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POSITIVE SOLUTIONS FOR A NEW SYSTEM OF HADAMARD FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS ON AN INFINITE INTERVAL

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Abstract. We study a new system of Hadamard fractional integro-differential equations, supplemented with multi-point boundary conditions which involve finite order Hadamard fractional derivatives on an infinite interval. By applying a fixed point theorem in ordered Banach spaces, the existence and uniqueness of positive solutions for this system are obtained. Moreover, we give convergent sequences to approximate the unique solution. In addition, an interesting example is provided to illustrate our main result.

Keywords. Hadamard fractional derivative; Infinite interval; Integro-differential equation; Multi-point boundary conditions.

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1. Introduction

In this paper, we investigate the existence and uniqueness of positive solutions for the following new system of Hadamard fractional integro-differential equations with multi-point boundary conditions:

$$\begin{cases} D_{1^+}^{\alpha}x(t)+p_1(t)u(t,x(t),(Sy)(t))=0, \omega-1<\alpha\leq\omega, t\in(1,\infty),\\ D_{1^+}^{\beta}y(t)+p_2(t)v(t,x(t),(Qy)(t))=0, \mu-1<\beta\leq\mu, t\in(1,\infty),\\ x(1)=x'(1)=\cdots=x^{(\omega-2)}(1)=0, D_{1^+}^{\alpha-1}x(\infty)=\Sigma_{i=1}^{n_1}\eta_i D_{1^+}^{m_1}x(\omega_i),\\ y(1)=y'(1)=\cdots=y^{(\mu-2)}(1)=0, D_{1^+}^{\beta-1}y(\infty)=\Sigma_{j=1}^{n_2}\zeta_j D_{1^+}^{m_2}y(\mu_j), \end{cases} \tag{1.1}$$

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where $\omega,\mu \in N, \omega,\mu \geq 3, D^{v}_{1^+}$ is Hadamard fractional derivative(HFD for short) of order $v \in \{\alpha,\beta,m_1,m_2\}, m_1 \in [0,\alpha-1], m_2 \in [0,\beta-1]$,

$$(Sy)(t) = \int_1^t k(t,s)y(s)ds,\ (Qy)(t) = \int_1^\infty l(t,s)y(s)ds,$$

$k,l \in C([1,\infty)\times[1,\infty),[0,\infty)), \eta_i \geq 0 (i=1,2,\dots n_1), \zeta_j \geq 0 (j=1,2,\dots n_2), 1<\omega_1<\omega_2<\cdots<\omega_{n_1}<\infty$, and $1<\mu_1<\mu_2<\cdots<\mu_{n_2}<\infty$.

First, we list the following assumptions:

(A_1) $u,v \in C([1,\infty)\times[0,\infty)\times[0,\infty),[0,\infty)), u(t,0,0)\not\equiv 0$, and $v(t,0,0)\not\equiv 0$ on any subinterval of $[1,\infty)$; when x and y are bounded, $u(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y)$ and $v(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y)$ are bounded on $[1,\infty)$.

(A_2) $p_1,p_2 : [1,\infty)\to[0,\infty)$ are not identically zero on any closed subinterval of $[1,\infty)$ and

$$0<\int_1^\infty p_1(s)\frac{ds}{s}<\infty, 0<\int_1^\infty p_2(s)\frac{ds}{s}<\infty.$$

(A_3) $k^* := \sup\limits_{t\in[1,\infty)} \int_1^t k(t,s)ds<\infty, l^* := \sup\limits_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\beta-1}} \int_1^\infty l(t,s)[1+(\log s)^{\beta-1}]ds<\infty.$

Fractional calculus has existed for nearly three decades. Compared with integer-order differential equations, fractional-order differential equations have more significant advantages in describing real world problems, so they are quite useful in many disciplines. Fractional differential equations have been relatively mature, and various real-world applications were investigated.

For an in-depth understanding of the theory of fractional differential equations and its related applications, we refer to [8, 9, 10, 12, 13]. It is known that HFD is a special form of fractional derivative, which is different from Riemann-Liouville and Caputo fractional derivatives. In addition, the existence and uniqueness of solutions for fractional differential equations under boundary conditions has aroused great interest; see, e.g., [2, 3, 4, 6, 11, 14, 18, 19, 20] for some results on Hadamard fractional problems. Although most of the study of fractional calculus is concerned with finite intervals, the study on infinite intervals has developed considerably. In [5], Benhamida et al. studied the Hadamard fractional problem:

$$\begin{cases} D^r_{1^+}y(t) = f(t,y(t)), 1<r\leq 2, t\in J=[1,T], \\ y(1)=0, D^p_{1^+}y(T)=\sum_{i=1}^n \lambda_i D^p_{1^+}y(\mu_i), 0<p<1, \end{cases} \tag{1.2}$$

where $\mu_i \in [1,T], \lambda_i \in \mathbf{R}$ for $i=1,2,\dots n, n\geq 2, f:[1,T]\times\mathbf{R}\longrightarrow\mathbf{R}$. The existence and uniqueness of solutions for (1.2) was given by using Banach's fixed point theorem, Schaefer's fixed point theorem, and Leray Schauder nonlinear alternative.

In [16], Xu et al. studied a system of Hadamard fractional multi-point problem:

$$\begin{cases} D^q_{1^+}u(t)+f_1(t,u(t),v(t))=0, t\in(1,e), \\ D^q_{1^+}v(t)+f_2(t,u(t),v(t))=0, t\in(1,e), \\ u(1)=\delta u(1)=0, u(e)=\Sigma_{i=1}^{m-1}a_i u(\xi_i), \\ v(1)=\delta v(1)=0, v(e)=\Sigma_{j=1}^{n-1}b_j v(\eta_j), \end{cases} \tag{1.3}$$

where $q\in(2,3]$ is a real number, $D^q_{1^+}$ is the q order HFD, and δ means the delta derivative, i.e., $\delta u(1)=t\frac{du}{dt}|_{t=1}, \delta v(1)=t\frac{dv}{dt}|_{t=1}$. The constants $a_i,b_j,\xi_i,\eta_j (i=1,2,\dots,m-1, j=1,2,\dots,n-1, m,n\geq 2)$ and f_1,f_2 satisfy the conditions:

$(1) a_i,b_j\geq 0, \xi_i,\eta_j\in(1,e)$ with $\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}\in[0,1)$ and $\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}\in[0,1)$;

(2) $f_i \in C([1,e)\times \mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+), \mathbf{R}^+ = [0,\infty), i = 1,2.$

By using a generalization of the Leggett-Williams fixed point theorem, they obtained the existence of three positive solutions for (1.3).

