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THE UNIQUENESS OF FIXED POINTS FOR TWO NEW CLASSES OF CONTRACTIVE MAPPINGS OF INTEGRAL TYPE IN b-METRIC SPACES

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Abstract. In this paper, the concepts of the Sehgal-Guseman contraction of integral type and the φ-E-contraction of integral type are introduced to establish some fixed point results. The existence and uniqueness of fixed points of these contractions of integral type under special conditions in b-metric spaces are studied. Furthermore, two examples are given to prove the feasibility of our results.
Keywords. $b-$metric; Contractive mapping of integral type; Fixed point.
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1. INTRODUCTION

In 1993, Czerwik [6] first introduced the concept of $b-$metric space by changing the third condition of the definition of metric spaces and generalized the celebrated Banach contraction principle [4] in the setting of $b-$metric spaces. Since then, authors have extended the Banach contraction principle in various aspects. In 1969, Sehgal [16] generalized the Banach contraction principle and presented a new inequality. Later, Matkowski [13] generalized some results of Sehgal [16], Khazanchi [10], and Iseki [9]. In 2008, Raja and Vaezpour [14] proved some fixed point results for Sehgal-Guseman-type theorems in cone metric spaces. After that, more scholars promoted the Sehgal-Guseman type in various spaces. Among them, the notion of Geraghty contraction of type E was introduced by Fulga and Proca [7]. Lang et al. [11] and Alqahtani et al. [3] introduced the $\alpha-\varphi_E-$Geraghty contraction mapping and Sehgal type contractions in $b-$metric space respectively. In particular, Guan et al. [8] investigated triangular $\alpha-$orbital admissible mappings and fixed point theorems in $b-$metric spaces as follows:

Theorem 1.1. *([8]) Let $(\Omega,\hbar)$ be a complete rectangular $b-$metric space with $s\geq 1$. Let $\mathbb{T}:\Omega\to\Omega$ be a continuous injectivity, $\alpha:\Omega\times\Omega\to[0,+\infty)$, and $p>0$. Assume that, for any*

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$\varepsilon \in \Omega$, *there exists a positive number* $n(\varepsilon)$ *with*

$$\forall \partial \in \Omega, \alpha(\varepsilon,\partial) \geq s^p \Rightarrow \alpha(\varepsilon,\partial)\hbar(\mathbb{T}^{n(\varepsilon)}\varepsilon,\mathbb{T}^{n(\varepsilon)}\partial) \leq \Phi(\hbar(\varepsilon,\partial),\hbar(\varepsilon,\mathbb{T}^{n(\varepsilon)}\varepsilon),\hbar(\varepsilon,\mathbb{T}^{n(\varepsilon)}\partial)),$$

where $\Phi \in \Theta$ *and*

(1) $\lim_{\varepsilon\to\infty}(\varepsilon - s\varphi(\varepsilon)) = \infty$,

(2) for all $\varepsilon > 0$, $\lim_{m\to\infty}\varphi^m(\varepsilon) = 0$.

Suppose that

(i) there exists an ε_0 *in* Ω *such that* $\alpha(\varepsilon_0,\mathbb{T}\varepsilon_0) \geq s^p$,

(ii) $\mathbb{T}$ *is triangular* α_{s^p} *orbital admissible,*

(iii) If $\{\partial_n\}$ *is* Ω *satisfies* $\alpha(\partial_n,\partial_{n+1}) \geq s^p(\forall n \in N)$ *and* $\partial_n \to \partial \in \Omega(n\to\infty)$, *then one can choose a subsequence* $\{\partial_{n_k}\}$ *of* $\{\partial_n\}$ *with* $\alpha(\partial_{n_k},\partial) \geq s^p, \forall k \in \mathbb{N}$,

(iv) for all $\varepsilon \in \Omega$ *with* $\mathbb{T}^{n(\varepsilon)}\varepsilon = \varepsilon$, $\alpha(\varepsilon,\partial) \geq s^p$ *for any* $\partial \in \Omega$.

Then, $\mathbb{T}$ *has a unique fixed point* $\varepsilon^* \in \Omega$. *Further, for each* $\varepsilon \in \Omega$, $\mathbb{T}^n\varepsilon$ *converges to* ε^*.

In this paper, we give the fixed point theorem of two integral type contractive mappings in $b-$metric spaces. Furthermore, two examples are given to prove the feasibility of the theorem.

2. PRELIMINARIES

Throughout this paper, we assume that $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, where $\mathbb{N}$ stands for the set of positive integers and

$$\Theta = \{\xi | \xi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies that } \xi \text{ is Lebesgue integrable, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^\delta \xi(t)\,dt > 0 \text{ for each } \delta > 0\},$$

We introduce the following definitions and lemmas.

Definition 2.1. ([6]) Let Ξ be a nonempty set and $s \geq 1$ be a given real number. A mapping $\varpi : \Xi \times \Xi \to [0,+\infty)$ is said to be a $b-$metric if and only if, for all $\kappa,\mu,\nu \in \Xi$, the following conditions are satisfied:

(i) $\varpi(\kappa,\mu) = 0$ if and only if $\kappa = \mu$;

(ii) $\varpi(\kappa,\mu) = \varpi(\mu,\kappa)$;

(iii) $\varpi(\kappa,\mu) \leq s(\varpi(\kappa,\nu)+\varpi(\mu,\nu))$.

