J. Nonlinear Funct. Anal. 2024 (2024) 29 https://doi.org/10.23952/jnfa.2024.29

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

FINITE-TIME BLOW-UP AND SOLVABILITY FOR A SEMILINEAR PARABOLIC PROBLEM WITH NONLINEAR INTEGRAL CONDITIONS

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Abstract. This paper is devoted to the study of a super-linear non-local problem with Neumann condition modeling by the integral condition of second type for a class of reaction-diffusion equations. We show the existence of the weak solution by developing the method of Fadeo-Galarkin to avoid the complexities produced by the existence of the integral condition. Then, by applying an a priori estimate, we prove the uniqueness of the weak solution to the problem. We also study the finite-time blow-up solution.
Keywords. Integral condition; Fadeo-Galarkin method; Nonlinear equations; Parabolic equation.
2020 MSC. 35K58, 35K61, 35B45, 35A01.

1. Introduction

Nonlinear diffusion equations are an important class of parabolic equations that come from a variety of diffusion phenomena appearing widely in nature [1, 2]. The complexity of nonlinear evolution equations and challenges in their theoretical study has attracted a lot of interest from researchers in nonlinear sciences [3, 4, 5, 6, 7]. Natural phenomena can be modeled by partial differential equations with non-local conditions [8, 9, 10, 11]. However, many phenomena can better be described by integral conditions, which are of growing interest [12, 13]. A large number of modern physics and technology problems were stated by using non-local and integral

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Received May 3, 2024; Accepted September 6, 2024.

conditions for partial differential equations; see, e.g., [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The first type of such conditions can be given by

$$\int_{\Omega} k(x,t)u(x,t)dx = E(t),$$

where $t \in (0,T)$, $\Omega \subset \mathbb{R}^n$, and k is a given function. On the other hand, the second type of such conditions, where the Dirichlet or the Neumann condition can be modeled by the integral condition, can be given by

$$u(x,t)|_{\partial\Omega} = \int k(x,t)u(x,t)dx.$$

Actually, the above condition can be used when it is impossible to directly measure the sought quantity on the border, its total value, or its average is known. The study of the problems of nonlinear evolution equations with different boundary conditions types (classical and non-classical condition) was solved by many powerful methods in nonlinear analysis, i.e., fixed-point theorems, semi-group methods, Galerkin, and monotone operator methods; see, e.g., [24, 25, 26, 27, 28, 29]. Motivated by this, we study a super-linear parabolic equation with a classical Dirichlet condition and an integral condition of the second type, which can be considered as a more general case than classical integral conditions, where we show the existence and uniqueness of the weak solution for the linear problem by the method of the Faedo-Galerkin method. Then, by applying an iterative process based on the results obtained for the linear problem, we prove the existence and the uniqueness of the weak solution of the semi-linear problem. Also, we study the blow-up solution theoretically, where we focus on profiling the finite time blow-up solution of the main problem under the effect of integral condition.

2. Formulation of the Problem

Let $\Omega = (0,l)$ be a bounded open of $\mathbb{R}$ and $Q = \Omega \times (0,T)$. We consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + u^p - bu = f(x,t) & \forall (x,t) \in \overline{Q} \\ u(x,0) = \varphi(x) & \forall x \in (0,l) \\ \dfrac{\partial u}{\partial x}(0,t) = \int_0^l k_1(x,t)u(x,t)dx & \forall t \in [0,T] \\ \dfrac{\partial u}{\partial x}(l,t) = \int_0^l k_2(x,t)h(u(x,t))dx & \forall t \in [0,T] \end{cases}, \qquad (P_1)$$

where a, b, and p are positive odd integers and $p \geq 1$. The purpose of this paper is to study a function $u = u(x,t)$ which solves problem (P_1) under certain assumptions (H), where $x \in \Omega$ and $t \in [0,T]$. In order to properly pose the problem and to find the tools to solve it, we need to introduce some concepts and some functional spaces that are needed later. We define space V by

$$V = \left\{ u \in H^1(\Omega) \cap L^{p+1}(\Omega) \right\},$$

where the space V is provided with the norm

$$\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^{p+1}(\Omega)},$$

with the scalar product of $H^1(\Omega)$. We are now able to precisely formulate problem (P_1). To this end, we need the following hypothesis:

$$(H): \begin{cases} f \in L^2\left(0,T;\ L^2(\Omega)\right) & (H.1) \\ \varphi \in H^1(\Omega) \cap L^{p+1}(\Omega) & (H.2) \\ k_i \in L^\infty((0,T)\times\Omega)\ \ \forall i \in \{1,2\} & \end{cases}.$$

Definition 2.1. The weak solution of problem (P_1) is a function that verifies

(i) $u \in L^2\left(0,T;\ H^1(\Omega)\right) \cap L^\infty\left(0,T;\ H^1(\Omega)\right)$.

(ii) u admits a strong derivative $\dfrac{\partial u}{\partial t} \in L^2\left(0,T;\ H^1(\Omega)\right)$.

(iii) $u(0) = \varphi$.

(iv) for all $v \in V$ and $t \in [0,T]$, we have

$$(u_t, v) + a(u_x, v_x) + (u^p, v) - b(u,v) = (f,v) + av(l,t)\int_0^l k(x,t)u(x,t)dx.$$

3. Existence of the Solution of the Semi-linear Problem (P_1)

3.1. **Variational formulation.** By multiplying the equation

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + u^p - bu = f(x,t) \tag{3.1}$$

and an element $v \in V$, and then integrating the result over Ω, we obtain

$$\int_\Omega \frac{\partial u}{\partial t}\cdot v dx - a\int_\Omega \frac{\partial^2 u}{\partial x^2}\cdot v dx + \int_\Omega u^p \cdot v dx - b\int_\Omega u\cdot v dx = \int_\Omega f\cdot v dx.$$

It follows that

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t}\cdot v dx + a\int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}dx + \int_\Omega u^p\cdot v dx - b\int_\Omega u\cdot v dx - av(l)u_x(l,t) + av(0)u_x(0,t) \\ = \int_\Omega f\cdot v dx. \end{aligned} \tag{3.2}$$

By using the boundary conditions and Green's, formula (3.2) becomes

$$(u_t, v) + a(u_x, v_x) + (u^p, v) - b(u,v) - av(l)u_x(l,t) + av(0)u_x(0,t) = (f,v) \quad \forall v \in V, \tag{3.3}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