In [15], Tariboon et al. studied the following fractional system of Hadamard differential equations subject to the fractional integral boundary conditions:

$$\begin{cases} D^p_{1^+}x(t)+a(t)f(t,x(t),y(t))=0, 1<p\le 2, t\in(1,\infty),\\ D^q_{1^+}y(t)+b(t)g(t,x(t),y(t))=0, 1<q\le 2, t\in(1,\infty),\\ x(1)=0, D^{p-1}_{1^+}x(\infty)=\sum_{i=1}^m \lambda_i I^{\alpha_i}y(\eta),\\ y(1)=0, D^{q-1}_{1^+}y(\infty)=\sum_{j=1}^n \sigma_j I^{\beta_j}x(\xi),\end{cases} \tag{1.4}$$

where $f,g\in C([1,\infty)\times\mathbf{R}^2_+,\mathbf{R}_+), \mathbf{R}_+=[0,\infty), I^{\alpha_i}, I^{\beta_i}$ are Hadamard fractional integrals of order $\alpha_i,\beta_i\ge 1, \lambda_i,\sigma_j>0, i=1,\dots m, j=1,\dots n$, and the fixed constants $0<\eta<\xi<1$. They discussed the existence of solutions for (1.4) via Guo-Krasnoselskii's and Leggett-Williams fixed point theorems.

In [22], Zhang and Ni investigated a Hadamard fractional differential equation with more general boundary conditions on infinite interval as follows:

$$\begin{cases} D^\alpha_{1+}x(t)+a(t)f(t,x(t))=0, t\in(1,+\infty),\\ x(1)=x'(1)=0, D^{\alpha-1}_{1+}x(+\infty)=\sum_{i=1}^m \alpha_i I^{\beta_i}_{1+}x(\eta)+\rho\sum_{j=1}^n \sigma_j x(\xi_j),\end{cases} \tag{1.5}$$

where $2<\alpha<3$; $I^{\beta_i}_{1+}$ is Hadamard-type fractional integral of order $\beta_i>0 (i=1,2,...,m)$; $1<\eta<\xi_1<\xi_2<\cdots<\xi_n<+\infty$; $\rho,\alpha_i,\sigma_j\ge 0 (i=1,2,...,m; j=1,2,...,n)$ are given constants with

$$\Gamma(\alpha)-\sum_{i=1}^m \alpha_i \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\ln\eta)^{\alpha+\beta_i-1}-\rho\sum_{j=1}^n \sigma_j(\ln\xi_j)^{\alpha-1} := \Delta>0,$$

$f(t,x): J\times\mathbf{R}^+\to\mathbf{R}^+, J=[1,\infty)$ satisfy the Carathéodory conditions. By using the generalized Avery-Henderson fixed point theorem, they gave the existence of three positive solutions for (1.5).

In this paper, motivated greatly by the results mentioned above, we consider the existence and uniqueness of positive solutions for system (1.1) by using a new method. To prove our main results, we present some definitions and related lemmas in Section 2. We also list some properties of the corresponding Green's function for system (1.1). In Section 3, we define some nonlinear operators, give some properties of these operators, and obtain some sufficient conditions for the existence and uniqueness of positive solutions for system (1.1) by using a recent fixed point theorem in ordered Banach spaces. To demonstrate our result, we give an interesting example in Section 4, the last section.

2. Preliminaries

In this section, we recall some basic concepts, notations, and related lemmas.

Definition 2.1. [1, 8] For a function c: $[1,\infty)\to\mathbf{R}$, the HFD of fractional order p is defined as

$$D^p_{1^+}c(t)=\frac{1}{\Gamma(n-p)}(t\frac{d}{dt})^n\int_1^t(\log\frac{t}{s})^{n-p-1}c(s)\frac{ds}{s},\ n-1<p<n,$$

where $[p]$ denotes the integer part of the real number $p, n=[p]+1$ and $\log(.)=\log_e(.)$.

Definition 2.2. [1, 8] For a function c: $[1,\infty)\to\mathbf{R}$, the Hadamard fractional integral of order p is defined as

$$I^p_{1^+}c(t)=\frac{1}{\Gamma(p)}\int_1^t(\log\frac{t}{s})^{p-1}c(s)\frac{ds}{s},v>0,$$

provided that the integral exists.

Lemma 2.3. [7] *Let $\gamma\in C([1,\infty))$ with $\int_1^\infty\gamma(s)\frac{ds}{s}<\infty$ and*

$$\Omega_1:=\Gamma(\alpha)-\sum_{i=1}^{n_1}\frac{\eta_i\Gamma(\alpha)}{\Gamma(\alpha-m_1)}(\log\omega_i)^{\alpha-m_1-1}>0.$$

Then, the unique solution of the following problem:

$$\begin{cases}D^\alpha_{1^+}x(t)+\gamma(t)=0,\omega-1<\alpha\le\omega,t\in(1,\infty),\\ x(1)=x'(1)=\cdots=x^{(\omega-2)}(1)=0,D^{\alpha-1}_{1^+}x(\infty)=\Sigma_{i=1}^{n_1}\eta_iD^{m_1}_{1^+}x(\omega_i),\end{cases}\tag{2.1}$$

is given by the integral equation

$$x(t)=\int_1^\infty H(t,s)\gamma(s)\frac{ds}{s},t\in[1,\infty),$$

where

$$H(t,s)=H_1(t,s)+\frac{(\log t)^{\alpha-1}}{\Omega_1}\sum_{i=1}^{n_1}\eta_iH_2(\omega_i,s),$$

$$H_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\log t)^{\alpha-1}-(\log\frac{t}{s})^{\alpha-1}, & 1\le s\le t<\infty,\\ (\log t)^{\alpha-1}, & 1\le t\le s<\infty,\end{cases}\tag{2.2}$$

and

$$H_2(\omega_i,s)=\frac{1}{\Gamma(\alpha-m_1)}\begin{cases}(\log\omega_i)^{\alpha-m_1-1}-(\log\frac{\omega_i}{s})^{\alpha-m_1-1}, & 1\le s\le\omega_i<\infty,\\ (\log\omega_i)^{\alpha-m_1-1}, & 1\le\omega_i\le s<\infty.\end{cases}\tag{2.3}$$