Generally, (Ξ,ϖ) is called a $b-$metric space with parameter $s \geq 1$.

Remark 2.2. Every metric space is a $b-$metric space with $s = 1$. One can discover several examples of $b-$metric spaces which are not metric spaces(see [5]).

Example 2.3. ([2]) Let (Ξ,d) be a metric space, and $\varpi(\kappa,\mu) = (d(\kappa,\mu))^q$, where $q > 1$ is a real number. Then $\varpi(\kappa,\mu)$ is a $b-$metric with $s = 2^{q-1}$.

Definition 2.4. ([15]) Let (Ξ,ϖ) be a $b-$metric space with parameter $s \geq 1$. Then a sequence $\{\kappa_n\}$ in Ξ is said to be:

(i) $b-$convergent if and only if there exists $\kappa \in \Xi$ such that $\varpi(\kappa_n,\kappa) \to 0$ as $n \to +\infty$.

(ii) a Cauchy sequence if and only if $\varpi(\kappa_n,\kappa_m) \to 0$ when $n,m \to +\infty$.

As usual, a $b-$metric space is called complete if and only if each Cauchy sequence in this space is $b-$convergent.

Definition 2.5. ([8]) Let Ξ be a nonempty set, $s \geq 1$ and $p > 0$ be constants and $\alpha : \Xi \times \Xi \to [0, +\infty)$ be a function. The self-mapping $T : \Xi \to \Xi$ is said to be triangular α_{s^p} orbital admissible if

(i) $\alpha(x, Tx) \geq s^p \Rightarrow \alpha(Tx, T^2x) \geq s^p$,

(ii) $\alpha(x, y) \geq s^p$ and $\alpha(y, Ty) \geq s^p$ implies $\alpha(x, Ty) \geq s^p$, for all $x, y \in \Xi$.

Definition 2.6. ([1]) The mapping $\phi \in \Theta$ is said to be integral subadditive if, for each $a, b > 0$,

$$\int_0^{a+b} \phi(t)\,dt \leq \int_0^a \phi(t)\,dt + \int_0^b \phi(t)\,dt.$$

One denotes by Θ_s the class of all integral subadditive functions $\phi \in \Theta$.

The following lemmas play a key role in our conclusion.

Lemma 2.7. *([8]) Let Ξ be nonempty set and $\alpha : \Xi \times \Xi \to [0, +\infty)$ be a function. Let $T : \Xi \to \Xi$ be a self-mapping such that T is a triangular α_{s^p} orbital admissible mapping, $s \geq 1, p > 0$. Assume that there exists $x_0 \in \Xi$ such that $\alpha(x_0, Tx_0) \geq s^p$. Define a sequence $\{x_n\}$ in Ξ by $x_1 = T^{n(x_0)}x_0, \cdots, x_{n+1} = T^{n(x_n)}x_n \cdots$. Then, for $m \in \mathbb{N}_0$, $\alpha(x_m, T^k x_m) \geq s^p, k = 0, 1, 2, \cdots$.*

Lemma 2.8. *([8]) Let $\varphi : [0, +\infty) \to [1, +\infty)$ be nondecreasing. Then, for every $t > 0$, one has $\lim_{n\to\infty} \varphi^n(t) = 0$ implies $\varphi(t) < t$ where φ^n denotes the nth iterate of φ.*

Lemma 2.9. *([2]) Let (Ξ, ϖ) be a $b-$metric space with parameter $s \geq 1$. Assume that $\{x_\nu\}$ and $\{y_\nu\}$ are $b-$convergent to x and y, respectively. Then,*

$$\frac{1}{s^2}\varpi(x, y) \leq \liminf_{\nu\to+\infty} \varpi(x_\nu, y_\nu) \leq \limsup_{\nu\to+\infty} \varpi(x_\nu, y_\nu) \leq s^2\varpi(x, y).$$

In particular, if $x = y$, then $\lim_{\nu\to+\infty} \varpi(x_\nu, y_\nu) = 0$. Moreover, for each $z \in \Xi$,

$$\frac{1}{s}\varpi(x, z) \leq \liminf_{\nu\to+\infty} \varpi(x_\nu, z) \leq \limsup_{\nu\to+\infty} \varpi(x_\nu, z) \leq s\varpi(x, z).$$

Lemma 2.10. *([12]) Let $\phi \in \Theta$ and $\{\mu_\nu\}_{\nu\in\mathbb{N}}$ be a nonnegative sequence. $\lim_{\nu\to+\infty} \int_0^{\mu_\nu} \phi(t)\,dt = 0$ if and only if $\lim_{\nu\to+\infty} \mu_\nu = 0$. If $\lim_{\nu\to+\infty} \mu_\nu = \mu$, $\lim_{\nu\to+\infty} \int_0^{\mu_\nu} \phi(t)\,dt = \int_0^{\mu} \phi(t)\,dt$.*

3. Main Results

In this section, we introduce the concept of α admissible and other definitions, which are used to prove the fixed point theorems of integral type in $b-$ metric spaces. Moreover, we also provide two examples to support our results.