3.2. **Study of the existence of the solution of problem (P_1).**

Theorem 3.1. *Let*

$$\|h(u)\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le \|u\|_{L^2(Q)}$$

and

$$\|h(u)\|_{L^2(Q)} \le \|u\|_{L^2(Q)}.$$

Then, for given initial data φ, there exist $T > 0$ and a weak solution u of problem (P_1) on Q such that

$$\begin{array}{rcl} u & \in & L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t & \in & L^2\left(0,T;L^2(\Omega)\right) \end{array}.$$

Proof. The demonstration of the existence of the solution of problem (P_1) is based on the method of Faedo-Galerkin, which consists in carrying out the following manner: The space V is separable, and then there exists a sequence $w_1, w_2, \ldots, w_m$ with the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, \ldots, w_m & \text{are linearly independent}, \\ V_m = \langle \{w_1, w_2, \ldots, w_m\} \rangle & \text{is dense in } V. \end{cases} \qquad (3.4)$$

In particular, we have

$$\forall \varphi \in V \implies \exists (\alpha_{km})_m \in IN^*,\ \varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \qquad (3.5)$$

Faedo Galerkin's approximation consists in searching a function

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\, w_i(x),$$

for any integer $m \geq 1$. The approximate solution satisfies the following identities:

$$\begin{cases} ((u_m(t))_t, w_k) + a(\Delta u_m(t), w_k) + \left(u_m^p(t) - b u_m(t), w_k\right) = (f(t), w_k) \quad \forall k = \overline{1,m} \\ (u_m(0), w_k) = \alpha_{km} \end{cases}, \qquad (P_2)$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Thus

$$((u_m(t))_t, w_k) = \left(\left(\sum_{i=1}^{m} g_{im}(t)\, w_i \right)_t, w_k \right) = \sum_{i=1}^{m} (w_i, w_k) \frac{\partial g_{im}}{\partial t}(t) \qquad (3.6)$$

and

$$\begin{aligned} & a(\Delta u_m(t), w_k) \\ & = a \sum_{i=1}^{m} g_{im}(t) \left[\frac{\partial w_i}{\partial x}(l) w_k(l) - \int_{\Omega} \frac{\partial w_i}{\partial x} \frac{\partial w_k}{\partial x} dx \right] \\ & = -a \sum_{i=1}^{m} g_{im}(t) \int_{\Omega} \frac{\partial w_i(x)}{\partial x} \frac{\partial w_k(x)}{\partial x} dx + a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(l) w_k(l) - a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(0) w_k(0) \\ & = -\sum_{i=1}^{m} a((w_i)_x, (w_k)_x)\, g_{im}(t) + a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(l) w_k(l) - a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(0) w_k(0). \end{aligned} \qquad (3.7)$$

Also, we have

$$u_m(0) = \sum_{i=1}^{m} g_{im}(0)\, w_i(x) = \varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k(x).$$

The existence of such α_{km} follows from $u_0 \in V \cap L^{P+1}(\Omega)$ and the fact that $\{w_k,\ k \in \mathbb{N}\}$ is the base of $V \cap L^{P+1}(\Omega)$. Thus (P_1) is reduced to the initial value problem for a system of first-order differential equations with respect to g_{im}, i.e.,

$$\left\{\begin{array}{c} \sum\limits_{i=1}^{m}(w_i,w_k)\dfrac{\partial g_{im}}{\partial t}(t)+a\sum\limits_{i=1}^{m}((w_i)_x,(w_k)_x)g_{im}(t) \\ -a\sum\limits_{i=1}^{m}g_{im}(t)\frac{\partial w_i}{\partial x}(l)w_k(l)+a\sum\limits_{i=1}^{m}g_{im}(t)\frac{\partial w_i}{\partial x}(0)w_k(0)+\left(u_m^p-bu_m,w_k\right)=(f(t),w_k) \\ g_{km}(0)=\alpha_{km} \qquad \forall k=\overline{1,m}. \end{array}\right. . \quad (P_3)$$

Now, we consider the vector

$$g_m=(g_{1m}(t),\ldots,g_{mm}(t)),f_m=((f,w_1),\ldots,(f,w_m))$$

as well as the matrices

$$B_m=\left((w_i,w_j)\right)_{\substack{1\le i\le m\\1\le j\le m}},A_m=\left(\left(\frac{\partial w_i}{\partial x},\frac{\partial w_j}{\partial x}\right)\right)_{\substack{1\le i\le m\\1\le j\le m}},$$

$$C_m = \left(\frac{\partial w_i}{\partial x}(l)\cdot w_j(l)\right)_{\substack{1\le i\le m\\1\le j\le m}},D_m=\left(\frac{\partial w_i}{\partial x}(0)\cdot w_j(0)\right)_{\substack{1\le i\le m\\1\le j\le m}},$$

and

$$G(g) = \left(\left(\left(\sum_{i=1}^{m}g_{im}(t)w_i\right)^p,w_j\right)\right)_{\substack{1\le i\le m\\1\le j\le m}}.$$

Thus, we can write problem (P_4) in the matrix form to obtain

$$\left\{\begin{array}{c} B_m\frac{\partial g_m}{\partial t}(t)+aA_mg_m-bB_mg_m+G(g)=f_m+aC_mg_m-aD_mg_m \\ g_m(0)=(\alpha_{im})_{1\le i\le m} \end{array}\right. . \quad (3.8)$$

Now, by using the Carathéodory's existence theorem used for ordinary differential equations, we can conclude that there exists a t_m depends only on $|\alpha_{im}|$ such that problem (3.8) admits a unique local solution $g_m(t)\in C[0,t_m]$ in $[0,t_m]$ such that $g_m'(t)\in L^2[0,T]$. □

Now, we intend to study the priori estimates for the approximate solution $u_m(x,t)$ obtained in the previous step. To this end, we introduce the subsequent result.