For $\phi\in C([1,\infty))$ with $\int_1^\infty\phi(s)\frac{ds}{s}<\infty$ and $\Omega_2:=\Gamma(\beta)-\Sigma_{j=1}^{n_2}\frac{\zeta_j\Gamma(\beta)}{\Gamma(\beta-m_2)}(\log\mu_j)^{\beta-m_2-1}>0$, the general solution of

$$\begin{cases}D^\beta_{1^+}y(t)+\phi(t)=0,\mu-1<\beta\le\mu,t\in(1,\infty),\\ y(1)=y'(1)=\cdots=y^{(\mu-2)}(1)=0,D^{\beta-1}_{1^+}y(\infty)=\Sigma_{j=1}^{n_2}\zeta_jD^{m_2}_{1^+}y(\mu_j),\end{cases}\tag{2.4}$$

can be written by

$$y(t)=\int_1^\infty H^*(t,s)\phi(s)\frac{ds}{s},t\in[1,\infty),$$

where

$$H^*(t,s)=H^*_1(t,s)+\frac{(\log t)^{\beta-1}}{\Omega_2}\sum_{j=1}^{n_2}\zeta_jH^*_2(\mu_j,s),$$

$$H^*_1(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}(\log t)^{\beta-1}-(\log\frac{t}{s})^{\beta-1}, & 1\le s\le t<\infty,\\ (\log t)^{\beta-1}, & 1\le t\le s<\infty,\end{cases}\tag{2.5}$$

and

$$H^*_2(\mu_j,s)=\frac{1}{\Gamma(\beta-m_2)}\begin{cases}(\log\mu_j)^{\beta-m_2-1}-(\log\frac{\mu_j}{s})^{\beta-m_2-1}, & 1\le s\le\mu_j<\infty,\\ (\log\mu_j)^{\beta-m_2-1}, & 1\le\mu_j\le s<\infty.\end{cases}\tag{2.6}$$

Then $K(t,s) = (H(t,s), H^*(t,s))$ is called the Green's function of the system (1.1).

Lemma 2.4. [7] *The functions $H(t,s), H^*(t,s)$ have the following properties:*
(1) $H(t,s), H^*(t,s)$ are continuous and $H(t,s), H^*(t,s) \geq 0$ for $(t,s) \in [1,\infty) \times [1,\infty)$;
(2) $H(t,s) \leq (\log t)^{\alpha-1} \frac{1}{\Omega_1}$ for all $(t,s) \in [1,\infty) \times [1,\infty)$;
(3) $H^*(t,s) \leq (\log t)^{\beta-1} \frac{1}{\Omega_2}$ for all $(t,s) \in [1,\infty) \times [1,\infty)$.

Lemma 2.5. *(1) $H(t,s) \geq \Sigma_{i=1}^{n_1} \eta_i H_2(\omega_i,s)(\log t)^{\alpha-1} \frac{1}{\Omega_1}$ for $(t,s) \in [1,\infty) \times [1,\infty)$;*
(2) $H^*(t,s) \geq \Sigma_{j=1}^{n_2} \zeta_j H_2(\mu_j,s)(\log t)^{\beta-1} \frac{1}{\Omega_2}$ for $(t,s) \in [1,\infty) \times [1,\infty)$.

Proof. (1) for $(t,s) \in [1,\infty) \times [1,\infty)$, we have $H_1(t,s) \geq 0$. Thus

$$\begin{aligned} H(t,s) &= H_1(t,s) + \sum_{i=1}^{n_1} \eta_i H_2(\omega_i,s)(\log t)^{\alpha-1} \frac{1}{\Omega_1} \\ &\geq \sum_{i=1}^{n_1} \eta_i H_2(\omega_i,s)(\log t)^{\alpha-1} \frac{1}{\Omega_1}. \end{aligned}$$

Similarly, we can obtain (2) easily. □

Next, we give abstract theory which plays crucial role in our discussion.

$(E, \|\cdot\|)$ is a Banach space and it is partially ordered by a cone $P \subset E$, i.e., $x \leq y$, if and only if $y - x \in P$. θ is the zero element in E. If there is a constant $N > 0$ such that, for $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, then P is called normal with $\inf\{N\}$, the normality constant of P. An operator $T : E \to E$ is increasing if $x \leq y$ implies $Tx \leq Ty$. For $x, y \in E$, the notation $x \sim y$ means the there exist $\varepsilon > 0, \varsigma > 0$ such that $\varepsilon x \leq y \leq \varsigma x$. Clearly, $\sim$ is a equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), define the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Let Φ denote the class of those functions $\varphi : (0,1) \to (0,1)$ which satisfied the conditions $\varphi(t) > t$, for $t \in (0,1)$.

Lemma 2.6. [21] *Let P be a normal cone in E and $h > \theta$. $T : P \to P$ is an increasing operator satisfying*
(i) there exists $h_0 \in P_h$ such that $Th_0 \in P_h$;
(ii) for any $z \in P$ and $t \in (0,1)$, there exists $\varphi \in \Phi$ such that $T(tz) \geq \varphi(t)Tz$.
Then
(1) T has a unique fixed z^* in P_h;
(2) for any initial value $z_0 \in P_h$, constructing successively the sequence $z_n = Tz_{n-1}, n = 1,2,\ldots,$ one has $z_n \to z^*$ as $n \to \infty$.

3. Main Results

In this section, we establish the existence and uniqueness of nontrivial solutions for system (1.1). For convenience, let

$$X = \{x \in C[1,\infty) : \sup_{t\in[1,\infty)} \frac{|x(t)|}{1 + (\log t)^{\alpha-1}} < \infty\}$$

and

$$Y = \{y \in C[1,\infty) : \sup_{t\in[1,\infty)} \frac{|y(t)|}{1 + (\log t)^{\beta-1}} < \infty\}.$$

For $x \in X, y \in Y$, define the norms:

$$\|x\|_X = \sup_{t\in[1,\infty)} \frac{|x(t)|}{1+(\log t)^{\alpha-1}}, \|y\|_Y = \sup_{t\in[1,\infty)} \frac{|y(t)|}{1+(\log t)^{\beta-1}}$$

and $\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$ for $(x,y) \in X\times Y$. It is clear that $(X\times Y, \|(\cdot,\cdot)\|)$ is a Banach space. Let

$$K = \{(x,y) \in X\times Y | x(t) \ge 0, y(t) \ge 0, t \in [1,\infty)\},$$
$$P_1 = \{x \in X | x(t) \ge 0, t \in [1,\infty)\}, P_2 = \{y \in Y | y(t) \ge 0, t \in [1,\infty)\}.$$

