensive inequalities utilizing local fractional integrals via parameters. Using particular parameter values, we can derive some generalizations of midpoint-type, trapezoidal-type, and Simpson-type inequalities.

2. PRELIMINARIES

In order to describe various non-differentiable issues that arise in fractal engineering, a new concept of the local fractional derivative was recently developed by Yang [36, 37]. In this light, several recent papers on the damped wave equation in fractal strings and the diffusion equation's and the heat-conduction equation on non-differentiable solution on Cantor sets were raised. Yang's theory can be understood from the references [36, 37]. Let Υ^ℓ be a Yang's fractional set, Υ be the base set, and $\ell\,(0<\ell\leq 1)$ be the dimension of cantor set. The ℓ-type set of integers $\mathbb{Z}^\ell$ is defined as $\mathbb{Z}^\ell=\{0^\ell,\pm 1^\ell,\pm 2^\ell,\pm 3^\ell,\ldots\}$. We may express the ℓ-type $\mathbb{Q}^\ell$ as

$$\mathbb{Q}^\ell=\{m^\ell=\left(\frac{r}{s}\right)^\ell;\ r,s\in\mathbb{Z} \text{ where } s\neq 0\},$$

the ℓ-type $\mathfrak{I}^\ell$ as

$$\mathfrak{I}^\ell=\{m^\ell\neq\left(\frac{r}{s}\right)^\ell;\ r,s\in\mathbb{Z},s\neq 0\},$$

and the ℓ-type $\mathbb{R}^\ell$ as

$$\mathbb{R}^\ell=\mathbb{Q}^\ell\cup\mathfrak{I}^\ell.$$

Some characteristics of real fractal numbers are listed below:

If $r^\ell,s^\ell,t^\ell\in\mathbb{R}^\ell$, then

- $r^\ell+s^\ell\in\mathbb{R}^\ell, r^\ell s^\ell\in\mathbb{R}^\ell$,
- $r^\ell+s^\ell=s^\ell+r^\ell=(s+r)^\ell=(r+s)^\ell$,
- $r^\ell+(s^\ell+t^\ell)=(r+s)^\ell+t^\ell$,
- $r^\ell s^\ell=s^\ell r^\ell=(rs)^\ell=(sr)^\ell$,
- $r^\ell(s^\ell t^\ell)=(r^\ell s^\ell)t^\ell$,
- $r^\ell(s^\ell+t^\ell)=r^\ell s^\ell+r^\ell t^\ell$,
- $0^\ell+r^\ell=r^\ell+0^\ell=r^\ell$, and $r^\ell 1^\ell=1^\ell r^\ell=r^\ell$
- If $r^\ell<s^\ell$, then $r^\ell+t^\ell<s^\ell+t^\ell$,
- If $0^\ell<r^\ell, 0^\ell<s^\ell$, then $0^\ell<r^\ell.s^\ell$,
- $(-r)^\ell$ is called the reverse element of $r^\ell, (-r)^\ell=-r^\ell$,
- $(r-s)^\ell=r^\ell-s^\ell$.

Definition 2.1. ([36, 37]) Let $\mathcal{H}:\mathbb{R}\to\mathbb{R}^\ell, y\to\mathcal{H}(y)$ be a non-differentiable function. Then $\mathcal{H}$ is local fractional continuous at y_0 if, given $\varepsilon>0$, there exists $\delta>0$ such that $|\mathcal{H}(y)-\mathcal{H}(y_0)|<\varepsilon^\ell$ whenever $|y-y_0|<\delta$. We write $\mathcal{H}(y)\in C_\ell(c,d)$ whenever $\mathcal{H}(y)$ is local continuous on (c,d).

There are various definitions of local fractional derivatives and integrals, also known as fractal calculus. The ones that follow are the Gao-Yang-Kang definitions.

Definition 2.2. ([36, 37]) Local fractional derivative of the function $\mathcal{H}(y)$ of order ℓ at $y=y_0$ can be defined as

$$\mathcal{H}^{(\ell)}(y_0)=\left.\frac{d^\ell\mathcal{H}(y)}{dy^{(\ell)}}\right|_{y=y_0}=\lim_{y\to y_0}\frac{\Gamma(1+\ell)(\mathcal{H}(y)-\mathcal{H}(y_0))}{(y-y_0)^\ell}.$$

The ℓ-local derivative set is denoted by $D_\ell(c,d)$. We write $\mathcal{H} \in D_{(n+1)\ell}(I)$, $n = 0,1,2,\ldots$, if there exists $\mathcal{H}^{((n+1)\ell)}(y) = \overbrace{D_y^\ell \ldots D_y^\ell}^{(n+1)times} \mathcal{H}(y)$ for any $y \in \mathbb{I} \subseteq \mathbb{R}$.

Definition 2.3. ([36, 37]) Let $\mathcal{H}(y) \in C_\ell[c,d]$. Local fractional integral of $\mathcal{H}(y)$ is defined by

$$ {}_cI_d^{(\ell)}\mathcal{H}(y) = \frac{1}{\Gamma(1+\ell)} \int_c^d \mathcal{H}(\mu)(d\mu)^\ell = \frac{1}{\Gamma(1+\ell)} \lim_{\Delta\mu \to 0} \sum_{j=0}^{N-1} \mathcal{H}(\mu_j)(\Delta\mu_j)^\ell, $$

where $c = \mu_0 < \mu_1 < \ldots < \mu_{N-1} < \mu_N = d$, $[\mu_j, \mu_{j+1}]$ is partition of $[c,d]$, $\Delta\mu_j = \mu_{j+1} - \mu_j, \Delta\mu = \max\{\Delta\mu_0, \Delta\mu_1, \ldots, \Delta\mu_{N-1}\}$.

Note that ${}_cI_c^{(\ell)}\mathcal{H}(y) = 0$ and ${}_cI_d^{(\ell)}\mathcal{H}(y) = -{}_dI_c^{(\ell)}\mathcal{H}(y)$ if $c < d$. We write $\mathcal{H}(y) \in I_y^{(\ell)}[c,d]$ if there exists ${}_cI_y^{(\ell)}\mathcal{H}(y)$ for any $y \in [c,d]$.

Lemma 2.4. *([36, 37])*

(1) Let $\mathcal{H}(y) = g^{(\ell)}(y) \in C_\ell[c,d]$. Then ${}_cI_d^{(\ell)}\mathcal{H}(y) = g(d) - g(c)$.