Theorem 3.2. *Leet $\varphi\in H^1(\Omega)\cap L^{p+1}(\Omega)$ and $f\in L^2\left(0,T,\ L^2(\Omega)\right)$, for all $\frac{p}{2}\ge b$. Then problem (P_1) admits a solution u such that $u\in L^2(0,T;\ V)\cap L^\infty\left(0,T;\ H^1(\Omega)\right)$ and $u'\in L^2(0,T;\ L^2(\Omega))$.*

Proof. For all $m\in\mathbb{N}^*$, we multiply both sides of equations in (P$_2$) by $g_{im}(t)$, and then make the summing the result with respect to k to obtain

$$\sum_{k=1}^{m}((u_m)_t,w_k)g_{km}(t)-a\sum_{k=1}^{m}(\Delta u_m,w_k)g_{km}(t)+\sum_{k=1}^{m}(u_m^p-bu_m,w_k)g_{km}(t)=\sum_{k=1}^{m}(f,w_k)\cdot g_{km}(t).$$

It follows that $((u_m)_t,u_m)-a(\Delta u_m,u_m)+\left(u_m^p-bu_m,u_m\right)=(f,u_m)$ and

$$((u_m)_t,u_m)+a\left(\frac{\partial u_m}{\partial x},\frac{\partial u_m}{\partial x}\right)+(u_m^p-bu_m,u_m)$$
$$=(f,u_m)+a\frac{\partial u_m}{\partial x}(l,t)u_m(l,t)-a\frac{\partial u_m}{\partial x}(0,t)u_m(0,t).$$

Thus

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_m\|^2_{L^2(\Omega)}+a\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}+\|u_m\|^{p+1}_{L^{p+1}(\Omega)}$$
$$=(f,u_m)+b\|u_m\|^2_{L^2(\Omega)}+a\frac{\partial u_m}{\partial x}(l,t)u_m(l,t)-a\frac{\partial u_m}{\partial x}(0,t)u_m(0,t).$$

By integrating the above equality from 0 to t, we see that

$$\frac{1}{2}\int_0^t\frac{\partial}{\partial t}\|u_m\|^2_{L^2(\Omega)}+a\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}+\int_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}$$
$$=\int_0^t(f,u_m)+b\int_0^t\|u_m\|^2_{L^2(\Omega)}+a\int_0^t\frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau-a\int_0^t\frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau,$$

which yields

$$\frac{1}{2}\|u_m\|^2_{L^2(\Omega)}+a\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}d\tau+\int_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}d\tau$$
$$=\int_0^t(f,u_m)d\tau+b\int_0^t\|u_m\|^2_{L^2(\Omega)}d\tau+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+a\int_0^t\frac{\partial u_m}{\partial x}(l,\tau)\cdot u_m(l,\tau)d\tau$$
$$-a\int_0^t\frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau.$$

Consequently,

$$\int_0^t\frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau-\int_0^t\frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau$$
$$=\int_0^t\left(\int_\Omega k_2(x,t)h(u_m(x,t))dx\right)\left(u_m(0,\tau)+\int_\Omega\frac{\partial u_m}{\partial x}(x,t)\,dx\right)d\tau$$
$$-\int_0^t\left(\int_\Omega k_1(x,t)\,u_m(x,t)\,dx\right)u_m(0,\tau)d\tau,$$

or

$$\int_0^t\frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau-\int_0^t\frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau$$
$$=\int_0^t\left(\int_\Omega k_2(x,t)h(u_m(x,t))dx-\int_\Omega k_1(x,t)\,u_m(x,t)\,dx\right)\left(u_m(x,t)-\int_\Omega\frac{\partial u_m}{\partial x}(x,t)\,dx\right)$$
$$+\int_0^t\left(\int_\Omega k_2(x,t)\,h(u_m(x,t))dx\right)\int_\Omega\frac{\partial u_m}{\partial x}(x,t)\,dxd\tau.$$

It follows that

$$\int_0^t \frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau - \int_0^t \frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau$$

$$\leq \frac{1}{2\varepsilon}\int_0^t \left(\int_\Omega k_2(x,t)\,h(u_m(x,t))dx - \int_\Omega k_1(x,t)\,u_m(x,t)\,dx\right)^2 d\tau$$

$$+\frac{\varepsilon}{2}\int_0^t \left(u_m(x,t) - \int_\Omega \frac{\partial u_m}{\partial x}(x,t)\,dx\right)^2 d\tau + \frac{1}{2\delta}\int_0^t \left(\int_\Omega k_2(x,t)\,h(u_m(x,t))dx\right)^2 d\tau$$

$$+\frac{\delta}{2}\int_0^t \left(\int_\Omega \frac{\partial u_m}{\partial x}(x,t)\,dxd\tau\right)^2 d\tau.$$

This directly gives

$$\int_0^t \frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau - \int_0^t \frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau$$

$$\leq \frac{K}{\varepsilon}\int_0^t \int_\Omega (h(u_m(x,t)))^2\,dxd\tau + \frac{K}{\varepsilon}\int_0^t \int_\Omega u_m^2(x,t)\,dxd\tau$$

$$+\frac{\varepsilon}{mes(\Omega)^2}\int_0^t \int_\Omega (u_m(x,t))^2\,d\tau + \varepsilon\int_0^t \int_\Omega \left(\frac{\partial u_m}{\partial x}(x,t)\right)^2 dx$$

$$+\frac{K}{2\delta}\int_0^t \int_\Omega (h(u_m(x,t)))^2\,dxd\tau + \frac{\delta}{2}\int_0^t \int_\Omega \left(\frac{\partial u_m}{\partial x}(x,t)\right)^2 dxd\tau,$$

or

$$\int_0^t \frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau - \int_0^t \frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau$$

$$\leq \left(\frac{K}{\varepsilon}(C+1) + \frac{\varepsilon}{mes(\Omega)^2} + \frac{KC}{2\delta}\right)\int_0^t \int_\Omega (u_m(x,t))^2\,dxd\tau$$

$$+\left(\varepsilon + \frac{\delta}{2}\right)\int_0^t \int_\Omega \left(\frac{\partial u_m}{\partial x}(x,t)\right)^2 dxd\tau,$$

which easily follows from the equality $u(l,t)=\int_x^l u_x(x,t)\,dx+u(x,t)$, where the constant $K=\max_{\tau\in[0,T]}\int_\Omega k_i^2(x,t)\,dxdt$, for $i=1,2$. Thus

$$\begin{aligned}
&\frac{1}{2}\|u_m\|^2_{L^2(\Omega)}+a\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}+\int_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}\\
&\quad\leq\frac{1}{2}\int_0^t\|f\|^2_{L^2(\Omega)}+\frac{1}{2}\int_0^t\|u_m\|^2_{L^2(\Omega)}+b\int_0^t\|u_m\|^2_{L^2(\Omega)}+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}\\
&\quad+\left(\frac{K}{\varepsilon}(C+1)+\frac{\varepsilon}{mes(\Omega)^2}+\frac{KC}{2\delta}\right)\int_0^t\|u_m\|^2_{L^2(\Omega)}d\tau+\left(\varepsilon+\frac{\delta}{2}\right)\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}d\tau.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\|u_m\|^2_{L^2(\Omega)}+\left(a-\left(\varepsilon+\frac{\delta}{2}\right)\right)\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}+2\int_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}\\
&\quad\leq\int_0^t\|f\|^2_{L^2(\Omega)}+\left(\frac{1}{2}+b+\frac{K}{\varepsilon}(C+1)+\frac{\varepsilon}{mes(\Omega)^2}+\frac{KC}{2\delta}\right)\int_0^t\|u_m\|^2_{L^2(\Omega)}+\|\varphi_m\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using the Lemma of Gronwall yields