These spaces X, Y are equipped with partial orders:

$$x \le y, x, y \in X \Leftrightarrow x(t) \le y(t), t \in [1,\infty);$$

$$x \le y, x, y \in Y \Leftrightarrow x(t) \le y(t), t \in [1,\infty);$$

and if $0 \le x(t) \le y(t)$, then

$$\sup_{t\in[1,\infty)} \frac{x(t)}{1+(\log t)^{\alpha-1}} \le \sup_{t\in[1,\infty)} \frac{y(t)}{1+(\log t)^{\alpha-1}} \Rightarrow \|x\|_X \le \|y\|_X.$$

Thus P_1 is a normal cone in X, and the same reason shows that P_2 is also a normal cone in Y. The cone $K \subset X\times Y$ and $K = P_1 \times P_2$ is also normal. The space $X\times Y$ can be equipped with a partial order:

$$(x_1,y_1) \le (x_2,y_2) \Longleftrightarrow x_1(t) \le x_2(t), y_1(t) \le y_2(t), t \in [1,\infty).$$

Lemma 3.1. *Let $(A_1)-(A_3)$ hold. Then $(x,y) \in X\times Y$ is a solution to system (1.1) if and only if $(x,y) \in X\times Y$ is a solution to the integral equations:*

$$\begin{cases} x(t) = \int_1^\infty H(t,s)p_1(s)u(s,x(s),Sy(s))\frac{ds}{s}, t \in [1,\infty), \\ y(t) = \int_1^\infty H^*(t,s)p_2(s)v(s,x(s),Qy(s))\frac{ds}{s}, t \in [1,\infty). \end{cases} \tag{3.1}$$

Proof. By Lemma 2.3, we only need to prove

$$\int_1^\infty p_1(s)u(s,x(s),Sy(s))\frac{ds}{s} < \infty, \int_1^\infty p_2(s)v(s,x(s),Qy(s))\frac{ds}{s} < \infty.$$

First, for $(x,y) \in X\times Y$, we have

$$\|x\|_X = \sup_{t\in[1,\infty)} \frac{|x(t)|}{1+(\log t)^{\alpha-1}} < \infty, \|y\|_Y = \sup_{t\in[1,\infty)} \frac{|y(t)|}{1+(\log t)^{\beta-1}} < \infty.$$

Hence,

$$\begin{aligned} \|Sy\|_Y &= \sup_{t\in[1,\infty)} \frac{|(Sy)(t)|}{1+(\log t)^{\beta-1}} \le \sup_{t\in[1,\infty)} \int_1^t k(t,s)\frac{|y(s)|}{1+(\log t)^{\beta-1}}ds \\ &= \sup_{t\in[1,\infty)} \int_1^t k(t,s)\frac{1+(\log s)^{\beta-1}}{1+(\log t)^{\beta-1}} \cdot \frac{|y(s)|}{1+(\log s)^{\beta-1}}ds \\ &\le \|y\|_Y \sup_{t\in[1,\infty)} \int_1^t k(t,s)ds = \|y\|_Y k^* < \infty \end{aligned}$$

and

$$\begin{aligned}\|Qy\|_Y &= \sup_{t\in[1,\infty)} \frac{|(Qy)(t)|}{1+(\log t)^{\beta-1}} \leq \sup_{t\in[1,\infty)} \int_1^\infty l(t,s)\frac{|y(s)|}{1+(\log t)^{\beta-1}}ds \\ &= \sup_{t\in[1,\infty)} \int_1^\infty l(t,s)\frac{1+(\log s)^{\beta-1}}{1+(\log t)^{\beta-1}} \cdot \frac{|y(s)|}{1+(\log s)^{\beta-1}}ds \\ &\leq \|y\|_Y \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\beta-1}} \int_1^\infty l(t,s)(1+(\log s)^{\beta-1})ds \\ &= \|y\|_Y l^* < \infty.\end{aligned}$$

From (A_1), we find that there exists $M_{xy} > 0$ such that

$$\begin{aligned}u(s,x(s),Sy(s)) &= u(s,(1+(\log s)^{\alpha-1})\frac{x(s)}{1+(\log s)^{\alpha-1}},(1+(\log s)^{\beta-1})\frac{Sy(s)}{1+(\log s)^{\beta-1}}) \\ &\leq M_{xy}\end{aligned}$$

and

$$\begin{aligned}v(s,x(s),Qy(s)) &= v(s,(1+(\log s)^{\alpha-1})\frac{x(s)}{1+(\log s)^{\alpha-1}},(1+(\log s)^{\beta-1})\frac{Qy(s)}{1+(\log s)^{\beta-1}}) \\ &\leq M_{xy}\end{aligned}$$

for all $s \in [1,\infty)$. It follows from (A_2) that

$$\int_1^\infty p_1(s)u(s,x(s),Sy(s))\frac{ds}{s} \leq M_{xy}\int_1^\infty p_1(s)\frac{ds}{s} < \infty$$

and

$$\int_1^\infty p_2(s)v(s,x(s),Qy(s))\frac{ds}{s} \leq M_{xy}\int_1^\infty p_2(s)\frac{ds}{s} < \infty.$$

□

For $(x,y) \in X \times Y$, define three operators T_1, T_2, and T by

$$\begin{cases} T_1(x,y)(t) = \int_1^\infty H(t,s)p_1(s)u(s,x(s),Sy(s))\frac{ds}{s}, \\ T_2(x,y)(t) = \int_1^\infty H^*(t,s)p_2(s)v(s,x(s),Qy(s))\frac{ds}{s}, \end{cases} \tag{3.2}$$

and

$$T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t)).$$

Then we can easily prove that $T_1 : X\times Y \to X, T_2 : X\times Y \to Y$ and $T : X\times Y \to X\times Y$. Evidently, (x,y) is a solution to (1.1) if and only if (x,y) is a fixed point of operator T.