(2) Let $\mathcal{H}(y), g(y) \in D_\ell[c,d]$ and $\mathcal{H}^{(\ell)}(y), g^{(\ell)}(y) \in C_\ell[c,d]$. Then

$$ {}_cI_d^{(\ell)}\mathcal{H}(y)g^{(\ell)}(y) = \mathcal{H}(y)g(y)\big|_c^d - {}_cI_d^{(\ell)}\mathcal{H}^{(\ell)}(y)g(y). $$

Lemma 2.5. *([36, 37]) The ℓ-local fractional derivative and integral of the function $\mathcal{H}(y) = y^{\tau\ell}$, $\tau \in \mathbb{R}$, is given by*

$$ \begin{aligned} \frac{d^\ell y^{\tau\ell}}{dy^\ell} &= \frac{\Gamma(1+\tau\ell)}{\Gamma(1+(\tau-1)\ell)} y^{(\tau-1)\ell}, \\ \frac{1}{\Gamma(\ell+1)} \int_c^d y^{\tau\ell}(dy)^\ell &= \frac{\Gamma(1+\tau\ell)}{\Gamma(1+(\tau+1)\ell)} (d^{(\tau+1)\ell} - c^{(\tau+1)\ell}). \end{aligned} $$

Lemma 2.6. *([36, 37])*

$$ {}_cI_d^{(\ell)} 1^\ell = \frac{(d-c)^\ell}{\Gamma(1+\ell)}. $$

The following Generalized Hölder's inequality for local fractional integral established by Yang plays a significant role in our work; see [36, 37]).

Lemma 2.7. *Let $\mathcal{H}(y), g(y) \in C_\ell[c,d]$, and $p,q > 1$ with $p^{-1} + q^{-1} = 1$. Then*

$$ \frac{1}{\Gamma(\ell+1)} \int_c^d \left|\mathcal{H}(y)g(y)\right|(dy)^\ell \leq \left(\frac{1}{\Gamma(\ell+1)} \int_c^d \left|\mathcal{H}(y)\right|^p (dy)^\ell \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\ell+1)} \int_c^d \left|g(y)\right|^q (dy)^\ell \right)^{\frac{1}{q}}. $$

3. Main Results

We begin with the following generalized definition of the $(h-m)$-preinvex mapping.

Definition 3.1. Let $\mathbb{J}$ be any interval in $\mathbb{R}$ with $(0,1) \subseteq \mathbb{J}$, and let $h : \mathbb{J} \to \mathbb{R}$ be nonnegative function. Suppose that $\mathbb{B} \subseteq [0, b^*]$, $b^* > 0$, is an invex set w.r.t. σ. Then a nonnegative function $\mathcal{H} : \mathbb{B} \to \mathbb{R}^\ell$ is said to be *generalized $(h-m)$-preinvex* if

$$ \mathcal{H}(um + \mu\sigma(v, um)) \leq h^\ell(\mu)\mathcal{H}(v) + m^\ell h^\ell(1-\mu)\mathcal{H}(u) \tag{3.1} $$

holds for all $u, v \in \mathbb{B}$, $\mu \in [0,1]$ and $m \in (0,1]$. The function $\mathcal{H}$ called generalized $(h-m)$-preconcave if we reverse the inequality.

Remark 3.2. It is worth noting that this definition addresses a wide range of other special cases of convexity and preinvexity. This highlights that the results obtained in the current work are still valid for such categories of convex and preinvex functions and their variant forms. In particular:

(1) If $\sigma(v,um) = (v-um)$ and $\ell = 1$, then Definition 3.1 simplifies the concept of $(h-m)$-convex functions [23].
(2) If $\sigma(v,um) = (v-um)$, then Definition 3.1 simplifies the concept of generalized $(h-m)$-convex functions [3].
(3) If $\sigma(v,um) = (v-um)$ and $h(\mu) = \mu$, then Definition 3.1 simplifies the concept of generalized m-convex functions. [13].
(4) If $\sigma(v,um) = (v-um)$ and $h(\mu) = \mu^{s\ell}$, then Definition 3.1 simplifies the concept of generalized $(s-m)$-convex functions [1].
(5) If $m = 1$, $\ell = 1$, and $h(\mu) = \mu$, then Definition 3.1 simplifies the concept of classical-preinvex functions [33].
(6) If $\ell = 1$ and $h(\mu) = \mu^s$, then Definition 3.1 simplifies the concept of generalized (s,m)-preinvex functions [14].
(7) If $m = 1$, then Definition 3.1 simplifies the concept of generalized h-preinvex functions [27].
(8) If $\ell = 1$, then Definition 3.1 simplifies the concept of generalized (m,h)-preinvex functions [39].

Remark 3.3. If $\mu = \frac{1}{2}$, then generalized Jensen-type $(h-m)$-preinvex function is as

$$\mathcal{H}\left(um + \frac{1}{2}\sigma(v,um)\right) \leq h^{\ell}\left(\frac{1}{2}\right)[\mathcal{H}(v) + m^{\ell}\mathcal{H}(u)]. \tag{3.2}$$

Now, we investigate some algebraic characteristics of the definition.

Proposition 3.4. *Let h_1 and h_1 be positive functions defined on $\mathbb{J} \subseteq \mathbb{R}$, with $h_2(\mu) \leq h_1(\mu)$, $\mu \in [0,1]$. If $\mathcal{H}$ is generalized (h_2-m)-preinvex function on fractal sets, then $\mathcal{H}$ is a generalized (h_1-m)-preinvex function on fractal sets.*

Proof.

$$\begin{aligned}\mathcal{H}(um + \mu\sigma(v,um)) &\leq h_2^{\ell}(\mu)\mathcal{H}(v) + m^{\ell}h_2^{\ell}(1-\mu)\mathcal{H}(u)\\ &\leq h_1^{\ell}(\mu)\mathcal{H}(v) + m^{\ell}h_1^{\ell}(1-\mu)\mathcal{H}(u),\end{aligned}$$

for all $u, v \in [0, b^*]$, $\mu \in [0,1]$, and $m \in (0,1]$. □

Proposition 3.5. *If $g, \mathcal{H}$ are generalized $(h-m)$- preinvex functions on fractal sets, and $\kappa^{\ell} > 0^{\ell}$, then*

(1) $g + \mathcal{H}$ is a generalized $(h-m)$-preinvex function on fractal sets;
(2) $\kappa^{\ell} g$ is a generalized $(h-m)$-preinvex function on fractal sets.