$$\begin{aligned}
&\|u_m\|^2_{L^\infty(0,T,\,L^2(\Omega))}+\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)}+\|u_m\|^{p+1}_{L^{p+1}(0,T,\,L^{p+1}(\Omega))}\\
&\quad\leq\frac{\exp\left(\left(\frac{1}{2}+b+\frac{K}{\varepsilon}(C+1)+\frac{\varepsilon}{mes(\Omega)^2}+\frac{KC}{2\delta}\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)}+\|\varphi_m\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

So, by putting

$$C_T=\frac{\exp\left(\left(\frac{1}{2}+b+\frac{K}{\varepsilon}(C+1)+\frac{\varepsilon}{mes(\Omega)^2}+\frac{KC}{2\delta}\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)}+\|\varphi_m\|^2_{L^2(\Omega)}\right),\tag{3.9}$$

we obtain

$$\|u_m\|^2_{L^\infty(0,T,\,L^2(\Omega))}+\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)}+\|u_m\|^{p+1}_{L^{p+1}(0,T,\,L^{p+1}(\Omega))}\leq C_T,\tag{3.10}$$

where C_T is a positive constant depending only on $\int_0^T\|f(\tau)\|^2_{L^2(\Omega)}$, $\|\varphi_m\|^2_{L^2(\Omega)}$ and T. It follows from (3.10) that $\|u_m(t)\|^2_{L^2(\Omega)}\leq C_T$. This implies that the solution to the initial value problem for the system of ODE (3.8) can be extended to $[0,T]$. As a result, on such an interval, we have the following uniform priori estimates:

$$\begin{cases}
u_m \text{ uniformly bounded in } L^\infty(0,T;\,L^2(\Omega))\\
u_m \text{ uniformly bounded in } L^2(0,T;\,H^1(\Omega))\\
u_m \text{ uniformly bounded in } L^{p+1}\left(0,T;\,L^{p+1}(\Omega)\right)
\end{cases}.$$

Now we could find more priori estimates. By using the same formulation variational by $g'_{km}(t)$ and then summing the result over k, we obtain

$$\sum_{k=1}^{m}\left((u_m)_t, w_k\right) g'_{km}(t)+\sum_{k=1}^{m} a\left(\Delta u_m, w_k\right) g'_{km}(t)+\sum_{k=1}^{m}\left(u_m^p-b u_m, w_k\right) g'_{km}(t)=\sum_{k=1}^{m}\left(f, w_k\right) g'_{km}(t). \tag{3.11}$$

So, it comes

$$\left((u_m)_t,(u_m)_t\right)-a\left(\frac{\partial^2 u_m}{\partial x^2},(u_m)_t\right)+\left(u_m^p-b u_m,(u_m)_t\right)=\left(f,(u_m)_t\right).$$

By integration over $(0,t)$, one has

$$\int_0^t\left((u_m)_t,(u_m)_t\right)-a\int_0^t\left(\frac{\partial^2 u_m}{\partial x^2},(u_m)_t\right)+\int_0^t\left(u_m^p-b u_m,(u_m)_t\right)=\int_0^t\left(f,(u_m)_t\right).$$

It follows that

$$\begin{aligned}
&\|(u_m)_t\|^2_{L^2(Q)}+\frac{a}{2}\|(u_m)_x\|^2_{L^2(\Omega)}+\frac{1}{p+1}\|u_m\|^{p+1}_{L^{p+1}(\Omega)}\\
&\qquad\leq\int_0^t\left(f+b u_m,(u_m)_t\right)+\frac{a}{2}\|(\varphi_m)_x\|^2_{L^2(\Omega)}+\frac{1}{p+1}\|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)}\\
&\qquad+a\int_0^t(u_m)_x(l,t)\cdot(u_m)_t(l,t)\,dx-a\int_0^t(u_m)_x(0,t)\cdot(u_m)_t(0,t)\,dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^t\left(\int_\Omega k_2(x,t)\,h(u_m(x,t))dx\right)(u_m)_t(l,t)\,d\tau\\
&+\int_0^t\left(\int_\Omega k_1(x,t)\,u_m(x,t)dx\right)\left(-(u_m)_t(l,t)+\frac{d}{dt}\int_\Omega\frac{\partial u_m}{\partial x}dx\right)dt\\
&=\int_0^t\left(\int_\Omega k_2(x,t)\,h(u_m(x,t))dx-\int_\Omega k_1(x,t)\,u_m(x,t)dx\right)\\
&\left((u_m)_t(x,t)+\frac{d}{dt}\int_x^l\frac{\partial u_m}{\partial\eta}d\eta\right)d\tau+\int_0^t\left(\int_\Omega k_1(x,t)\,u_m(x,t)dx\right)\left(\frac{d}{dt}\int_\Omega\frac{\partial u_m}{\partial x}dx\right)dt
\end{aligned}$$

$$\leq \left\| \int_{\Omega} k_2(x,t)\, h(u_m(x,t))dx - \int_{\Omega} k_1(x,t)\, u_m(x,t)dx \right\|_{\infty}$$
$$\left(\int_0^t (u_m)_t(x,t)\, d\tau + \int_0^l \frac{\partial u_m}{\partial x} dx - \int_0^l \frac{\partial \varphi_m}{\partial x} dx \right)$$
$$+ \left\| \int_{\Omega} k_1(x,t)\, u_m(x,t)dx \right\|_{\infty} \left(\int_{\Omega} \frac{\partial u_m}{\partial x} dx - \int_{\Omega} \frac{\partial \varphi_m}{\partial x} dx \right).$$
$$\leq \frac{K}{\varepsilon} \left[\|h(u_m)\|^2_{L^\infty(0,T,L^2(\Omega))} + \|u_m\|^2_{L^\infty(0,T,L^2(\Omega))} \right] + \frac{1}{\varepsilon} \|(u_m)_t\|_2^2$$
$$+ \frac{1}{\varepsilon} \|(u_m)_x\|_2^2 + \frac{1}{\varepsilon} \|(\varphi_m)_x\|_2^2 + \frac{K}{2\delta} \|u_m\|^2_{L^\infty(0,T,L^2(\Omega))} + \frac{K}{\delta} \left(\|(u_m)_x\|_2^2 + \|(\varphi_m)_x\|_2^2 \right)$$
$$\leq \left(\frac{2K}{\varepsilon} + \frac{K}{2\delta} \right) \|u_m\|^2_{L^\infty(0,T,L^2(\Omega))} + \frac{1}{\varepsilon} \|(u_m)_t\|_2^2 + \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \|(u_m)_x\|_2^2 + \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \|(\varphi_m)_x\|_2^2 .$$