Lemma 3.2. *If $(A_1)-(A_3)$ hold, then $T_1 : K \longrightarrow P_1, T_2 : K \longrightarrow P_2, T : K \longrightarrow K$.*

Proof. For $(x,y) \in K = P_1 \times P_2$, we have $x(t), y(t) \geq 0, t \in [1,\infty)$ and

$$\|x\|_X = \sup_{t\in[1,\infty)} \frac{|x(t)|}{1+(\log t)^{\alpha-1}} < \infty, \|y\|_Y = \sup_{t\in[1,\infty)} \frac{|y(t)|}{1+(\log t)^{\beta-1}} < \infty.$$

By Lemma 3.1, we have $\|Sy\|_Y \leq k^*\|y\|_Y < \infty$ and $\|Qy\|_Y \leq l^*\|y\|_Y < \infty$, so there exist constants $L_x, L_y, L_S, L_Q > 0$ such that $\|x\|_X \leq L_x$, $\|y\|_Y \leq L_y$, $\|Sy\|_Y \leq L_S$, and $\|Qy\|_Y \leq L_Q$. From (A_1), we see that there exists $M_1>0$ defined by $M_1=\sup\{u(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y):$ $t\in[1,\infty),x\in[0,L_x),y\in[0,L_S)\}$. By Lemma 2.4, we have

$$\begin{aligned}&\frac{T_1(x,y)(t)}{1+(\log t)^{\alpha-1}}\\&=\int_1^\infty\frac{H(t,s)}{1+(\log t)^{\alpha-1}}p_1(s)u(s,x(s),Sy(s))\frac{ds}{s}\\&\le\frac{1}{\Omega_1}\int_1^\infty p_1(s)u(s,(1+(\log s)^{\beta-1})\frac{x(s)}{1+(\log s)^{\beta-1}},(1+(\log s)^{\beta-1})\frac{(Sy)(s)}{1+(\log s)^{\beta-1}})\frac{ds}{s}\\&\le\frac{M_1}{\Omega_1}\int_1^\infty p_1(s)\frac{ds}{s}<\infty.\end{aligned}$$

By (A_1), (A_2), and Lemma 2.4, we find $T_1(x,y)(t)\ge 0,t\in[1,\infty)$. Hence $T_1(x,y)\in P_1$. Similarly, we can prove $T_2(x,y)\in P_2$, so $T:K\to K$. Therefore, the conclusions hold. □

Remark 3.3. For $h=(h_1,h_2)\in K$, by [17], $K_h=P_{1,h_1}\times P_{2,h_2}$ with $P_{1,h_1}=\{x\in X|x\sim h_1\},P_{2,h_2}=\{y\in Y|y\sim h_2\}$.

Theorem 3.4. *Assume that $(A_1)-(A_3)$ and the following conditions hold:*
(B_1) $u(t,x_1,y_1)\le u(t,x_2,y_2),v(t,x_1,y_1)\le v(t,x_2,y_2)$ *for any* $t\in[1,\infty),x_2\ge x_1\ge 0$ *and* $y_2\ge y_1\ge 0$ *;*
(B_2) *For* $\lambda\in(0,1)$, *there exist* $\varphi_1,\varphi_2\in\Phi$ *such that*

$$\begin{cases}u(t,\lambda x,\lambda y)\ge\varphi_1(\lambda)u(t,x,y),t\in[1,\infty),\quad x,y\in[0,\infty),\\ v(t,\lambda x,\lambda y)\ge\varphi_2(\lambda)v(t,x,y),t\in[1,\infty),\quad x,y\in[0,\infty).\end{cases}\tag{3.3}$$

Then: (1) system (1.1) has a unique solution (x^*,y^*) *in* K_h, *where*

$$h(t)=(h_1(t),h_2(t))=((\log t)^{\alpha-1},(\log t)^{\beta-1}),t\in[1,\infty);$$

(2) for a given point $(x_0,y_0)\in K_h$ *and the two sequences defined by:*

$$\begin{cases}x_{n+1}(t)=\int_1^\infty H(t,s)p_1(s)u(s,x_n(s),(Sy_n)(s))\frac{ds}{s},t\in[1,\infty),\\ y_{n+1}(t)=\int_1^\infty H^*(t,s)p_2(s)v(s,x_n(s),(Qy_n)(s))\frac{ds}{s},t\in[1,\infty),\end{cases}\tag{3.4}$$

$n=0,1,2,\dots$, $x_n(t)\longrightarrow x^*(t),y_n(t)\longrightarrow y^*(t)$ *as* $n\longrightarrow\infty$.

Proof. Firstly, we prove that $T:K\to K$ is increasing. For $(x_1,y_1),(x_2,y_2)\in K$ with $(x_1,y_1)\le(x_2,y_2)$, we know that $x_1(t)\le x_2(t),y_1(t)\le y_2(t),t\in[1,\infty)$. By using Lemma 2.4 and (B_1), we have

$$\begin{aligned}T_1(x_1,y_1)(t)&=\int_1^\infty H(t,s)p_1(s)u(s,x_1(s),(Sy_1)(s))\frac{ds}{s}\\&\le\int_1^\infty H(t,s)p_1(s)u(s,x_2(s),(Sy_2)(s))\frac{ds}{s}=T_1(x_2,y_2)(t)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}T_2(x_1,y_1)(t)&=\int_1^\infty H^*(t,s)p_2(s)v(s,x_1(s),(Qy_1)(s))\frac{ds}{s}\\&\le\int_1^\infty H^*(t,s)p_2(s)v(s,x_2(s),(Qy_2)(s))\frac{ds}{s}=T_2(x_2,y_2)(t).\end{aligned}\tag{3.6}$$

Thus

$$T(x_1,y_1)(t) = (T_1(x_1,y_1)(t),T_2(x_1,y_1)(t)) \le (T_1(x_2,y_2)(t),T_2(x_2,y_2)(t)) = T(x_2,y_2)(t),$$

which shows that $T : K \to K$ is increasing. In the sequel, we show that T satisfies the two conditions of Lemma 2.6. From (B_2), for any $\lambda \in (0,1)$ and $(x,y)\in K$, we have

$$\begin{aligned}
T_1(\lambda x,\lambda y))(t) &= \int_1^\infty H(t,s)p_1(s)u(s,\lambda x(s),(S\lambda y)(s))\frac{ds}{s}\\
&= \int_1^\infty H(t,s)p_1(s)u(s,\lambda x(s),\lambda(Sy)(s))\frac{ds}{s}\\
&\ge \varphi_1(\lambda)\int_1^\infty H(t,s)p_1(s)u(s,x(s),(Sy)(s))\frac{ds}{s}\\
&= \varphi_1(\lambda)T_1(x,y)(t),\\
T_2(\lambda x,\lambda y))(t) &= \int_1^\infty H^*(t,s)p_2(s)v(s,\lambda x(s),(Q\lambda y)(s))\frac{ds}{s}\\
&= \int_1^\infty H^*(t,s)p_2(s)v(s,\lambda x(s),\lambda(Qy)(s))\frac{ds}{s}\\
&\ge \varphi_2(\lambda)\int_1^\infty H^*(t,s)p_2(s)v(s,x(s),(Qy)(s))\frac{ds}{s}\\
&= \varphi_2(\lambda)T_2(x,y)(t).
\end{aligned}$$

Thus

$$\begin{aligned}
T(\lambda(x,y))(t) &= (T_1(\lambda x,\lambda y)(t),T_2(\lambda x,\lambda y)(t))\\
&\ge (\varphi_1(\lambda)T_1(x,y)(t),\varphi_2(\lambda)T_2(x,y)(t)).
\end{aligned}$$

Let $\varphi(t) = \min\{\varphi_1(t),\varphi_2(t)\}, t \in (0,1)$. Then $\varphi \in \Phi$ and

$$\begin{aligned}
T(\lambda(x,y)) &\ge (\varphi(\lambda)T_1(x,y),\varphi(\lambda)T_2(x,y))\\
&= \varphi(\lambda)(T_1(x,y),T_2(x,y))\\
&= \varphi(\lambda)T(x,y), \lambda \in (0,1).
\end{aligned}$$

Hence, the second condition of Lemma 2.6 holds.