Proof. (1). Since the two functions are generalized $(h-m)$-preinvex, we have

$$\mathcal{H}(um + \mu\sigma(v,um)) \leq h^{\ell}(\mu)\mathcal{H}(v) + m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)$$

and

$$g(um+\mu\sigma(v,um)) \le h^{\ell}(\mu)g(v)+m^{\ell}h^{\ell}(1-\mu)g(u),$$

for all $u,v \in \mathbb{B}$, $m \in (0,1]$, and $\mu \in [0,1]$,

$$\begin{aligned}(g+\mathcal{H})(um+\mu\sigma(v,um) &\le h^{\ell}(\mu)(g(v)+\mathcal{H}(v))+m^{\ell}h^{\ell}(1-\mu)(g(u)+\mathcal{H}(u))\\ &\le h^{\ell}(\mu)(g(v)+\mathcal{H}(v))+m^{\ell}h^{\ell}(1-\mu)(g(u)+\mathcal{H}(u))\\ &\le h^{\ell}(\mu)((g+\mathcal{H})(v))+m^{\ell}h^{\ell}(1-\mu)((g+\mathcal{H})(u)).\end{aligned} \tag{3.3}$$

(2). Multiplying (3.1) by κ^{ℓ}, we have

$$\begin{aligned}\kappa^{\ell}\mathcal{H}(um+\mu\sigma(v,um)) &\le \kappa^{\ell}\left(h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)\right)\\ &\le h^{\ell}(\mu)\kappa^{\ell}\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\kappa^{\ell}\mathcal{H}(u)\\ &\le \left(h^{\ell}(\mu)(\kappa^{\ell}\mathcal{H})(v)+m^{\ell}h^{\ell}(1-\mu)(\kappa^{\ell}\mathcal{H})(u)\right).\end{aligned} \tag{3.4}$$

□

Proposition 3.6. *Let $\mathcal{H}:\mathbb{X}^{\ell}\to\mathbb{Y}^{\ell}$ be a non-decreasing function, and let $g:\mathbb{B}\to\mathbb{X}^{\ell}$ be a generalized $(h-m)$-preinvex function. Then the composition of both functions $g\circ\mathcal{H}$ is generalized $(h-m)$-preinvex.*

Proof. For $m\in(0,1]$, and every $u,v\in\mathbb{B}$, $\mu\in[0,1]$, we have

$$\begin{aligned}(g\circ\mathcal{H})(um+\mu\sigma(v,um)) &\le g\left(h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)\right)\\ &\le h^{\ell}(\mu)g(\mathcal{H}(v))+m^{\ell}h^{\ell}(1-\mu)g(\mathcal{H}(u))\\ &\le \left(h^{\ell}(\mu)((g\circ\mathcal{H})(v)+m^{\ell}h^{\ell}(1-\mu)(g\circ\mathcal{H})(u)\right).\end{aligned}$$

□

Proposition 3.7. *Let $g,\mathcal{H}:\mathbb{B}\subseteq[0,b^*]\to\mathbb{R}^{\ell}$, $b^*>0$, be two similarly ordered functions which are generalized $(h-m)$-preinvex on $\mathbb{B}$, and that $h^{\ell}(\mu)+m^{\ell}h^{\ell}(1-\mu)\le 1$. Then the product of the functions g and $\mathcal{H}$ is a generalized $(h-m)$-preinvex w.r.t. σ.*

Proof. Since $\mathcal{H}$and g are similarly ordered functions on $\mathbb{B}$, then

$$(\mathcal{H}(u)-\mathcal{H}(v))(g(u)-g(v))\ge 0,\ \text{ for every } u,v\in\mathbb{B}.$$

For $m\in(0,1]$ and $\mu\in[0,1]$, we have

$$\begin{aligned}&g(um+\mu\sigma(v,um))\mathcal{H}(um+\mu\sigma(v,um))\\ &\le [h^{\ell}(\mu)g(v)+m^{\ell}h^{\ell}(1-\mu)g(u)][h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)]\\ &= (h^{\ell}(\mu))^2g(v)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)h^{\ell}(1-\mu)g(v)\mathcal{H}(u)+m^{\ell}h^{\ell}(\mu)h^{\ell}(1-\mu)g(u)\mathcal{H}(v)\\ &\quad +m^{2\ell}(h^{\ell}(1-\mu))^2\mathcal{H}(u)g(u)\\ &\le [(h^{\ell}(\mu))^2g(v)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)h^{\ell}(1-\mu)g(v)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)h^{\ell}(1-\mu)g(u)\mathcal{H}(u)\\ &\quad +m^{2\ell}(h^{\ell}(1-\mu))^2\mathcal{H}(u)g(u)]\\ &\le [h^{\ell}(\mu)g\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)g\mathcal{H}(u)][[h^{\ell}(\mu)+m^{\ell}h^{\ell}(1-\mu)]\\ &\le [h^{\ell}(\mu)g\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)g\mathcal{H}(u)].\end{aligned}$$

This completes the proof. □

4. HH Inequalities for Generalized $(h-m)$-Preinvex Function

We introduce some new variations of local fractional inequalities of Hermite–Hadamard-type via generalized $(h-m)$-preinvexity in this section.

Theorem 4.1. *Let $(0,1)\subseteq \mathbb{J}\subseteq\mathbb{R}$ and $h:\mathbb{J}\to\mathbb{R}$ be nonnegative integrable function, $h(\frac{1}{2})\neq 0$, and $\mathbb{J}$ be any interval in $\mathbb{R}$. Let $\mathbb{B}\subseteq\mathbb{R}$ be an open invex subset w.r.t. $\sigma:\mathbb{B}\times\mathbb{B}\to\mathbb{R}$, $\sigma(v,u)\geqslant 0$ satisfies Condition C, and, for $u,v\in\mathbb{B}$, $m\in(0,1]$, $um<um+\sigma(v,um)$. Let $\mathcal{H}:\mathbb{B}\to\mathbb{R}^{\ell}$, $\ell\in(0,1)$, be a local differentiable function with $\mathcal{H}^{(\ell)}\in I_y^{(\ell)}[um,um+\sigma(v,um)]$. If $\mathcal{H}$ is generalized $(h-m)$-preinvex on $[um,um+\sigma(v,um)]$, then*

$$\begin{aligned}
&\frac{1^{\ell}}{h^{\ell}\left(\frac{1}{2}\right)\Gamma(1+\ell)}\mathcal{H}\left(um+\frac{1}{2}\sigma(v,um)\right)\\
&\qquad\leq\frac{1}{\sigma^{\ell}(v,um)}\left[{}_{um+\sigma(v,um)}I_{um}^{(\ell)}\mathcal{H}(y)+m^{\ell}\,{}_{um}I_{um+\sigma(v,um)}^{(\ell)}\mathcal{H}(x)\right]\\
&\qquad\leq\left[\left[\mathcal{H}(v)+m^{2\ell}\ \mathcal{H}(u)\right]{}_0I_1^{(\ell)}h^{\ell}(1-\mu)+\left[\mathcal{H}(v)+\mathcal{H}(u)\right]m^{\ell}\,{}_0I_1^{(\ell)}h^{\ell}(\mu)\right].
\end{aligned}\tag{4.1}$$

Proof. Since $\mathcal{H}$ is generalized $(h-m)$-preinvex, by Remark 3.3, we have

$$\mathcal{H}\left(am+\frac{1}{2}\sigma(b,am)\right)\leq h^{\ell}\left(\frac{1}{2}\right)\left[\mathcal{H}(b)+m^{\ell}\mathcal{H}(a)\right].$$