Consequently,

$$\|(u_m)_t\|^2_{L^2(Q)} + \frac{a}{2} \|(u_m)_x\|^2_{L^2(\Omega)} + \frac{1}{p+1} \|u_m\|^{p+1}_{L^{p+1}(\Omega)}$$
$$\leq \frac{1}{2} \|(u_m)_t\|^2_{L^2(Q)} + \frac{1}{2} \|f\|^2_{L^2(Q)} + \frac{b}{2} \|u_m\|^2_{L^2(\Omega)} + \frac{a}{2} \|(\varphi_m)_x\|^2_{L^2(\Omega)}$$
$$+ \frac{1}{p+1} \|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)} - \frac{b}{2} \|\varphi_m\|^2_{L^2(\Omega)} + a \left(\frac{2K}{\varepsilon} + \frac{K}{2\delta} \right) \|u_m\|^2_{L^\infty(0,T,L^2(\Omega))}$$
$$+ \frac{a}{\varepsilon} \|(u_m)_t\|_2^2 + a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \|(u_m)_x\|_2^2 + a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \|(\varphi_m)_x\|_2^2 .$$

Hence, we find

$$\left(\frac{1}{2} - \frac{1}{\varepsilon} \right) \|(u_m)_t\|^2_{L^2(Q)} + \left(\frac{a}{2} - a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \right) \|(u_m)_x\|^2_{L^2(\Omega)} + \frac{1}{p+1} \|u_m\|^{p+1}_{L^{p+1}(\Omega)}$$
$$\leq \frac{1}{2} \|f\|^2_{L^2(Q)} + \left(\frac{a}{2} + a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \right) \|(\varphi_m)_x\|^2_{L^2(\Omega)}$$
$$+ \left(\frac{b}{2} + a \left(\frac{2K}{\varepsilon} + \frac{K}{2\delta} \right) \right) \|u_m\|^2_{L^\infty(0,T,L^2(\Omega))} + \frac{1}{p+1} \|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)} - \frac{b}{2} \|\varphi_m\|^2_{L^2(\Omega)} .$$

Now, for all $\varepsilon > 0$ and $\delta > 0$, we put

$$C_1 = \min \left\{ \left(\frac{1}{2} - \frac{1}{\varepsilon} \right), \frac{a}{2} - a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right), \frac{1}{(p+1)} \right\},$$

and

$$C_2 = \max \left\{ \frac{1}{2}, \left(\frac{a}{2} + a \left(\frac{1}{\varepsilon} + \frac{K}{\delta} \right) \right), \frac{1}{p+1}, -\frac{b}{2} \right\}.$$

Then, we have

$$C_1 \left[\|(u_m)_t\|^2_{L^2(Q)} + \|(u_m)_x\|^2_{L^2(Q)} + \int_0^t \|u_m\|^{p+1}_{L^{p+1}(\Omega)} \right]$$
$$\leq C_2 \left[\|f\|^2_{L^2(Q)} + \|(\varphi_m)_x\|^2_{L^2(\Omega)} + \|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)} \right] + \left(\frac{b}{2} + a \left(\frac{2K}{\varepsilon} + \frac{K}{2\delta} \right) \right) C_T . \tag{3.12}$$

Hence, $\|(u_m)_t\|^2_{L^2(Q)}$ is bounded, so we can get the following further priori estimates:

$$\begin{cases} u_m \text{ uniformly bounded in } L^{p+1}(0,T;\ L^{P+1}(\Omega)) \\ u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t \text{ uniformly bounded in } L^2\left(0,T;\ L^2(\Omega)\right) \end{cases}. \tag{3.13}$$

Thus, by Lemma 1.2, there is a subsequence of u_m, still denoted by u_m, such that

$$\begin{cases} u_m \longrightarrow u \text{ weakly in } L^{p+1}(0,T;\ L^{P+1}(\Omega)) \\ u_m \longrightarrow u \text{ weakly in } L^2(0,T;\ H^1(\Omega)) \\ u_m \longrightarrow u \text{ weakly in } L^2\left(0,T;\ L^2(\Omega)\right) \end{cases}. \tag{3.14}$$

We deduce from Lemma 1.2 that there are subsequences denoted by (u_{m_k}) and $\left(\frac{\partial u_{m_k}}{\partial t}\right)$ of (u_m) and $(u_m)_t$, respectively, such that

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T;\ H^1(\Omega)) \tag{3.15}$$

and

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \qquad \text{in } L^2\left(0,T;\ L^2(\Omega)\right) \ . \tag{3.16}$$

We know, according to Relikh-Kondrachoff's theorem, that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. Also, like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. Thus

$$u_{m_k} \longrightarrow u \qquad \text{in } L^2(Q)\ . \tag{3.17}$$

On the other hand, from Lemma 1.3, we see that there exists a subsequence of $(u_{m_k})_k$ that is still denoted by u_{m_k} and converges almost everywhere to u such that

$$u_{m_k} \longrightarrow u \qquad \text{almost everywhere } Q\ . \tag{3.18}$$

By Lemma 1.3, there exists a subsequence of u_m, still denoted by u_m, such that u_m almost everywhere converges to u in $Q_T = \Omega \times [0,T]$. It turns out that

$$(u_m)^p \text{ almost everywhere converges to } u^p \text{in } Q_T.$$

On the other hand, (3.13) implies that $(u_m)^p$ is uniformly bounded in $L^{\frac{p+1}{p}}(Q_T)$. Therefore, we infer, from Lemma 1.4, that

$$u_m^p \rightharpoonup u^p \qquad \text{weakly in } L^{\frac{p+1}{p}}\left(0,T\ , L^{\frac{p+1}{p}}(\Omega)\right),$$

which remains to demonstrate that $w = \frac{\partial u}{\partial t}$. For this purpose, it suffices to prove