Next, we take $h_0(t) = h(t) = (h_1(t),h_2(t))$, where $h_1(t) = (\log t)^{\alpha-1}, h_2(t) = (\log t)^{\beta-1}, t \in [1,\infty)$. We show $h_0 \in K_h$. That is, we need to prove $h_1 \in P_{1,h_1}, h_2 \in P_{2,h_2}$. In fact, $h_1(t) \ge 0, h_2(t) \ge 0, t \in [1,\infty)$. Moreover,

$$\sup_{t\in[1,\infty)} \frac{h_1(t)}{1+(\log t)^{\alpha-1}} = \sup_{t\in[1,\infty)} \frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}} = 1 < \infty$$

and

$$\sup_{t\in[1,\infty)} \frac{h_2(t)}{1+(\log t)^{\beta-1}} = \sup_{t\in[1,\infty)} \frac{(\log t)^{\beta-1}}{1+(\log t)^{\beta-1}} = 1 < \infty.$$

Thus $h_1 \in P_1, h_2 \in P_2$. Further, $h_1 \in P_{1,h_1}, h_2 \in P_{2,h_2}$. Next we mainly show that $Th \in K_h$. Let

$$M_2 = \sup\{u(s,(1+(\log s)^{\alpha-1}),(1+(\log s)^{\beta-1})k^*) : s \in [1,\infty)\}.$$

From (A_1), we have that $M_2 > 0$ exists and $u(s,(1+(\log s)^{\alpha-1}),(1+(\log s)^{\beta-1})k^*) \le M_2$. Let

$$l_1 := \int_1^\infty \frac{1}{\Omega_1}\sum_{i=1}^{n_1}\eta_i H_2(\omega_i,s)p_1(s)u(s,0,0)\frac{ds}{s}$$

and

$$l_2 := \int_1^\infty \frac{1}{\Omega_1} p_1(s) u(s, 1+(\log s)^{\alpha-1}, (1+(\log s)^{\beta-1})k^*) \frac{ds}{s}.$$

We know that

$$\begin{aligned} l_2 &:= \int_1^\infty \frac{1}{\Omega_1} p_1(s) u(s, 1+(\log s)^{\alpha-1}, (1+(\log s)^{\beta-1})k^*) \frac{ds}{s} \\ &\leq \int_1^\infty \frac{1}{\Omega_1} p_1(s) M_2 \frac{ds}{s} = \frac{M_2}{\Omega_1} \int_1^\infty p_1(s) \frac{ds}{s} < \infty. \end{aligned}$$

From Remark 3.3, we only need to prove $T_1(h_1,h_2)(t) \in P_{1,h_1}, T_2(h_1,h_2)(t) \in P_{2,h_2}$. By using Lemma 2.5 and (B_1), we have

$$\begin{aligned} T_1(h_1,h_2)(t) &= \int_1^\infty H(t,s) p_1(s) u(s, h_1(s), Sh_2(s)) \frac{ds}{s} \\ &\geq \int_1^\infty \sum_{i=1}^{n_1} \eta_i H_2(\omega_i, s) h_1(t) \frac{1}{\Omega_1} p_1(s) u(s, (\log s)^{\alpha-1}, S(\log s)^{\beta-1}) \frac{ds}{s} \\ &\geq h_1(t) \int_1^\infty \frac{1}{\Omega_1} \sum_{i=1}^{n_1} \eta_i H_2(\omega_i, s) p_1(s) u(s,0,0) \frac{ds}{s} \\ &= h_1(t) l_1 \end{aligned}$$

and

$$\begin{aligned} & T_1(h_1,h_2)(t) \\ &= \int_1^\infty H(t,s) p_1(s) u(s, h_1(s), Sh_2(s)) \frac{ds}{s} \\ &\leq \int_1^\infty (\log t)^{\alpha-1} \frac{1}{\Omega_1} p_1(s) u(s, (\log s)^{\alpha-1}, Sh_2(s)) \frac{ds}{s} \\ &= h_1(t) \int_1^\infty \frac{1}{\Omega_1} p_1(s) u(s, (1+(\log s)^{\alpha-1}) \frac{(\log s)^{\alpha-1}}{1+(\log s)^{\alpha-1}}, (1+(\log s)^{\beta-1}) \frac{Sh_2(s)}{1+(\log s)^{\beta-1}}) \frac{ds}{s} \\ &\leq h_1(t) \int_1^\infty \frac{1}{\Omega_1} p_1(s) u(s, 1+(\log s)^{\alpha-1}, (1+(\log s)^{\beta-1})k^*) \frac{ds}{s} \\ &= h_1(t) l_2. \end{aligned}$$

From (A_1) and (A_2), we obtain

$$u(s, 1+(\log s)^{\alpha-1}, (1+(\log s)^{\beta-1})k^*) \geq u(s,0,0) \geq 0, s \in [1,\infty)$$

and

$$\begin{aligned} \sum_{i=1}^{n_1} \eta_i H_2(\omega_i, s) &\leq \sum_{i=1}^{n_1} \eta_i \frac{1}{\Gamma(\alpha - m_1)} (\log \omega_i)^{\alpha - m_1 - 1} \\ &= \frac{\Gamma(\alpha) - \Omega_1}{\Gamma(\alpha)} = 1 - \frac{\Omega_1}{\Gamma(\alpha)} < 1. \end{aligned}$$

Moreover, from $u(s,0,0) \not\equiv 0$, (A_2), Lemma 2.4, and Lemma 2.5, we have $0 < l_1 \leq l_2$ and $l_1(h_1,h_2)(t) \leq T_1(h_1,h_2)(t) \leq l_2(h_1,h_2)(t), t \in [1,\infty)$. Thus $T_1(h_1,h_2) \in P_{1,h_1}$. Similarly, by using Lemma 2.5, (A_1), and (A_2), we can also prove $T_2(h_1,h_2) \in P_{2,h_2}$. Therefore,

$$Th = T(h_1,h_2) = (T_1(h_1,h_2), T_2(h_1,h_2)) \in P_{1,h_1} \times P_{2,h_2} = K_h.$$