Lemma 3.1. *Let $\phi \in \Theta$ and $\{\mu_\iota\}_{\iota\in\mathbb{N}}$ be a nonnegative sequence. If $\limsup_{\iota\to+\infty} \mu_\iota = \mu$, then $\int_0^{\mu} \phi(t)\,dt \leq \limsup_{\iota\to+\infty} \int_0^{\mu_\iota} \phi(t)\,dt$. If $\liminf_{\iota\to+\infty} \mu_\iota = \mu$, then*

$$\liminf_{\iota\to+\infty} \int_0^{\mu_\iota} \phi(t)\,dt \leq \int_0^{\mu} \phi(t)\,dt.$$

Proof. In view of $\limsup_{\iota\to+\infty} \mu_\iota = \mu$, we can choose a subsequence $\{\mu_{\iota_\varsigma}\}$ of $\{\mu_\iota\}$ such that $\lim_{\varsigma\to+\infty} \mu_{\iota_\varsigma} = \mu$. It follows from Lemma 2.10 that

$$\int_0^{\mu} \phi(t)\,dt = \lim_{\varsigma\to+\infty} \int_0^{\mu_{\iota_\varsigma}} \phi(t)\,dt \leq \limsup_{\iota\to+\infty} \int_0^{\mu_\iota} \phi(t)\,dt.$$

Similarly, one can prove the other inequality. □

Theorem 3.2. *Let (Ξ,ϖ) be a complete $b-$metric space with parameter $s\geq 1$, and let $\mathbb{T}:\Xi\to\Xi$ be a continuous mapping. Let $\alpha:\Xi\times\Xi\to[0,+\infty)$ and $p\geq 1$. Assume that, for all $\mathfrak{p}\in\Xi$, there exists a positive integer $n(\mathfrak{p})$ such that, for any $\mathfrak{q}\in\Xi$,*

$$\alpha(\mathfrak{p},\mathfrak{q})\geq s^p\Rightarrow$$

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt\leq\Phi(\int_0^{\varpi(\mathfrak{p},\mathfrak{q})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt),\tag{3.1}$$

where $\Phi:[0,+\infty)^3\to[0,+\infty)$ and $\phi\in\Theta_s$ satisfy that

(1) Φ is nondecreasing and continuous for each variable;

(2) $\lim\limits_{t\to\infty}(\int_0^t\phi(t)\,dt-\varphi(\int_0^t\phi(t)\,dt))=\infty,\varphi(t)=\Phi(t,t,t)$;

(3) $\lim\limits_{n\to\infty}\varphi^n(\int_0^t\phi(t)\,dt)=0,\forall t>0$.

Suppose also that

(i) $\mathbb{T}$ is triangular α_{s^p} orbital admissible,

(ii) there exists $\mathfrak{p}_0\in\Xi$ satisfying $\alpha(\mathfrak{p}_0,\mathbb{T}\mathfrak{p}_0)\geq s^p$,

(iii) if $\{\mathfrak{p}_n\}$ is a sequence in Ξ satisfying $\alpha(\mathfrak{p}_n,\mathfrak{p}_{n+1})\geq s^p$ for all $n\in\mathbb{N}$ and $\mathfrak{p}_n\to\mathfrak{p}$ as $n\to\infty$, then there exists a subsequence $\{\mathfrak{p}_{n(k)}\}$ of $\{\mathfrak{p}_n\}$ with $\alpha(\mathfrak{p}_{n(k)},\mathfrak{p})\geq s^p$ for $k\in\mathbb{N}$,

(iv) for any fixed point $\mathfrak{p}$ of $\mathbb{T}^{n(\mathfrak{p})}$, $\alpha(\mathfrak{p},\mathfrak{q})\geq s^p$ for any $\mathfrak{q}\in\Xi$.

Then $\mathbb{T}$ has a unique fixed point $\mathfrak{p}^*$ in Ξ. For any $\mathfrak{p}\in\Xi$, $\{\mathbb{T}^n\mathfrak{p}\}$ converges to the point $\mathfrak{p}^*$.

Proof. It follows from condition (ii) that there exists an $\mathfrak{p}_0\in\Xi$ satisfying $\alpha(\mathfrak{p}_0,\mathbb{T}\mathfrak{p}_0)\geq s^p$. If $\mathfrak{p}_0$ is a fixed point of $\mathbb{T}$ and $\mathfrak{q}_0$ is the other one, then $\mathfrak{p}_0=\mathbb{T}\mathfrak{p}_0=\cdots=\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0=\cdots$ and $\mathfrak{q}_0=\mathbb{T}\mathfrak{q}_0=\cdots=\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{q}_0=\cdots$. By (iv), we have $\alpha(\mathfrak{p}_0,\mathfrak{q}_0)\geq s^p$ and

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt\leq&\int_0^{\alpha(\mathfrak{p}_0,\mathfrak{q}_0)\varpi(\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{q}_0)}\phi(t)\,dt\\ \leq&\Phi(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{q}_0)}\phi(t)\,dt)\\ =&\varphi(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt).\end{aligned}$$

In view of Lemma 2.8, one has $\varphi(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt)<\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt$. It follows that

$$\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt\leq\varphi(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt)<\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt,$$

which is a contradiction. Thus p_0 is the unique fixed point of $\mathbb{T}$. Therefore, we assume that p_0 is not the fixed point of $\mathbb{T}$ in subsequent discussion. Define a sequence $\{\mathfrak{p}_n\}$ in Ξ by $\mathfrak{p}_1=\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0,\cdots,\mathfrak{p}_{n+1}=\mathbb{T}^{n(\mathfrak{p}_n)}\mathfrak{p}_n$ for $n\in\mathbb{N}$.