Set $am=um+\mu\sigma(v,um)$ and $b=um+(1-\mu)\sigma(v,um))$. Since σ satisfies Condition C, we have

$$\sigma(um+(1-\mu)\sigma(v,um),um+\mu\sigma(v,um))=(1-2\mu)\sigma(v,um).$$

Thus $am+\frac{1}{2}\sigma(b,am)=um+\frac{1}{2}\sigma(v,um)$. Consequently,

$$\mathcal{H}\left(um+\frac{1}{2}\sigma(v,um)\right)\leq h^{\ell}\left(\frac{1}{2}\right)\left[\mathcal{H}(um+(1-\mu)\sigma(v,um))+m^{\ell}\mathcal{H}(um+\mu\sigma(v,um))\right].$$

By local fractional integration w.r.t. μ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h^{\ell}\left(\frac{1}{2}\right)\Gamma(1+\ell)}\int_0^1\mathcal{H}\left(um+\frac{1}{2}\sigma(v,um)\right)(d\mu)^{\ell}\\
&\leq\frac{1}{\Gamma(1+\ell)}\int_0^1\mathcal{H}(um+(1-\mu)\sigma(v,um))(d\mu)^{\ell}\\
&\quad+m^{\ell}\frac{1}{\Gamma(1+\ell)}\int_0^1\mathcal{H}(um+\mu\sigma(v,um))(d\mu)^{\ell}.
\end{aligned}\tag{4.2}$$

From

$$\frac{1}{\Gamma(1+\ell)}\int_0^1\mathcal{H}(um+\mu\sigma(v,um))(d\mu)^{\ell}=\frac{1}{\Gamma(1+\ell)}\int_{um}^{um+\sigma(v,um)}\mathcal{H}(x)\frac{(dx)^{\ell}}{\sigma^{\ell}(v,um)}$$

and

$$\frac{1}{\Gamma(1+\ell)}\int_0^1\mathcal{H}(um+(1-\mu)\sigma(v,um))(d\mu)^{\ell}=\frac{1}{\Gamma(1+\ell)}\int_{um}^{um+\sigma(v,um)}\mathcal{H}(y)\frac{(dy)^{\ell}}{\sigma^{\ell}(v,um)},$$

we see that inequality (4.2) becomes

$$\begin{aligned}
&\frac{1^{\ell}}{h^{\ell}\left(\frac{1}{2}\right)\Gamma(1+\ell)}\mathcal{H}\left(um+\frac{1}{2}\sigma(v,um)\right)\\
&\leq \frac{1}{\sigma^{\ell}(v,um)}\left[\frac{1}{\Gamma(1+\ell)}\int_{um+\sigma(v,um)}^{um}\mathcal{H}(y)(dy)^{\ell}\right.\\
&\qquad\left.+m^{\ell}\frac{1}{\Gamma(1+\ell)}\int_{um}^{um+\sigma(v,um)}\mathcal{H}(x)(dx)^{\ell}\right]\\
&\leq \frac{1}{\sigma^{\ell}(v,um)}\left[{}_{um+\sigma(v,um)}I_{um}^{(\ell)}\mathcal{H}(y)+m^{\ell}\,{}_{um}I_{um+\sigma(v,um)}^{(\ell)}\mathcal{H}(x)\right].
\end{aligned}$$

As a result, the first half of the inequality is established. We employ the generalized $(h-m)$-preinvexity of $\mathcal{H}$ on fractal sets to show the second part. Observe that

$$\begin{aligned}
&\mathcal{H}(um+(1-\mu)\sigma(v,um))+m^{\ell}\mathcal{H}(um+\mu\sigma(v,um))\\
&\leq h^{\ell}(1-\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)\mathcal{H}(u)+m^{\ell}[h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)].\\
&=h^{\ell}(1-\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)\mathcal{H}(u)+m^{\ell}h^{\ell}(\mu)\mathcal{H}(v)+m^{2\ell}h^{\ell}(1-\mu)\mathcal{H}(u).
\end{aligned}$$

Integrating local fractionally over $[0,1]$, one has

$$\begin{aligned}
&\frac{1^{\ell}}{\sigma^{\ell}(v,um)}\left[{}_{um+\sigma(v,um)}I_{um}^{(\ell)}\mathcal{H}(y)+m^{\ell}\,{}_{um}I_{um+\sigma(v,um)}^{(\ell)}\mathcal{H}(x)\right]\\
&\leq[\mathcal{H}(v)+m^{2\ell}\mathcal{H}(u)]\frac{1^{\ell}}{\Gamma(1+\ell)}\int_0^1 h^{\ell}(1-\mu)(d\mu)^{\ell}\\
&+[\mathcal{H}(v)+\mathcal{H}(u)]\frac{m^{\ell}}{\Gamma(1+\ell)}\int_0^1 h^{\ell}(\mu)(d\mu)^{\ell}.
\end{aligned}$$

Hence the proof is done. □

Theorem 4.2. *Let $\mathcal{H}:\mathbb{B}\to\mathbb{R}^{\ell}$, $\ell\in(0,1)$, be a local differentiable function with $\sigma(v,um)\geq 0$, with $u,v\in\mathbb{B}, um<um+\sigma(v,um)$ and $m\in(0,1]$. If $0\leq u<v<\infty$ and $\mathcal{H}\in I_y^{(\ell)}[um,um+\sigma(v,um)]$ be generalized $(h-m)$-preinvex, then*

$$\begin{aligned}
&\frac{1^{\ell}}{\sigma^{\ell}(v,um)}\left[{}_{um}I_{u+\sigma(v,um)}^{(\ell)}\mathcal{H}(x)+{}_{u+\sigma(v,um)}I_{um}^{(\ell)}\mathcal{H}(y)\right]\\
&\leq\Big[\left[\mathcal{H}(v)+m^{\ell}\mathcal{H}(u)\right]\left[{}_0I_1^{(\ell)}h^{\ell}(\mu)+{}_0I_1^{(\ell)}h^{\ell}(1-\mu)\right].
\end{aligned}$$

Proof. Using the generalized $(h-m)$-preinvexity of $\mathcal{H}$, we have

$$\mathcal{H}(um+\mu\sigma(v,um))\leq h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)$$

and

$$\mathcal{H}(um+(1-\mu)\sigma(v,um))\leq h^{\ell}(1-\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)\mathcal{H}(u)).$$

Adding the above two inequalities yields

$$\begin{aligned}
&\mathcal{H}(um+\mu\sigma(v,um))+\mathcal{H}(um+(1-\mu)\sigma(v,um))\\
&\leq h^{\ell}(\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(1-\mu)\mathcal{H}(u)+h^{\ell}(1-\mu)\mathcal{H}(v)+m^{\ell}h^{\ell}(\mu)\mathcal{H}(u))\\
&\leq[\mathcal{H}(v)+m^{\ell}\mathcal{H}(u)][h^{\ell}(\mu)+h^{\ell}(1-\mu)].
\end{aligned}\tag{4.3}$$