Consequently, by Lemma 2.6, there exists a unique $z^* \in K_h$ such that $Tz^* = z^*$. For any $z_0 \in K_h$, construct a sequence $z_{n+1} = Tz_n, n = 0,1,2,\ldots$. Then $z_n \to z^*$ as $n \to \infty$. Set $z^* = (x^*, y^*), z_0 = (x_0, y_0)$, and $z_1 = Tz_0 = T(x_0,y_0) = (T_1(x_0,y_0), T_2(x_0,y_0))$. Let $x_1 = T_1(x_0,y_0), y_1 = T_2(x_0,y_0)$, $\cdots, x_{n+1} = T_1(x_n,y_n), y_{n+1} = T_2(x_n,y_n)$. Then we see that (x^*,y^*) is the unique positive solution of system (1.1) in K_h, and

$$x_{n+1}(t) = \int_1^\infty H(t,s)p_1(s)u(s,x_n(s),(Sy_n)(s))\frac{ds}{s} \to x^*(t),$$
$$y_{n+1}(t) = \int_1^\infty H^*(t,s)p_2(s)v(s,x_n(s),(Qy_n)(s))\frac{ds}{s} \to y^*(t),$$

as $n \to \infty$. □

From Lemma 3.1, Lemma 3.2, and Theorem 3.4, we have the following result.

Corollary 3.5. *Assume that* $(A_1)-(A_3), (B_1)$, *and* (B_2) *are satisfied. Then the following system:*

$$\begin{cases} D_{1^+}^{\alpha}x(t) + p_1(t)u(t,x(t),y(t)) = 0, \omega-1<\alpha\le\omega, t\in(1,\infty),\\ D_{1^+}^{\beta}y(t) + p_2(t)v(t,x(t),y(t)) = 0, \mu-1<\beta\le\mu, t\in(1,\infty),\\ x(1)=x'(1)=\cdots=x^{(\omega-2)}(1)=0, D_{1^+}^{\alpha-1}x(\infty) = \Sigma_{i=1}^{n_1}\eta_i D_{1^+}^{m_1}x(\omega_i),\\ y(1)=y'(1)=\cdots=y^{(\mu-2)}(1)=0, D_{1^+}^{\beta-1}y(\infty) = \Sigma_{j=1}^{n_2}\zeta_j D_{1^+}^{m_2}y(\mu_j), \end{cases}$$

has a unique positive solution (x^*,y^*) *in* K_h, *where* $h=(h_1,h_2)$ *with* $h_1(t) = (\log t)^{\alpha-1}, h_2(t) = (\log t)^{\beta-1}, t\in[1,\infty)$. *In addition, for* $(x_0,y_0)\in K_h$, *the sequences*

$$x_{n+1}(t) = \int_1^\infty H(t,s)p_1(s)u(s,x_n(s),y_n(s))\frac{ds}{s} \to x^*(t),$$
$$y_{n+1}(t) = \int_1^\infty H^*(t,s)p_2(s)v(s,x_n(s),y_n(s))\frac{ds}{s} \to y^*(t),$$

as $n\to\infty$.

4. AN EXAMPLE

To illustrate the main result, we consider the following Hadamard fractional differential system

$$\begin{cases} D_{1^+}^{2.5}x(t) + \frac{1}{t^2}\{e^{-t} + \frac{x^{\tau_1}(t)}{1+(\log t)^{\frac{3}{2}}} + \frac{1}{1+(\log t)^{\frac{3}{2}}}(\int_1^t \frac{y(s)}{(t+s)^2}ds)^{\tau_1}\} = 0,\\ D_{1^+}^{2.5}y(t) + \frac{1}{t}\{e^{-2t} + \frac{x^{\tau_2}(t)}{1+(\log t)^{\frac{3}{2}}} + \frac{1}{1+(\log s)^{\frac{3}{2}}}(\int_1^\infty \frac{y(s)}{(1+s^2)(1+(\log t)^{\frac{3}{2}})}ds)^{\tau_2}\} = 0,\\ x(1)=x'(1)=0, D_{1^+}^{1.5}x(\infty) = \frac{1}{10}D_{1^+}^{0.5}x(e^2) + \frac{1}{10}D_{1^+}^{0.5}x(e^3),\\ y(1)=y'(1)=0, D_{1^+}^{1.5}y(\infty) = \frac{1}{10}D_{1^+}^{0.5}y(e^2) + \frac{1}{10}D_{1^+}^{0.5}y(e^3), \end{cases} \tag{4.1}$$

where $\alpha=\beta=\frac{5}{2}, \omega=\mu=3, n_1=n_2=2, m_1=m_2=\frac{1}{2}, \eta_1=\eta_2=\zeta_1=\zeta_2=\frac{1}{10}, \omega_1=\mu_1=e^2, \omega_2=\mu_2=e^3, \tau_1,\tau_2\in(0,1), k(t,s)=\frac{1}{(t+s)^2}, l(t,s)=\frac{1}{(1+s^2)(1+(\log t)^{\frac{3}{2}})}$.

$$p_1(t)=\frac{1}{t^2}, p_2(t)=\frac{1}{t},$$
$$u(t,x,y) = e^{-t} + \frac{x^{\tau_1}}{1+(\log t)^{\frac{3}{2}}} + \frac{y^{\tau_1}}{1+(\log t)^{\frac{3}{2}}},$$

$$v(t,x,y) = e^{-2t} + \frac{x^{\tau_2}}{1+(\log t)^{\frac{3}{2}}} + \frac{y^{\tau_2}}{1+(\log t)^{\frac{3}{2}}},$$

$$(Sy)(t) = \int_1^t \frac{y(s)}{(t+s)^2}ds, (Qy)(t) = \int_1^\infty \frac{y(s)}{(1+s^2)(1+(\log t)^{\frac{3}{2}})}ds.$$

Clearly,

$$0 < \int_1^\infty p_1(s)\frac{ds}{s} = \int_1^\infty \frac{1}{s^2}\frac{ds}{s} = \frac{1}{2} < \infty, 0 < \int_1^\infty p_2(s)\frac{ds}{s} = \int_1^\infty \frac{1}{s}\frac{ds}{s} = 1 < \infty.$$