Now, we prove that $\{\mathbb{T}^i\mathfrak{p}_0\}_{i=0}^{\infty}$ is bounded. To this end, we fix an integer $\mathfrak{l}$ such that $0\leq\mathfrak{l}<n(\mathfrak{p}_0)$. Let $\mathfrak{u}_j=\varpi(\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)$, $j=0,1,2\cdots$, and $\mathfrak{h}=\max\{\mathfrak{u}_0,\varpi(\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0)\}$. According to (2), there exists $c>\mathfrak{h}$ such that $\int_0^t\phi(t)\,dt-\varphi(\int_0^t\phi(t)\,dt)>\int_0^{s\mathfrak{h}}\phi(t)\,dt,t\geq c$. It is obvious that $\mathfrak{u}_0\leq\mathfrak{h}<c$. Suppose that there exists a positive integer j such that $\mathfrak{u}_j\geq c$. Evidently, we assume that $\mathfrak{u}_i<c,\forall i<j$. Also, we suppose that $\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0$ are different from each other. Otherwise, the conclusion can be easily proved. It is easy to obtain $\alpha(\mathfrak{p}_0,\mathbb{T}^k\mathfrak{p}_0)\geq s^p,\forall k\in\mathbb{N}$.

By Lemma 2.7 and integral subadditive, we have

$$\begin{aligned}
&\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)}\phi(t)\,dt\\
\leq&\int_0^{s\mathfrak{h}+\alpha(\mathfrak{p}_0,\mathbb{T}^{(j-1)n(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)\varpi(\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)}\phi(t)\,dt\\
\leq&\int_0^{s\mathfrak{h}}\phi(t)\,dt+\int_0^{\alpha(\mathfrak{p}_0,\mathbb{T}^{(j-1)n(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)\varpi(\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)}\phi(t)\,dt\\
\leq&\int_0^{s\mathfrak{h}}\phi(t)\,dt+\Phi(\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{(j-1)n(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{n(\mathfrak{p}_0)}\mathfrak{p}_0)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l}}\mathfrak{p}_0)}\phi(t)\,dt)\\
\leq&\int_0^{s\mathfrak{h}}\phi(t)\,dt+\varphi(\int_0^{\varpi(\mathfrak{p}_0,\mathbb{T}^{jn(\mathfrak{p}_0)+\mathfrak{l})}\mathfrak{p}_0)}\phi(t)\,dt).
\end{aligned}$$

That is, $\int_0^{\mathfrak{u}_j}\phi(t)\,dt-\varphi(\int_0^{\mathfrak{u}_j}\phi(t)\,dt)\leq\int_0^{s\mathfrak{h}}\phi(t)\,dt$, a contradiction. It follows that $\mathfrak{u}_j<c$ for $j=0,1,2,\cdots$ and $\{\mathbb{T}^i\mathfrak{p}_0\}_{i=0}^\infty$ is bounded. If there exists some $n_0\in\mathbb{N}$ such that $\mathfrak{p}_{n_0}=\mathfrak{p}_{n_0+1}=\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{p}_{n_0}$, then $\mathfrak{p}_{n_0}$ is a fixed point of $\mathbb{T}^{n(\mathfrak{p}_{n_0})}$. Assume that there exists $\mathfrak{q}\in\Xi$ such that $\mathfrak{q}=\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{q}$ and $\mathfrak{q}\neq\mathfrak{p}_{n_0}$, by condition (iv), we obtain $\alpha(\mathfrak{p}_{n_0},\mathfrak{q})\geq s^p$ and

$$\begin{aligned}
\int_0^{\varpi(\mathfrak{p}_{n_0},\mathfrak{q})}\phi(t)\,dt\leq&\int_0^{\alpha(\mathfrak{p}_{n_0},\mathfrak{q})\varpi(\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{p}_{n_0},\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{q})}\phi(t)\,dt\\
\leq&\Phi(\int_0^{\varpi(\mathfrak{p}_{n_0},\mathfrak{q})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n_0},\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{p}_{n_0})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n_0},\mathbb{T}^{n(\mathfrak{p}_{n_0})}\mathfrak{q})}\phi(t)\,dt)\\
\leq&\varphi(\int_0^{\varpi(\mathfrak{p}_{n_0},\mathfrak{q})}\phi(t)\,dt)<\int_0^{\varpi(\mathfrak{p}_{n_0},\mathfrak{q})}\phi(t)\,dt,
\end{aligned}$$

which is impossible. From this, we see that $\mathfrak{p}_{n_0}$ is the unique fixed point of $\mathbb{T}^{n(\mathfrak{p}_{n_0})}$. In what follows, we suppose that $\mathfrak{p}_n\neq\mathfrak{p}_{n+1}$ for all $n\in\mathbb{N}$.