$$u(t) = \varphi + \int_0^t w(\tau)d\tau \tag{3.19}$$

as $u_{m_k} \rightharpoonup u$ in $L^2(0,T\ ; L^2(\Omega))$. In this regard, we find that the proof of (3.19) is equivalent to demonstrate that $u_{m_k} \rightharpoonup \varphi + \chi$ in $L^2(0,T\ ; L^2(\Omega))$, which means

$$\lim\,(u_{m_k} - \varphi - \chi, v)_{L^2(0,T\ ; L^2(\Omega))} = 0,\ \forall v \in L^2(0,T\ ; L^2(\Omega)),$$

as $\chi(t)=\int_0^t w(\tau)d\tau$. By using the equality,

$$u_{m_k}-\varphi_{m_k}=\int_0^t \frac{\partial u_{m_k}}{\partial \tau}d\tau, \text{ for all } t\in[0,T],$$

which results from $u_{m_k}\in L^2(0,T;V_{m_k})$ and $(u_{m_k})_t\in L^2(0,T;V_{m_k})$, we obtain

$$\begin{aligned}
&\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}\\
&=\left(u_{m_k}-\varphi_{m_k}-\int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}+(\varphi_{m_k}-\varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}\\
&=\left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau}-w(\tau)\right)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}+(\varphi_{m_k}-\varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}, \text{ for all } t\in[0,T].
\end{aligned}$$

Consequently, by virtue of **(ii)** of Lemma 1.6, it comes

$$\begin{aligned}
&\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}\\
&=\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau}-w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))}d\tau+(\varphi_{m_k}-\varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}, \text{ for all } t\in[0,T].
\end{aligned}$$

On the one hand, we have

$$\lim_{k\longrightarrow\infty}\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau}-w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))}d\tau=0, \text{ for } t\in[0,T]. \tag{3.20}$$

Also, we obtain

$$\lim_{k\longrightarrow\infty}(\varphi_m-\varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}=0. \tag{3.21}$$

So, we have

$$\lim_{k\longrightarrow\infty}(u_{m_k}-\varphi-\chi, v)_{L^2(0,T\,;\,L^2(\Omega))}=0, \ \ \forall v\in L^2\left(0,T;L^2(\Omega)\right).$$

Finally, from (3.20) and (3.21), we arrive at

$$\lim_{k\longrightarrow\infty}(u_{m_k}-\varphi-\chi, v)_{L^2(0,T\,;\,L^2(\Omega))}=0, \ \ \forall v\in L^2\left(0,T;L^2(\Omega)\right).$$

Now, since each term on the left side of (P_2) is weakly convergent in $L^{\frac{p+1}{p}}(\Omega)$, by passing to the limit in (P_2), we obtain that the following equality holds in $L^{\frac{p+1}{p}}(\Omega)$:

$$((u_m(t))_t, w_k)+a(u_m(t), w_k)+(u_m^p(t)-bu_m(t), w_k)=(f(t), w_k) \quad \forall k=\overline{1,m}. \tag{3.22}$$

Since $\{w_j, \ \ j \in \mathbb{N}\}$ is a base in $L^{\frac{p+1}{p}}(\Omega)$, we infer from (3.22) that the following equality also holds in $L^{\frac{p+1}{p}}\left(0,T \ , L^{\frac{p+1}{p}}(\Omega)\right)$:

$$u' - a\Delta u + u^p - bu = f. \tag{3.23}$$

Since all $u', \Delta u$, and f belong to $L^2\left(0,T;L^2(\Omega)\right)$, then u^p also belongs to $L^2\left(0,T;L^2(\Omega)\right)$. Hence (3.23) also holds in $L^2\left(0,T;L^2(\Omega)\right)$. □

4. The Uniqueness of the Solution

In this section, we study the uniqueness of the solution only in the case that p is odd. For this purpose, we state and prove the subsequent theorem.

Theorem 4.1. *Let $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and $f \in L^2\left(0,T;L^2(\Omega)\right)$. Then problem (P_1) admits the unique solution u such that $u \in L^2\left(0,T, \ H^1(\Omega)\right) \cap \ L^{p+1}(0,T; \ L^{P+1}(\Omega))$ and $u' \in L^2(0,T, \ L^2(\Omega))$.*

Proof. Let p be odd. Multiplying the equation of problem (P_1) by the following multiplier Mu, $Mu = u$, and integrating the result over the domain $\Omega = (0,l)$, we obtain

$$\begin{aligned}\int_\Omega [u_t - a\Delta u + u^p - bu] \, .Mudx &= \int_\Omega [u_t(x,t) - a\Delta u(x,t) + u^p - bu] \cdot udx \\ &= \int_\Omega u_t(x,t)udx - a\int_\Omega \Delta u \cdot udx + \int_\Omega u^p udx - b\int_\Omega u^2 dx \\ &= \int_\Omega f(x,t)udx,\end{aligned}$$

where u_t denotes the partial derivative with respect to t. Thus

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} + a\int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dx + \int_\Omega u^{p+1}dx - b\int_\Omega u^2 dx \\ = \int_\Omega fudx + a\frac{\partial u}{\partial x}(l,t)u(l,t) - a\frac{\partial u}{\partial x}(0,t)u(0,t).\end{aligned}$$

Then, by integrating on $(0,\tau)$, where $\tau \in (0,T)$, and by doing the same steps to get (3.12), we obtain

$$\begin{aligned}&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + (a - a\varepsilon)\int_0^t \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)} d\tau + \left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{K}{2\varepsilon} + b\right)\frac{2}{p+1}\right)\int_0^t \|u\|^{p+1}_{L^{p+1}(\Omega)} d\tau \\ &\leq \frac{\exp\left(\left(\frac{1}{2} + b + \frac{K}{\varepsilon}(C+1) + \frac{\varepsilon}{mes(\Omega)^2} + \frac{KC}{2\delta}\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)} + \|\varphi_m\|^2_{L^2(\Omega)}\right).\end{aligned}$$

By putting

$$C_T = \frac{\exp\left(\left(\frac{1}{2} + b + \frac{K}{\varepsilon}(C+1) + \frac{\varepsilon}{mes(\Omega)^2} + \frac{KC}{2\delta}\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)} + \|\varphi_m\|^2_{L^2(\Omega)}\right), \tag{4.1}$$

we have

$$\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} d\tau + \int_0^t \|u_m\|^{p+1}_{L^{p+1}(\Omega)} d\tau \leq C_T. \tag{4.2}$$