On integrating local fractionally, we obtain the desired result immediately. □

Lemma 4.3. *Let $\mathbb{B} \subseteq [0, b^*]$ be an open invex subset with $\sigma : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$, $u, v \in \mathbb{B}$, $m \in (0,1]$ with $um < um + \sigma(v, um)$, and $\sigma(v, u) \geqslant 0$. Let $\mathcal{H} : \mathbb{B} \to \mathbb{R}^{\ell}(\ell \in (0,1))$ be a local differentiable function with $\mathcal{H}^{(\ell)} \in I_y^{(\ell)}[um, um + \sigma(v, um)]$. For $0 \le \kappa \le 1$ and $x \in [um, um + \sigma(v, um)]$, the following inequality holds:*

$$\begin{aligned}
&(\kappa-1)^{\ell}\mathcal{H}\left(um+\frac{1}{2}\sigma(v,um)\right)-\kappa^{\ell}\,\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\\
&+\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\\
&=\frac{\sigma^{\ell}(v,um)}{2^{\ell}}\left[\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}(\kappa-2t)^{\ell}\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))(d\mu)^{\ell}\right.\\
&\left.+\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}(2-\kappa-2t)^{\ell}\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))(d\mu)^{\ell}\right].
\end{aligned}\tag{4.4}$$

Proof. First, note that $um + \mu\sigma(v, um) \in I$ for $\mu \in (0,1]$, and $u, v \in I$ because I is invex w.r.t. σ. Applying local fractional integration by parts, then setting $x = um + \mu\sigma(v, um)$, we arrive at

$$\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}(\kappa-2t)^{\ell}\mathcal{H}^{(\ell)}\left(um+\mu\sigma(v,um)(d\mu)^{\ell}\right.\\
&=\frac{1}{\sigma^{\ell}(v,um)}\left[(\kappa-2t)^{\ell}\mathcal{H}(um+\mu\sigma(v,um))^{\ell}\Big|_0^{1/2}+2^{\ell}\Gamma(1+\ell)\right.\\
&\left.\times\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}\mathcal{H}\left(um+\mu\sigma(v,um)\right)(d\mu)^{\ell}\right]\\
&=\frac{1}{\sigma^{\ell}(v,um)}\left[(\kappa-1)^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^{\ell}\mathcal{H}(um)\right.\\
&\left.+2^{\ell}\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\frac{1}{2}\sigma(v,um)}\mathcal{H}(x)\right].
\end{aligned}\tag{4.5}$$

Similarly,

$$\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}(2-\kappa-2\mu)^{\ell}\mathcal{H}^{(\ell)}\left(um+\mu\sigma(v,um)\right)(d\mu)^{\ell}\\
&=\frac{1}{\sigma^{\ell}(v,um)}\left[(2-\kappa-2t)^{\ell}\mathcal{H}(um+\mu\sigma(v,um))\Big|_{1/2}^{1}+2^{\ell}\Gamma(1+\ell)\right.\\
&\left.\times\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}\mathcal{H}(um+\mu\sigma(v,um))(d\mu)^{\ell}\right]\\
&=\frac{1}{\sigma^{\ell}(v,um)}\left[(\kappa-1)^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^{\ell}\mathcal{H}(um+\sigma(v,um))\right.\\
&\left.+2^{\ell}\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um+\frac{1}{2}\sigma(v,um)}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\right].
\end{aligned}\tag{4.6}$$

From (4.5) and (4.6), we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}(\kappa-2t)^\ell\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))(d\mu)^\ell+\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}(2-\kappa-2t)^\ell\\
&\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))(d\mu)^\ell\\
&=\frac{1}{\sigma^\ell(v,um)}\Big[2^\ell(\kappa-1)^\ell\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^\ell[\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um)]\\
&+\frac{2^\ell\Gamma(1+\ell)}{\sigma^\ell(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big].
\end{aligned}
$$

□

In light of Lemma 4.3, we now introduce some new generalizations of the Hermite–Hadamard-type inequality for generalized $(h-m)$-preinvex functions.

Theorem 4.4. *Let the hypotheses of Lemma 4.3 hold. If $\left|\mathcal{H}^{(\ell)}\right|^q$ is a generalized $(h-m)$-preinvex function on I, then*

$$
\begin{aligned}
&\Big|(\kappa-1)^\ell\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^\ell\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um)}{2^\ell}\\
&+\frac{\Gamma(1+\ell)}{\sigma^\ell(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big|\\
&\le\left|\frac{\sigma^\ell(v,um)}{2^\ell}\right|\left[\frac{\Gamma(1+p\ell)}{2^\ell\Gamma(1+(1+p)\ell)}\left[\kappa^{(1+p)\ell}+(1-\kappa)^{(1+p)\ell}\right]\right]^{1/p}\\
&\times\left[\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}\left[h^\ell(\mu)|\mathcal{H}^{(\ell)}(v)|^q+m^\ell h^\ell(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^q\right](d\mu)^\ell\right)^{1/q}\right.\\
&\left.+\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}\left[h^\ell(\mu)|\mathcal{H}^{(\ell)}(v)|^q+m^\ell h^\ell(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^q\right](d\mu)^\ell\right)^{1/q}\right].
\end{aligned}
$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus of (4.4) then applying the Generalized Hölders Inequality, we have

$$
\begin{aligned}
&\Big|(\kappa-1)^\ell\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^\ell\,\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^\ell}\\
&\qquad+\frac{\Gamma(1+\ell)}{\sigma^\ell(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\Bigg[\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|(\kappa-2\mu)^{p\ell}(d\mu)^{\ell}\right)^{1/p}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))|^{q}(d\mu)^{\ell}\right)^{1/q}\\
&+\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}\left|(2-\kappa-2\mu)^{p\ell}\right|(d\mu)^{\ell}\right)^{1/p}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))|^{q}(d\mu)^{\ell}\right)^{1/q}\Bigg]. \qquad (4.7)
\end{aligned}
$$

From Lemma 2.5, we see that

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{p\ell}(d\mu)^{\ell}=\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{p\ell}(d\mu)^{\ell}\\
&=\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\left[\kappa^{(1+p)\ell}+(1-\kappa)^{(1+p)\ell}\right]. \qquad (4.8)
\end{aligned}
$$

Since $\left|\mathcal{H}^{(\ell)}\right|^{q}$ is a generalized $(h-m)$-preinvex, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))|^{q}(d\mu)^{\ell}\\
&\leq\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}\left[h^{\ell}(\mu)|\mathcal{H}^{(\ell)}(v)|^{q}+m^{\ell}h^{\ell}(1-\mu)|\mathcal{H}^{(\ell)}(u)|^{q}\right](d\mu)^{\ell}. \qquad (4.9)
\end{aligned}
$$