Now, we prove that $\{\mathfrak{p}_n\}$ is a Cauchy sequence. Let i,n be any positive integer. It is obvious that $\alpha(\mathfrak{p}_{n-1},\mathbb{T}^k\mathfrak{p}_{n-1})\geq s^p$ for all $k\in\mathbb{N}$. Then,

$$\begin{aligned}
&\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+i})}\phi(t)\,dt\\
\leq&\Phi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}_{n+i-1})+n(\mathfrak{p}_{n+i-2})+\cdots+n(\mathfrak{p}_n)}\mathfrak{p}_{n-1})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1})}\phi(t)\,dt,\\
&\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}_{n+i-1})+n(\mathfrak{p}_{n+i-2})+\cdots+n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1})}\phi(t)\,dt)\\
\leq&\varphi(\sup\{\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_{n-1}\}_{m=0}^\infty\}).
\end{aligned}$$

For each $\{\mathbb{T}^m\mathfrak{p}_{n-1}\}_{m=0}^{\infty}$, we deduce

$$\begin{aligned}&\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{v})}\phi(t)\,dt\\ \leq&\int_0^{\alpha(\mathfrak{p}_{n-2},\mathbb{T}^m\mathfrak{p}_{n-2})\varpi(\mathbb{T}^{n(\mathfrak{p}_{n-2})}\mathfrak{p}_{n-2},\mathbb{T}^{m+n(\mathfrak{p}_{n-2})}\mathfrak{p}_{n-2})}\phi(t)\,dt\\ \leq&\Phi(\int_0^{\varpi(\mathfrak{p}_{n-2},\mathbb{T}^m\mathfrak{p}_{n-2})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n-2},\mathbb{T}^{n(\mathfrak{p}_{n-2})}\mathfrak{p}_{n-2})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n-2},\mathbb{T}^{n(\mathfrak{p}_{n-2})+m}\mathfrak{p}_{n-2})}\phi(t)\,dt)\\ \leq&\varphi(\sup\{\int_0^{\varpi(\mathfrak{p}_{n-2},\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_{n-2}\}_{m=0}^{\infty}\}).\end{aligned}$$

Thus

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+i})}\phi(t)\,dt\leq&\varphi(\sup\{\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_{n-1}\}_{m=0}^{\infty}\})\\ \leq&\cdots\\ \leq&\varphi^n(\sup\{\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_0\}_{m=0}^{\infty}\})\to 0(n\to\infty).\end{aligned}$$

That is, $\{\mathfrak{p}_n\}$ is a Cauchy sequence. Since Ξ is complete, there exists a point $\mathfrak{p}^*\in\Xi$ such that $\lim_{n\to\infty}\mathfrak{p}_n=\mathfrak{p}^*$. We might as well let $\mathfrak{p}_n\neq\mathfrak{p}^*$ and $\mathfrak{p}_n\neq\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*$. Otherwise, we have $\mathfrak{p}^*=\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*$ according to the continuity of $\mathbb{T}$. Note that

$$\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)}\phi(t)\,dt\leq\int_0^{s\varpi(\mathfrak{p}^*,\mathfrak{p}_n)+s^2\varpi(\mathfrak{p}_n,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_n)+s^2\varpi(\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_n,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)}\phi(t)\,dt. \tag{3.2}$$

On the other hand, one has

$$\begin{aligned}&\int_0^{\varpi(\mathfrak{p}_n,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_n)}\phi(t)\,dt\\ \leq&\int_0^{\alpha(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_{n-1})\varpi(\mathbb{T}^{n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}^*)+n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1})}\phi(t)\,dt\\ \leq&\Phi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(p)^*}\mathfrak{p}_{n-1})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}_{n-1},\mathbb{T}^{n(\mathfrak{p}^*)+n(\mathfrak{p}_{n-1})}\mathfrak{p}_{n-1})}\phi(t)\,dt)\\ \leq&\varphi(\sup\{\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_{n-1}\}_{m=0}^{\infty}\})\\ \leq&\cdots\\ \leq&\varphi^n(\sup\{\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{v})}\phi(t)\,dt|\mathfrak{v}\in\{\mathbb{T}^m\mathfrak{p}_0\}_{m=0}^{\infty}\})\to 0(n\to\infty).\end{aligned}$$

According to Lemma 2.8, one sees that $\lim_{n\to\infty}\varpi(\mathfrak{p}_n,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_n)=0$. The continuity of $\mathbb{T}$ ensures that $\lim_{n\to\infty}\varpi(\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}_n,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)=0$. Taking the limit as $n\to\infty$ in(3.2), we find by Lemma 2.1 that $\lim_{n\to\infty}\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)=0$. Assume that there exists $\mathfrak{q}^*\neq\mathfrak{p}^*$ such that $\mathfrak{q}^*=\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{q}^*$.