Now, we put

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u\|^2_{L^2(0,T;L^2(\Omega))} + \|u\|^{p+1}_{L^{p+1}(0,T\,;L^{p+1}(\Omega))} \equiv \|u\|_B.$$

Also, we let u_1 and u_2 be two solutions to problem (P$_1$) such that

$$\begin{cases} Lu_1 = \mathscr{F} \\ Lu_2 = \mathscr{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

where L is the differential operator of the main semi-linear problem. Then, $L(u_1-u_2)=0$. As a result, $\|u_1-u_2\|_B \leq c\,\|0\|_F = 0$, which gives $u_1 = u_2$. □

5. Finite Time Blow-up Solution

Theorem 5.1. *Let $h(u) = u^s$, where $s = \frac{p+1}{2}$ in problem (P_1). Then, for all $\varepsilon > 0$, the solution u of (P_1) blows up in a finite time*

$$T = \frac{1}{K}\ln\left(\frac{\frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)}{(\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}}\right),$$

where $K = \max_{i\in\{1,2\},\forall(x,t)\in Q} |k_i(x,t)|$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By putting $\Pi(t) = \int_\Omega u^2 dx$ and by integrating the equation over Ω with null source, we have

$$\begin{aligned} u_t - a\Delta u - bu &= u^p \\ \int_\Omega u_t u dx - a\int_\Omega \Delta u.u dx - \int_\Omega bu^2 dx &= \int_\Omega u^{p+1} dx. \end{aligned}$$

By using Green's formula, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - \int_\Omega bu^2 dx + \int_\Omega u^{p+1} dx &\leq a\left[u_x(l,t)u(l,t) - u_x(0,t)u(0,t)\right] \\ \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - \int_\Omega bu^2 dx + \int_\Omega u^{p+1} dx &\leq a\left[\begin{array}{l} \int_\Omega k_2(x,t)g(u(x,t))dx u(l,t) \\ -\int_\Omega k_1(x,t)u(x,t)dx u(0,t) \end{array}\right]. \\ \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - b\int_\Omega u^2 dx + \int_\Omega u^{p+1} dx &\leq aK\left[\int_\Omega u^s dx u(l,t) - \int_\Omega u dx u(0,t)\right], \end{aligned}$$

which yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - \int_\Omega bu^2 dx + \int_\Omega u^{p+1} dx \\ &\quad \leq aK\left(\frac{1}{\varepsilon} + \frac{\varepsilon}{2l} + \frac{1}{2\delta}\right)\left[\int_\Omega u^2\right] + aK\left(\frac{\varepsilon}{2} + \frac{\delta}{2}\right)\left[\int_\Omega u_x^2\right] + \frac{1}{\varepsilon}\int_\Omega u^{2s} dx - \int_\Omega u^{p+1} dx. \end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\left(a+aK\left(\frac{\varepsilon}{2}+\frac{\delta}{2}\right)\right)\int_\Omega u_x^2-\left[aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right]\int_\Omega u^2dx$$
$$\leq-\int_\Omega u^{p+1}dx+\frac{1}{\varepsilon}\int_\Omega u^{2s}dx,\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx-\left[aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right]\int_\Omega u^2dx$$
$$\leq-\int_\Omega u^{p+1}dx+\frac{1}{\varepsilon}\int_\Omega u^{2s}dx.$$

It follows from the Jensen inequality that

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)\int_\Omega u^2dx$$
$$\geq+\int_\Omega u^{p+1}dx-\frac{1}{\varepsilon}\int_\Omega u^{2s}dx-\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)\int_\Omega u^2dx$$
$$\geq\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}-\frac{1}{\varepsilon}\right)\left(\int_\Omega u^2\right)^{\frac{p+1}{2}},$$

for $s=\frac{p+1}{2}$ and $\forall\varepsilon>0$. Thus, we get

$$-\frac{1}{2}\Pi'(t)+\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)\Pi(t)=\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}-\frac{1}{\varepsilon}\right)(\Pi(t))^{\frac{q+1}{2}}.$$

Therefore, we obtain

$$\Pi'(t)-2\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)\Pi(t)+2\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}-\frac{1}{\varepsilon}\right)(\Pi(t))^{\frac{p+1}{2}}=0. \quad (5.1)$$

To solve this equation, we use the following change of variable: $v=\Pi^{1-q}$, where $q=\frac{p+1}{2}$. Thus

$$\frac{1}{q-1}v'v^{\frac{q}{1-q}}-2\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)v^{\frac{1}{1-q}}+2\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}-\frac{1}{\varepsilon}\right)v^{\frac{q}{1-q}}=0. \quad (5.2)$$

By multiplying the equation (5.2) by $v^{\frac{-q}{1-q}}$, we obtain

$$v'-2(q-1)\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)v+2(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}-\frac{1}{\varepsilon}\right)=0. \quad (5.3)$$

So, we solve the following homogeneous equation:

$$v'+2(q-1)\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right)v=0.$$

To do so, we put

$$K=2(q-1)\left(aK\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{2l}+\frac{1}{2\delta}\right)+b\right).$$

Then, $v'+Kv=0$ and $v_1(t)=Be^{-Kt}$.

Now, we move to the non-homogeneous equation (5.3) by the method of constant variation, where we set $v_2(t) = B(t)e^{-Kt}$. Thus

$$v_2(t) = 2(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right),$$

which implies

$$v(t) = v_1(t) + v_2(t) = Be^{-Kt} + \frac{1}{K}2(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right).$$

Therefore, we obtain

$$\Pi(t) = \left(Be^{-Kt} + \frac{1}{K}2(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)\right)^{\frac{1}{1-q}}.$$

For $t = 0$, we get

$$\Pi(0) = \left(B + \frac{1}{K}2(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)\right)^{\frac{1}{1-q}},$$

which means

$$B = (\Pi(0))^{1-q} - \frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right).$$

Finally,

$$\Pi(t) = \left(Be^{-Kt} + \frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)\right)^{\frac{1}{1-q}}.$$

Then, we obtain

$$\Pi(t) = \left(\frac{1}{\left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)e^{-Kt} + \frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)}\right)^{\frac{1}{q-1}}$$

So, like $\dfrac{1}{q-1} > 0$, $\Pi \to \infty$ if

$$\left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)e^{-Kt} + \frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right) \to 0,$$

and hence we have

$$T = \frac{1}{K}\ln\left(\frac{\frac{1}{K}(q-1)\left(\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}} - \frac{1}{\varepsilon}\right)}{(\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}}\right).$$

□

6. Conclusion

The goal of this work is to analyze a class of reaction-diffusion equations' super-linear nonlocal problem with Neumann condition modeling by the integral condition of second type. By using the Fadeo-Galarkin method to avoid the complications caused by the integral condition's existence, we demonstrated the existence of the weak solution. Also, we demonstrated the uniqueness of the weak solution to the problem by using an a priori estimate as well as we examined the blow-up solution in the finite-time case.