By the substitution of (4.8) and (4.9) in (4.7), we see that inequality (4.7) is reduced to

$$
\begin{aligned}
&\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\Bigg[\left(\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\left[\kappa^{(1+p)\ell}+(1-\kappa)^{(1+p)\ell}\right]\right)^{1/p}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}[h^{\ell}(\mu)\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+m^{\ell}h^{\ell}(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^{q}](d\mu)^{\ell}\right)^{1/q}\\
&+\left(\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\left[\kappa^{(1+p)\ell}+(1-\kappa)^{(1+p)\ell}\right]\right)^{1/p}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}[h^{\ell}(\mu)\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+m^{\ell}h^{\ell}(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^{q}](d\mu)^{\ell}\right)^{1/q}\Bigg]\\
&\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\left(\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\left[\kappa^{(1+p)\ell}+(1-\kappa)^{(1+p)\ell}\right]\right)^{1/p}\\
&\times\Bigg[\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}\left[h^{\ell}(\mu)\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+m^{\ell}h^{\ell}(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right](d\mu)^{\ell}\right)^{1/q}\\
&+\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}\left[h^{\ell}(\mu)\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+m^{\ell}h^{\ell}(1-\mu)\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right](d\mu)^{\ell}\right)^{1/q}\Bigg].
\end{aligned}
$$

Hence the desired result is proved. $\square$

Corollary 4.5 (Mid Point Inequality). *If $\kappa = 0$ and $h(\mu) = \mu$ in Theorem 4.4, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x) - \mathcal{H}(um+\frac{1}{2}\sigma(v,um))\right| \\
&\leq \frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\left[\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\right]^{1/p}\left[\frac{\Gamma(1+\ell)}{4^{\ell}\Gamma(1+2\ell)}\right]^{1/q} \\
&\times\left[\left(\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + 3^{\ell}m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q} + \left(3^{\ell}\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q}\right].
\end{aligned}$$

Corollary 4.6 (Trapezoid Inequality). *If $\kappa = 1$ and $h(\mu) = \mu$ in Theorem 4.4, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x) - \frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\right| \\
&\leq \frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\left[\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\right]^{1/p}\left[\frac{\Gamma(1+\ell)}{4^{\ell}\Gamma(1+2\ell)}\right]^{1/q} \\
&\times\left[\left(\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + 3^{\ell}m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q} + \left(3^{\ell}\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q}\right].
\end{aligned}$$

Corollary 4.7 (Simpson Inequality). *If $\kappa = \frac{1}{3}$ and $h(\mu) = \mu$ in Theorem 4.4, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,u)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x) - \left(\frac{1}{3}\right)^{\ell}\left[2^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um)) + \frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(u,um))}{2^{\ell}}\right]\right| \\
&\leq \frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\left[\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\left(\left(\frac{1}{3}\right)^{(1+p)\ell} + \left(\frac{2}{3}\right)^{(1+p)\ell}\right)\right]^{1/p}\left[\frac{\Gamma(1+\ell)}{4^{\ell}\Gamma(1+2\ell)}\right]^{1/q} \\
&\times\left[\left(\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + 3^{\ell}m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q} + \left(3^{\ell}\left|\mathcal{H}^{(\ell)}(v)\right|^{q} + m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q}\right].
\end{aligned}$$

Theorem 4.8. *Let the assumptions of Lemma 4.3 be satisfied. If $|\mathcal{H}^{(\ell)}|^q$ is a generalized $(h-m)$-preinvex function on I for $q \geq 1$, then*

$$\begin{aligned}
&\Big|(\kappa-1)^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^{\ell}\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\\
&\qquad\qquad\qquad+\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big|\\
&\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\left(\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\left[\kappa^{2\ell}+(1-\kappa)^{2\ell}\right]\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg[\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}h^{\ell}(1-\mu)(d\mu)^{\ell}\right)\times\left|\mathcal{H}^{(\ell)}(um)\right|^q\\
&+\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}h^{\ell}(\mu)(d\mu)^{\ell}\right)\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^q\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\left(\frac{1}{\Gamma(1+\ell)}\int_1^{1/2}|2-\kappa-2\mu|^{\ell}h^{\ell}(1-\mu)(d\mu)^{\ell}\right)\times\left|\mathcal{H}^{(\ell)}(um)\right|^q\\
&+\left(\frac{1}{\Gamma(1+\ell)}\int_1^{1/2}|2-\kappa-2\mu|^{\ell}h^{\ell}(\mu)(d\mu)^{\ell}\right)\times m^{\ell}|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)|^q\Bigg]^{1/q}\Bigg).
\end{aligned}$$

Proof. Taking the modulus of the right side of equation (4.4) and applying the Power-Mean Inequality, we obtain

$$\begin{aligned}
&\Big|(\kappa-1)^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^{\ell}\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\\
&\qquad\qquad\qquad+\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big| \qquad (4.10)\\
&\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\Bigg[\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}(\kappa-2\mu)^{\ell}|\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|(d\mu)^{\ell}\\
&+\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}(2-\kappa-2\mu)^{\ell}|\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|(d\mu)^{\ell}\Bigg]\\
&\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}}\Bigg[\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}(d\mu)^{\ell}\right)^{1-1/q}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|^q(d\mu)^{\ell}\right)^{1/q}\\
&+\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}(d\mu)^{\ell}\right)^{1-1/q}\\
&\times\left(\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|^q(d\mu)^{\ell})\right)^{1/q}\Bigg]. \qquad (4.11)
\end{aligned}$$

From Lemma 2.5, it is easy to see that

$$\begin{aligned}\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}\left|\kappa-2\mu\right|^{\ell}(d\mu)^{\ell} &= \frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}\left|2-\kappa-2\mu\right|^{\ell}(d\mu)^{\ell} \\ &= \frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\left[\kappa^{2\ell}+(1-\kappa)^{2\ell}\right].\end{aligned}$$

Moreover, Condition C implies that

$$\sigma(a,a+\sigma(b,a)) = -\sigma(b,a). \tag{4.12}$$

Hence, as $|\mathcal{H}^{(\ell)}|^q$ is generalized $(h-m)$-preinvex, from equation (4.12), we have

$$\begin{aligned}\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|^q &= \left|\mathcal{H}^{(\ell)}\left(um+\sigma(v,um)+(1-\mu)\sigma(um,um+\sigma(v,um))\right)\right|^q \\ &\le h^{\ell}(1-\mu)\left|\mathcal{H}^{(\ell)}(um)\right|^q+m^{\ell}h^{\ell}(\mu)\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^q.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}\left|\mathcal{H}^{(\ell)}\left(um+\mu\sigma\left(v,um\right)\right)\right|^q(d\mu)^{\ell} \\ &\le \frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}h^{\ell}(1-\mu)(d\mu)^{\ell}\times\left|\mathcal{H}^{(\ell)}(um)\right|^q \\ &+\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}h^{\ell}(\mu)(d\mu)^{\ell}\times m^{\ell}|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)|^q,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}\left|\mathcal{H}^{(\ell)}(um+\mu\sigma(v,um))\right|^q(d\mu)^{\ell} \\ &\le \frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}h^{\ell}(1-\mu)(d\mu)^{\ell}\times\left|\mathcal{H}^{(\ell)}(um)\right|^q \\ &+\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}h^{\ell}(\mu)(d\mu)^{\ell}\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^q.\end{aligned}$$