According to condition (iv), $\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\geq s^p$, one has

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt\leq&\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{q}^*)}\phi(t)\,dt\\ \leq&\Phi(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{q}^*)}\phi(t)\,dt)\\ \leq&\varphi(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt)<\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt,\end{aligned}$$

which is a contradiction. Hence, $\mathfrak{p}^*$ is the unique fixed point of $\mathbb{T}^{n(\mathfrak{p}^*)}$. Since $\mathbb{T}\mathfrak{p}^*=\mathbb{T}\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*=\mathbb{T}^{n(\mathfrak{p}^*)}\mathbb{T}\mathfrak{p}^*$, we see that $\mathbb{T}\mathfrak{p}^*=\mathfrak{p}^*$. The uniqueness is easily proved.

Finally, we prove the last part of the theorem. To prove this assertion, we fix an integer $\mathfrak{l}$, $0\leq\mathfrak{l}<n(\mathfrak{p}^*)$, and let $\varpi_k=\varpi(\mathfrak{p}^*,\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p}),k=0,1,2,\cdots$ for $\mathfrak{p}\in\Xi$. If there exists $k\in\Xi$ satisfying $\varpi_k=0$, then $\varpi_{k+1}=\varpi(\mathfrak{p}^*,\mathbb{T}^{(k+1)n(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})=\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})=0$. The subsequent result is $\varpi_{k+2}=\varpi_{k+3}=\cdots=0$. Thus

$$\begin{aligned}&\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})}\phi(t)\,dt\\ \leq&\int_0^{\alpha(\mathfrak{p}^*,\mathbb{T}^{(k-1)n(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})\varpi(\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*,\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})}\phi(t)\,dt\\ \leq&\Phi(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{(k-1)n(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{n(\mathfrak{p}^*)}\mathfrak{p}^*)}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l})}\mathfrak{p})}\phi(t)\,dt)\\ =&\Phi(\int_0^{\varpi_{k-1}}\phi(t)\,dt,0,\int_0^{\varpi_k}\phi(t)\,dt).\end{aligned}$$

If $\varpi_k\geq\varpi_{k-1}$ for some $k\in\mathbb{N}$, then

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p})}\phi(t)\,dt\leq&\Phi(\int_0^{\varpi_k}\phi(t)\,dt,\int_0^{\varpi_k}\phi(t)\,dt,\int_0^{\varpi_k}\phi(t)\,dt)\\ =&\varphi(\int_0^{\varpi_k}\phi(t)\,dt)<\int_0^{\varpi_k}\phi(t)\,dt,\end{aligned}$$

which is impossible. Hence,

$$\begin{aligned}\int_0^{\varpi_k}\phi(t)\,dt\leq&\Phi(\int_0^{\varpi_{k-1}}\phi(t)\,dt,\int_0^{\varpi_{k-1}}\phi(t)\,dt,\int_0^{\varpi_{k-1}}\phi(t)\,dt)\\ =&\varphi(\int_0^{\varpi_{k-1}}\phi(t)\,dt)\\ \leq&\cdots\\ \leq&\varphi^n(\int_0^{\varpi_0}\phi(t)\,dt)\to0(k\to\infty).\end{aligned}$$

It follows that $\lim_{k\to\infty}\varpi_k=0$. That is, the sequence $\{\mathbb{T}^n\mathfrak{p}\}=\{\mathbb{T}^{kn(\mathfrak{p}^*)+\mathfrak{l}}\mathfrak{p}\}$ converges to the point $\mathfrak{p}^*$ for any $\mathfrak{p}\in\Xi$. □

Example 3.3. Let $\Xi=[0,1]$ and $\varpi(\mathfrak{p},\mathfrak{q})=(\mathfrak{p}-\mathfrak{q})^2$. Define mappings $\mathbb{T}:\Xi\to\Xi$, $\alpha:\Xi\times\Xi\to[0,+\infty)$ by $\mathbb{T}\mathfrak{p}=\frac{\mathfrak{p}}{4},\mathfrak{p}\in[0,1]$ and $\alpha(\mathfrak{p},\mathfrak{q})=2,\forall\mathfrak{p},\mathfrak{q}\in\Xi$. Define $\Phi(t_1,t_2,t_3)=\frac{\sqrt{2}}{16}(t_1+t_2+t_3)$ for

$t_i \in [0,+\infty)(i=1,2,3)$, $\varphi(t)=\frac{3\sqrt{2}}{16}t$, and $\phi:[0,+\infty)\to[0,+\infty)$ by $\phi(t)=\frac{1}{2}t^{-\frac{1}{2}}$. Let $n(\mathfrak{p})=2$ and $\mathfrak{p},\mathfrak{q}\in[0,1]$. Then

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt=\int_0^{2(\frac{\mathfrak{p}}{16}-\frac{\mathfrak{q}}{16})^2}\frac{1}{2}t^{-\frac{1}{2}}\,dt=(2(\frac{\mathfrak{p}-\mathfrak{q}}{16})^2)^{\frac{1}{2}}=\frac{\sqrt{2}|\mathfrak{p}-\mathfrak{q}|}{16}$$

and

$$\begin{aligned}&\Phi(\int_0^{\varpi(\mathfrak{p},\mathfrak{q})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt)\\ =&\Phi(\int_0^{(\mathfrak{p}-\mathfrak{q})^2}\phi(t)\,dt,\int_0^{(\mathfrak{p}-\frac{\mathfrak{p}}{16})^2}\phi(t)\,dt,\int_0^{(\mathfrak{p}-\frac{\mathfrak{q}}{16})^2}\phi(t)\,dt)\\ =&\frac{\sqrt{2}}{16}(|\mathfrak{p}-\mathfrak{q}|+\frac{15\mathfrak{p}}{16}+|\mathfrak{p}-\frac{\mathfrak{q}}{16}|).\end{aligned}$$