References

[1] I.M. Batiha, T.E. Oussaeif, A. Benguesmia, A.A. Abubaker, A. Ouannas, S. Momani, A study of a superlinear parabolic dirichlet problem with unknown coefficient, Int. J. Innovative Computing, Information & Control 20 (2024) 541-556.

[2] A. Fatma, I. Batiha, R. Imad, O. Takieddine, A. Ouannas, Solvability and weak controllability of fractional degenerate singular problem, J. Robotics and Control 5 (2024) 542-550.

[3] I.M. Batiha, O. Ogilat, I. Bendib, A. Ouannas, I.H. Jebril, N. Anakira, Finite-time dynamics of the fractional-order epidemic model: Stability, synchronization, and simulations, Chaos Solitons Fractals X 13 (2024) 100118.

[4] Z. Chebana, T.E. Oussaeif, S. Dehilis, A. Ouannas, I.M. Batiha, On nonlinear Neumann integral condition for a semilinear heat problem with blowup simulation, Palestine J. Math. 12 (2023) 390-394.

[5] I.M. Batiha, O. Ogilat, A. Hioual, A. Ouannas, N. Anakira, A.A. Amourah, S. Momani, On discrete FitzHugh–Nagumo reaction–diffusion model: Stability and simulations, Partial Differential Equations Appl. Math. 11 (2024) 100870.

[6] I.M. Batiha, B. Laouadi, I.H. Jebril, L. Benaoua, T.E. Oussaeif, S. Alkhazaleh, Some new fixed point theorems of rational type contraction with an application to coupled fixed point theory, Adv. Fixed Point Theory 14 (2024) 52.

[7] L.C. Ceng, S.Y. Cho, On approximate controllability for systems of fractional evolution hemivariational inequalities with Riemann-Liouville fractional derivatives, J. Nonlinear Var. Anal. 6 (2022) 421-438.

[8] I.M. Batiha, I.H. Jebril, Z. Chebana, T.E. Oussaeif, S. Dehilis, S. Alkhazaleh, Finite-time blow-up and solvability of a weak solution for a superlinear reaction-diffusion problem with integral conditions of the second type, Adv. Fixed Point Theory, 14 (2024) 45.

[9] I.M. Batiha, I. Rezzoug, T.E. Oussaeif, A. Ouannas, I.H. Jebril, Pollution detection for the singular linear parabolic equation, J. Appl. Math. Info. 41 (2023) 647-656.

[10] I.M. Batiha, N. Barrouk, A. Ouannas, A. Farah, A study on invariant regions, existence and uniqueness of the global solution for tridiagonal reaction-diffusion systems, J. Appl. Math. Info. 41 (2023) 893-906.

[11] I.M. Batiha, Z. Chebana, T.E. Oussaeif, A. Ouannas, I.H. Jebril, M. Shatnawi, Solvability of nonlinear wave equation with nonlinear integral Neumann conditions, Int. J. Anal. Appl. 21 (2023) 34.

[12] A. Jameel, N.A. Anakira, A.K. Alomari, N.H. Man, Solution and analysis of the fuzzy Volterra integral equations via homotopy analysis method, Comput. Modeling in Eng. Sci. 127 (2021) 875-99.

[13] A.F. Jameel, N.A. Anakira, A.K. Alomari, D.M. Alsharo, A. Saaban, New semi-analytical method for solving two point nth order fuzzy boundary value problem, Int. J. Math. Modelling Numer. Optim. 9 (2019) 12-31,

[14] J.R. Cannon, The solution of the heat equation anubject to the specification of energy, Quarterly Appl. Math. 21 (1963) 155-160.

[15] A.A. Samarskii, Some problems of the theory of differential equations, Differential'nye Uravneniya 16 (1980) 1925-1935.

[16] S. Dhelis, A. Bouziani, T.E. Oussaeif, Study of solution for a parabolic integrodifferential equation with the second kind integral condition, Int. J. Anal. Appl. 16 (2018) 569-593.

[17] T.E. Oussaeif, A. Bouziani, Inverse problem of a hyperbolic equation with an integral overdetermination condition, Electron. J. Differential Equations 2016 (2016) 1-7.

[18] A. Bouziani, T.E. Oussaeif, L. Benaoua, A mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator, J. Math. 2013 (2013), 457631.

[19] T.E. Oussaeif, A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, Electron. J. Differential Equations 2014 (2014) 1-10.
[20] T.E. Oussaeif, A. Bouziani, Mixed problem with an integral two-space-variables condition for a class of hyperbolic equations, Int. J. Anal. 2013 (2013) 957163.
[21] T.E. Oussaeif, A. Bouziani, Mixed problem with an integral two-space-variables condition for a parabolic equation, Int. J. Evolution Equations 9 (2014) 181-198.
[22] T.E. Oussaeif, A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, Electron. J. Differential Equations 2014 (2014) 1-10.
[23] O. Taki-Eddine, B. Abdelfatah, A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition, Chaos, Solitons & Fractals 103 (2017) 79-89.
[24] Y. Boukhatem, B. Benabderrahmane, A. Rahmoune, Méthode de Faedo-Galerkin pour un probléme aux limites non linéaire, Analele Universităţii Orade, Fasc. Matematica 16 (2009) 167-181.
[25] S.M. Zheng, Nonlinear Evolution Equations, Chapman & Hall/CRC, London, 2004.
[26] T. Oussaeif, A. Bouziani, Solvability of nonlinear Goursat type problem for hyperbolic equation with integral condition, Khayyam J. Math. 4 (2018) 198-213.
[27] R. Imad, O. Taki-Eddine, B. abdelouahab, Solvability of a solution and controllability for nonlinear fractional differential equations, Bull. Institute Math. 15 (2020), 237-249.
[28] T.E. Oussaeif, A. Bouziani, Solvability of nonlinear viscosity equation with a boundary integral condition, J. Nonlinear Evol. Equ. Appl. 2015 (2015) 31-45.
[29] B. Sihem, O. Taki Eddine, B. Abdelfatah, Galerkin finite element method for a semi-linear parabolic equation with purely integral conditions, Bol. Soc. Paran. Mat. 40 (2022) 1-15.