Evidently, $u, v \in C([1,\infty)\times[0,\infty)\times[0,\infty),[0,\infty))$ and $u(t,0,0) = e^{-t} \not\equiv 0, v(t,0,0) = e^{-2t} \not\equiv 0$. Note that $u(t,x,y)$ and $v(t,x,y)$ are increasing in $x \in [0,\infty), y \in [0,\infty)$. If $x, y \in [0,\infty)$ are bounded, then

$$\begin{aligned} u(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y) &= u(t,(1+(\log t)^{\frac{3}{2}})x,(1+(\log t)^{\frac{3}{2}})y) \\ &= e^{-t} + x^{\tau_1} + y^{\tau_1} \le \frac{1}{e} + x^{\tau_1} + y^{\tau_1} < \infty \end{aligned}$$

and

$$\begin{aligned} v(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y) &= v(t,(1+(\log t)^{\frac{3}{2}})x,(1+(\log t)^{\frac{3}{2}})y) \\ &= e^{-2t} + x^{\tau_2} + y^{\tau_2} \le \frac{2}{e} + x^{\tau_2} + y^{\tau_2} < \infty, \end{aligned}$$

which imply that $u(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y), v(t,(1+(\log t)^{\alpha-1})x,(1+(\log t)^{\beta-1})y)$ are bounded. In addition,

$$\begin{aligned} k^* &= \sup_{t\in[1,\infty)} \int_1^t \frac{1}{(t+s)^2}ds = \sup_{t\in[1,\infty)} (\frac{1}{t+1} - \frac{1}{2t}) = \frac{3}{2} - \sqrt{2} < \infty, \\ l^* &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\beta-1}} \int_1^\infty l(t,s)(1+(\log s)^{\beta-1})ds \\ &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\frac{3}{2}}} \int_1^\infty \frac{1}{(1+s^2)(1+(\log s)^{\frac{3}{2}})}(1+(\log s)^{\frac{3}{2}})ds \\ &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\frac{3}{2}}} \int_1^\infty \frac{1}{1+s^2}ds = \frac{\pi}{4} < \infty. \end{aligned}$$

Let $\varphi_1(\sigma) = \sigma^{\tau_1}$ and $\varphi_2(\sigma) = \sigma^{\tau_2}, \sigma \in (0,1)$. Then $\varphi_1, \varphi_2 \in \Phi$, for $\lambda \in (0,1)$ and $x \in [0,\infty), y \in [0,\infty)$,

$$\begin{aligned} u(t,\lambda x,\lambda y) &= e^{-t} + \frac{(\lambda x)^{\tau_1}}{1+(\log t)^{\frac{3}{2}}} + \frac{(\lambda y)^{\tau_1}}{1+(\log t)^{\frac{3}{2}}} \\ &\ge \lambda^{\tau_1}(e^{-t} + \frac{x^{\tau_1}}{1+(\log t)^{\frac{3}{2}}} + \frac{y^{\tau_1}}{1+(\log t)^{\frac{3}{2}}}) \\ &= \varphi_1(\lambda)u(t,x,y) \end{aligned}$$

and

$$\begin{aligned} v(t,\lambda x,\lambda y) &= e^{-2t} + \frac{(\lambda x)^{\tau_2}}{1+(\log t)^{\frac{3}{2}}} + \frac{(\lambda y)^{\tau_2}}{1+(\log t)^{\frac{3}{2}}} \\ &\geq \lambda^{\tau_2}(e^{-2t} + \frac{x^{\tau_2}}{1+(\log t)^{\frac{3}{2}}} + \frac{y^{\tau_2}}{1+(\log t)^{\frac{3}{2}}}) \\ &= \varphi_2(\lambda)v(t,x,y). \end{aligned}$$

Hence all the conditions of Theorem 3.4 are satisfied and then system (4.1) has a unique solution (x^*,y^*) in K_h, where

$$h(t) = (h_1(t),h_2(t)) = ((\log t)^{\frac{3}{2}},(\log t)^{\frac{3}{2}}),t \in [1,\infty).$$

For any given $(x_0,y_0) \in K_h$, define the sequences

$$x_{n+1}(t) = \int_1^\infty H(t,s)\frac{1}{s^2}\{e^{-s} + \frac{x_n^{\tau_1}(s)}{1+(\log t)^{\frac{3}{2}}} + \frac{1}{1+(\log s)^{\frac{3}{2}}}(\int_1^s \frac{y_n(\tau)}{(s+\tau)^2}d\tau)^{\tau_1}\}\frac{ds}{s},$$

$$y_{n+1}(t) = \int_1^\infty H^*(t,s)\frac{1}{s}\{e^{-2s} + \frac{x_n^{\tau_2}(s)}{1+(\log s)^{\frac{3}{2}}} + \frac{1}{1+(\log s)^{\frac{3}{2}}}(\int_1^\infty \frac{y_n(\tau)}{(1+\tau^2)(1+(\log s)^{\frac{3}{2}})}d\tau)^{\tau_2}\}\frac{ds}{s},$$

$n = 0,1,2,\ldots$, where

$$H(t,s) = H_1(t,s) + \frac{(\log t)^{\alpha-1}}{\Omega_1}\Sigma_{i=1}^{n_1}\eta_i H_2(\omega_i,s),$$

$$H^*(t,s) = H_1^*(t,s) + \frac{(\log t)^{\beta-1}}{\Omega_2}\Sigma_{j=1}^{n_2}\zeta_j H_2^*(\mu_j,s),$$

$$H_1(t,s) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} (\log t)^{\frac{3}{2}} - (\log\frac{t}{s})^{\frac{3}{2}}, & 1 \leq s \leq t < \infty, \\ (\log t)^{\frac{3}{2}}, & 1 \leq t \leq s < \infty, \end{cases}$$

$$H_1^*(t,s) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} (\log t)^{\frac{3}{2}} - (\log\frac{t}{s})^{\frac{3}{2}}, & 1 \leq s \leq t < \infty, \\ (\log t)^{\frac{3}{2}}, & 1 \leq t \leq s < \infty, \end{cases}$$

$$H_2(\omega_i,s) = \frac{1}{\Gamma(2)}\begin{cases} \log s, & 1 \leq s \leq \omega_i < \infty, \\ \log\omega_i, & 1 \leq \omega_i \leq s < \infty, \end{cases}$$

and

$$H_2^*(\mu_i,s) = \frac{1}{\Gamma(2)}\begin{cases} \log s, & 1 \leq s \leq \mu_i < \infty, \\ \log\mu_i, & 1 \leq \mu_i \leq s < \infty, \end{cases}$$

Then $x_n \to x^*, y_n \to y^*$ as $n \to \infty$.

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