Thus

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt\le\Phi(\int_0^{\varpi(\mathfrak{p},\mathfrak{q})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{p})}\phi(t)\,dt,\int_0^{\varpi(\mathfrak{p},\mathbb{T}^{n(\mathfrak{p})}\mathfrak{q})}\phi(t)\,dt).$$

Hence (3.1) holds. It follows that all conditions of Theorem 3.2 are satisfied with $s=2$. Here 0 is the fixed point of $\mathbb{T}$.

Theorem 3.4. *Let (Ξ,ϖ) be a complete b−metric space with parameter $s\ge1$ and let $\mathbb{T}:\Xi\to\Xi$ be a given self-mapping. Let $\alpha:\Xi\times\Xi\to[0,+\infty)$ and $p\ge2$. If*
(i) $\mathbb{T}$ is triangular α_{s^p} orbital admissible,
(ii) there exists $\mathfrak{p}_0\in\Xi$ satisfying $\alpha(\mathfrak{p}_0,\mathbb{T}\mathfrak{p}_0)\ge s^p$,
(iii) if $\{\mathfrak{p}_n\}$ is a sequence in Ξ satisfying $\mathfrak{p}_n\to\mathfrak{p}$, then $\alpha(\mathfrak{p}_n,\mathfrak{p})\ge s^p$ for all $n\in\mathbb{N}$,
(iv) for any fixed point $\mathfrak{p}$ of $\mathbb{T}$, $\alpha(\mathfrak{p},\mathfrak{q})\ge s^p$, for any $\mathfrak{q}\in\Xi$,
(v) there exists a function $\varphi:[0,\infty)\to[0,\frac{1}{s})$ such that, for each $\mathfrak{p},\mathfrak{q}\in\Xi$,

$$\alpha(\mathfrak{p},\mathfrak{q})\ge s^p\Rightarrow\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}\mathfrak{p},\mathbb{T}\mathfrak{q})}\phi(t)\,dt\le\varphi(\int_0^{E(\mathfrak{p},\mathfrak{q})}\phi(t)\,dt)\int_0^{E(\mathfrak{p},\mathfrak{q})}\phi(t)\,dt,\tag{3.3}$$

where $\phi\in\Theta$ and $E(\mathfrak{p},\mathfrak{q})=\varpi(\mathfrak{p},\mathfrak{q})+|\varpi(\mathfrak{p},\mathbb{T}\mathfrak{p})-\varpi(\mathfrak{q},\mathbb{T}\mathfrak{q})|$. Then $\mathbb{T}$ has a unique fixed point $\mathfrak{p}^*$ in Ξ.

Proof. Obviously, $E(\mathfrak{p},\mathfrak{q})=0$ if and only if $\mathfrak{p}=\mathfrak{q}$. Under condition (ii), there exists an $\mathfrak{p}_0\in\Xi$ satisfying $\alpha(\mathfrak{p}_0,\mathbb{T}\mathfrak{p}_0)\ge s^p$. If $\mathfrak{p}_0$ is a fixed point of $\mathbb{T}$ and $\mathfrak{q}_0$ is the other one, then $\mathfrak{p}_0=\mathbb{T}\mathfrak{p}_0=\cdots=\mathbb{T}^n\mathfrak{p}_0=\cdots$ and $\mathfrak{q}_0=\mathbb{T}\mathfrak{q}_0=\cdots=\mathbb{T}^n\mathfrak{q}_0=\cdots$. By (iv), we have $\alpha(\mathfrak{p}_0,\mathfrak{q}_0)\ge s^p$ and

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt\le&\int_0^{\alpha(\mathfrak{p}_0,\mathfrak{q}_0)\varpi(\mathbb{T}\mathfrak{p}_0,\mathbb{T}\mathfrak{q}_0)}\phi(t)\,dt\\ \le&\varphi(\int_0^{E(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt)\int_0^{E(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt,\end{aligned}\tag{3.4}$$

where

$$E(\mathfrak{p}_0,\mathfrak{q}_0)=\varpi(\mathfrak{p}_0,\mathfrak{q}_0)+|\varpi(\mathfrak{p}_0,\mathbb{T}\mathfrak{p}_0)-\varpi(\mathfrak{q}_0,\mathbb{T}\mathfrak{q}_0)|=\varpi(\mathfrak{p}_0,\mathfrak{q}_0).$$

It follows from (3.4) that

$$\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt \le \int_0^{\alpha(\mathfrak{p}_0,\mathfrak{q}_0)\varpi(\mathbb{T}\mathfrak{p}_0,\mathbb{T}\mathfrak{q}_0)}\phi(t)\,dt < \frac{1}{s}\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{q}_0)}\phi(t)\,dt,$$

a contradiction. That is, $\mathfrak{p}_0$ is the unique fixed point of $\mathbb{T}$. In subsequent discussion, we assume that $\mathfrak{p}_0$ is not the fixed point of $\mathbb{T}$. Define a sequence $\{\mathfrak{p}_n\}$ in Ξ by $\mathfrak{p}_1 = \mathbb{T}\mathfrak{p}_0, \cdots, \mathfrak{p}_{n+1} = \mathbb{T}\mathfrak{p}_n$ for $n \in \mathbb{N}$.