By direct calculation we obtain the desired inequality immediately. □

Corollary 4.9. *If $h(\mu)=\mu$ in Theorem 4.8, then*

$$\begin{aligned}
&\Big|(\kappa-1)^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))-\kappa^{\ell}\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\\
&\qquad\qquad\qquad+\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)\Big|\\
&\le\frac{\left|\sigma^{\ell}(v,um)\right|}{2^{\ell}}\left(\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\left[\kappa^{2\ell}+(1-\kappa)^{2\ell}\right]\right)^{1-\frac{1}{q}}\\
&\times\left[M_1(\kappa,\ell)\times\left|\mathcal{H}^{(\ell)}(um)\right|^{q}+M_2(\kappa,\ell)\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^{q}\right]^{1/q}\\
&+\left[M_3(\kappa,\ell)\times\left|\mathcal{H}^{(\ell)}(um)\right|^{q}+M_4(\kappa,\ell)\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^{q}\right]^{1/q},
\end{aligned}$$

where,

$$\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}(1-\mu)^{\ell}(d\mu)^{\ell}\\
&=\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\left[\kappa^{2\ell}+(1-\kappa)^{2\ell}\right]-\frac{\kappa^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}\left[\left(\frac{\kappa^2}{2}\right)^{\ell}-\left(\frac{1}{4}\right)^{\ell}\right]\\
&+\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\left[\left(\frac{\kappa^3}{2}\right)^{\ell}-\left(\frac{1}{4}\right)^{\ell}\right]:=M_1(\kappa,\ell).\\
&\frac{1}{\Gamma(1+\ell)}\int_0^{1/2}|\kappa-2\mu|^{\ell}t^{\ell}(d\mu)^{\ell}\\
&=\frac{\kappa^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}\left[\left(\frac{\kappa^2}{2}\right)^{\ell}-\left(\frac{1}{4}\right)^{\ell}\right]-\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\left[\left(\frac{\kappa^3}{2}\right)^{\ell}-\left(\frac{1}{4}\right)^{\ell}\right]\\
&:=M_2(\kappa,\ell),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}(1-\mu)^{\ell}(d\mu)^{\ell}\\
&=\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\left[\kappa^{2\ell}+(1-\kappa)^{2\ell}\right]-\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\left[\left(\frac{9}{4}\right)^{\ell}-\frac{(2-\kappa)^{3\ell}}{2^{\ell}}\right]:=M_3(\kappa,\ell),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\ell)}\int_{1/2}^{1}|2-\kappa-2\mu|^{\ell}t^{\ell}(d\mu)^{\ell}\\
&=(2-\kappa)^{\ell}\frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)}\left[\frac{(2-\kappa)^{2\ell}}{2^{\ell}}-\left(\frac{5}{4}\right)^{\ell}\right]+\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\left[\left(\frac{9}{4}\right)^{\ell}-\frac{(2-\kappa)^{3\ell}}{2^{\ell}}\right]:=M_4(\kappa,\ell).
\end{aligned}$$

By assigning particular values to the parameters κ and $h(\mu)$, we present some remarkable cases of Corollary 4.9 as corollaries and remarks that are vital and essential in analysis and numerical integration.

Remark 4.10. If $m=1$ in Corollary 4.9, we obtain [29, Theorem 5]

Corollary 4.11 (Mid Point Inequality)**.** *If $\kappa = 0$ in Corollary 4.9, then*

$$
\begin{aligned}
&\Big|\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)-\mathcal{H}(um+\frac{1}{2}\sigma(v,um))\Big|\\
&\leq \frac{\left|\sigma^{\ell}(v,um)\right|}{2^{\ell}}\left(\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\frac{\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}-\frac{\Gamma(1+2\ell)}{4^{\ell}\Gamma(1+3\ell)}\right)\left|\mathcal{H}^{(\ell)}(um)\right|^{q}\\
&+\frac{\Gamma(1+2\ell)}{4^{\ell}\Gamma(1+3\ell)}\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^{q}\Bigg]^{1/q}\\
&+\Bigg[\left(\frac{7^{\ell}\Gamma(1+2\ell)}{4^{\ell}\Gamma(1+3\ell)}-\frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)}\right)\times|\mathcal{H}^{(\ell)}(um)|^{q}\\
&+\left(\frac{3^{\ell}\Gamma(1+\ell)}{2^{\ell}\Gamma(1+2\ell)}-\frac{7^{\ell}\Gamma(1+2\ell)}{4^{\ell}\Gamma(1+3\ell)}\right)\times m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^{q}\Bigg]^{1/q}.
\end{aligned}
$$

Remark 4.12. If $m=1$ in Corollary 4.11 we have [29, Corollary 5].

Corollary 4.13 (Trapezoid Inequality)**.** *If $\kappa = 1$ in Corollary 4.9, then*

$$
\begin{aligned}
&\Big|\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I^{(\ell)}_{um+\sigma(v,um)}\mathcal{H}(x)-\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\Big|\\
&\leq \frac{|\sigma(v,um)|^{\ell}}{4^{\ell}\left(2^{\ell}\right)^{\frac{1}{q}}}\left(\frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)}\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg[\left(\frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)}+\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell))}\right)\times|\mathcal{H}^{(\ell)}(um)|^{q}\\
&+\left(\frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)}-\frac{\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\right)m^{\ell}\left|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)\right|^{q}\Bigg]^{1/q}\\
&+\Bigg[\left(\frac{5^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}-\frac{7^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\right)\times|\mathcal{H}^{(\ell)}(um)|^{q}\\
&+\left(\frac{7^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}-\frac{3^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}\right)m^{\ell}|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)|^{q}\Bigg]^{1/q}\Bigg).
\end{aligned}
$$

Remark 4.14. If $m=1$ in Corollary 4.13 we have [29, Corollary 6].

Corollary 4.15 (Simpson Inequality)**.** *If $\kappa = \frac{1}{3}$ in Corollary 4.9, then*

$$\left|\left(\frac{1}{3}\right)^{\ell}\left[2^{\ell}\mathcal{H}(um+\frac{1}{2}\sigma(v,um))+\frac{\mathcal{H}(um)+\mathcal{H}(um+\sigma(v,um))}{2^{\ell}}\right]\right.$$
$$\left.-\frac{\Gamma(1+\ell)}{\sigma^{\ell}(v,um)}{}_{um}I_{um+\sigma(v,um)}\mathcal{H}(x)\right|$$
$$\leq\frac{|\sigma^{\ell}(v,um)|}{2^{\ell}(108^{\ell})^{1/q}}\left(\frac{5^{\ell}\Gamma(1+\ell)}{18^{\ell}\Gamma(1+2\ell)}\right)^{1-\frac{1}{q}}\times\left(\left(\left(\frac{37^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}-\frac{25^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\right)\right.\right.$$
$$\left.\times|\mathcal{H}^{(\ell)}(um)|^{q}+\left(\frac{25^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}-\frac{7^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}\right)\times m^{\ell}|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)|^{q}\right)^{1/q}$$
$$+\left(\frac{5^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}+\frac{7^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\right)\times|\mathcal{H}^{(\ell)}(um)|^{q}$$
$$\left.\left.+\left(\frac{25^{\ell}\Gamma(1+\ell)}{\Gamma(1+2\ell)}-\frac{7^{\ell}\Gamma(1+2\ell)}{\Gamma(1+3\ell)}\right)m^{\ell}|\mathcal{H}^{(\ell)}((um+\sigma(v,um))/m)|^{q}\right]^{1/q}\right).$$

Remark 4.16. If $m=1$ in Corollary 4.15, we obtain [29, Corollary 7].

5. Applications to Numerical Quadrature Rule

To examine error estimates of quadrature rules, we explore applications of the local fractional integral inequalities obtained in this work.

Consider the division of the interval $[um,v]$, $0<u<v$, given by $I_n: um=\mu_0<\mu_1<\cdots<\mu_n=v$. The formula of numerical quadrature connected to local fractional integral is given by

$$ {}_{um}I_{v}^{(\ell)}\mathcal{H}(x)=Q(f,I_n)+E(f,I_n), \tag{5.1}$$

where $Q(f,I_n)$ is the ${}_{um}I_{v}^{(\ell)}\mathcal{H}(x)$, local fractional integral approximation, and the remainder term $E(f,I_n)$ is the approximation error.

Corollary 5.1. *Assume the hypotheses of Corollary 4.5 with $\sigma(v,um)=v-um$, and $u<v$. Then*

$$\left|\frac{\Gamma(1+\ell)}{(v-um)^{\ell}}{}_{um}I_{v}^{(\ell)}\mathcal{H}(x)-\mathcal{H}\left(\frac{v+um}{2}\right)\right|$$
$$\leq\frac{|v-um|^{\ell}}{2^{\ell}}\left(\frac{\Gamma(1+p\ell)}{2^{\ell}\Gamma(1+(1+p)\ell)}\right)^{\frac{1}{p}}\times\left(\frac{\Gamma(1+\ell)}{4^{\ell}\Gamma(1+2\ell)}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+3^{\ell}m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q}+\left(3^{\ell}\left|\mathcal{H}^{(\ell)}(v)\right|^{q}+m^{\ell}\left|\mathcal{H}^{(\ell)}(u)\right|^{q}\right)^{1/q}\right],$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proposition 5.2. *Let the hypotheses of Corollary 5.1 hold. Then the remainder term $E(f,I_n)$ in (5.1) satisfies*

$$
\begin{aligned}
|E(f,I_n)| &= \left| {}_{um}I_v^{(\ell)}\mathcal{H}(x) - Q(f,I_n) \right| \\
&\leq \left(\frac{\Gamma(1+p\ell)}{2^\ell\Gamma(1+(1+p)\ell)} \right)^{\frac{1}{p}} \times \left(\frac{\Gamma(1+\ell)}{4^\ell\Gamma(1+2\ell)} \right)^{\frac{1}{q}} \\
&\quad \times \sum_{i=0}^{n-1} \frac{(\mu_{i+1}-\mu_i)^{2\ell}}{2^\ell\Gamma(1+\ell)} \left[\left(\left| \mathcal{H}^{(\ell)}(\mu_{i+1}) \right|^q + 3^\ell m^\ell \left| \mathcal{H}^{(\ell)}(\mu_i) \right|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(3^\ell \left| \mathcal{H}^{(\ell)}(\mu_{i+1}) \right|^q + m^\ell \left| \mathcal{H}^{(\ell)}(\mu_i) \right|^q \right)^{\frac{1}{q}} \right],
\end{aligned}
$$

for every division I_n of $[um,v]$, for $p,q>1, \frac{1}{p}+\frac{1}{q}=1, Q(f,I_n)$ is the midpoint quadrature formula, and

$$
Q(f,I_n) = \frac{1}{\Gamma(1+\ell)} \sum_{i=0}^{n-1} \mathcal{H}\left(\frac{\mu_i+\mu_{i+1}}{2} \right) (\mu_{i+1}-\mu_i)^\ell .
$$

Proof. Applying Corollary 5.1 on the interval $[\mu_i,\mu_{i+1}]\,(i=0,\ldots,n-1)$, we obtain

$$
\begin{aligned}
&\left| {}_{\mu_i}I_{\mu_{i+1}}^{(\ell)}\mathcal{H}(x) - \frac{1}{\Gamma(1+\ell)}\mathcal{H}\left(\frac{\mu_i+\mu_{i+1}}{2} \right) \times (\mu_{i+1}-\mu_i)^\ell \right| \\
&\leq \frac{(\mu_{i+1}-\mu_i)^{2\ell}}{2^\ell\Gamma(1+\ell)} \left(\frac{\Gamma(1+p\ell)}{2^\ell\Gamma(1+(1+p)\ell)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+\ell)}{4^\ell\Gamma(1+2\ell)} \right)^{\frac{1}{q}} \\
&\times \left[\left(\left| \mathcal{H}^{(\ell)}(\mu_{i+1}) \right|^q + 3^\ell m^\ell \left| \mathcal{H}^{(\ell)}(\mu_i) \right|^q \right)^{\frac{1}{q}} + \left(3^\ell \left| \mathcal{H}^{(\ell)}(\mu_{i+1}) \right|^q + m^\ell \left| \mathcal{H}^{(\ell)}(\mu_i) \right|^q \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Taking the sum over i from 0 to $n-1$ and applying the triangle inequality, we obtain the desired estimation immediately. □

6. Conclusion

In this paper, we defined generalized $(h-m)$-preinvex functions on fractal sets and provided some algebraic properties. Utilizing the new concept, we derived HH-type inequalities for the latest function, provided a new identity for generalized $(h-m)$-preinvex function with variables related to local fractional integrals, and derived some general local fractional integral inequalities for generalized $(h-m)$-preinvex functions. Certain generalized midpoint-type, trapezoidal-type, and Simpson-type inequalities were obtained by employing particular values for the parameters. In addition, numerical integration error estimates for local fractional integrals were provided as applications.

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