If we suppose that for some $n_0 \in \mathbb{N}_0$ such that $\mathfrak{p}_{n_0} = \mathfrak{p}_{n_0+1}$, then the proof is completed, since $\mathfrak{p}_{n_0}$ is a fixed point of $\mathbb{T}$. Without loss of generality, we presume that $\mathfrak{p}_n \neq \mathfrak{p}_{n+1}$ for each $n \in \mathbb{N}$. By (i), (ii), and Lemma 2.7, we have $\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n) \ge s^p$. Applying (3.3) with $\mathfrak{p} = \mathfrak{p}_{n-1}, \mathfrak{q} = \mathfrak{p}_n$, one can obtain

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt &\le \int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{T}\mathfrak{p}_{n-1},\mathbb{T}\mathfrak{p}_n)}\phi(t)\,dt \\ &\le \varphi\Big(\int_0^{E(\mathfrak{p}_{n-1},\mathfrak{p}_n)}\phi(t)\,dt\Big)\int_0^{E(\mathfrak{p}_{n-1},\mathfrak{p}_n)}\phi(t)\,dt,\end{aligned} \tag{3.5}$$

where $E(\mathfrak{p}_{n-1},\mathfrak{p}_n) = \varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n) + |\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n) - \varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})|$. If $\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1}) \ge \varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)$, then (3.5) implies

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt &\le \varphi\Big(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt\Big)\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt \\ &\le \frac{1}{s}\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt,\end{aligned}$$

which is a contradiction. Hence, $\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n) > \varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})$. It follows that $\{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})\}$ is decreasing. Thus there exists $\mathfrak{r} \ge 0$ satisfying $\lim_{n\to\infty}\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1}) = \mathfrak{r}$. Suppose that $\mathfrak{r} > 0$. Letting the limit as $n \to \infty$ in (3.5), we have

$$\begin{aligned}\int_0^{\mathfrak{r}}\phi(t)\,dt &= \lim_{n\to\infty}\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})}\phi(t)\,dt \\ &\le \lim_{n\to\infty}\int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{T}\mathfrak{p}_{n-1},\mathbb{T}\mathfrak{p}_n)}\phi(t)\,dt \\ &\le \lim_{n\to\infty}\varphi\Big(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)}\phi(t)\,dt\Big)\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)}\phi(t)\,dt \le \frac{1}{s}\int_0^{\mathfrak{r}}\phi(t)\,dt,\end{aligned}$$

which is a contradiction. Thus $\mathfrak{r} = 0$, that is, $\lim_{n\to\infty}\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1}) = 0$.

Next, we prove that $\{\mathfrak{p}_n\}$ is a Cauchy sequence. Suppose that $\{\mathfrak{p}_n\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ and we can choose sequences $\{\mathfrak{p}_{m_k}\}$ and $\{\mathfrak{p}_{n_k}\}$ of $\{\mathfrak{p}_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) \ge \varepsilon, \varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k}) < \varepsilon.$$

It follows that

$$\varepsilon \le \varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) \le s\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{m_k-1}) + s\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k}) < s\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{m_k-1}) + s\varepsilon. \tag{3.6}$$

Taking the superior limit and inferior limit as $k \to \infty$ in (3.6), we obtain

$$\varepsilon \le \liminf_{k\to\infty}\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) \le \limsup_{k\to\infty}\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) \le s\varepsilon. \tag{3.7}$$

Similarly,

$$\varepsilon \leq \varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) \leq s\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{m_k-1}) + s^2\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) + s^2\varpi(\mathfrak{p}_{n_k-1},\mathfrak{p}_{n_k}), \tag{3.8}$$

$$\varepsilon \leq \varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \leq s\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{m_k}) + s^2\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k}) + s^2\varpi(\mathfrak{p}_{n_k},\mathfrak{p}_{n_k-1}). \tag{3.9}$$

In view of (3.7), (3.8), and (3.9), we have

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty}\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \leq \limsup_{k\to\infty}\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \leq s^3\varepsilon.$$

One can deduce $\alpha(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \geq s^p$. Taking $\mathfrak{p} = \mathfrak{p}_{m_k-1}, \mathfrak{q} = \mathfrak{p}_{n_k-1}$ in (3.3), we find by Lemma 2.10 that

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}_{m_k},\mathfrak{p}_{n_k})}\phi(t)\,dt \leq& \int_0^{\alpha(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})\varpi(\mathbb{T}\mathfrak{p}_{m_k-1},\mathbb{T}\mathfrak{p}_{n_k-1})}\phi(t)\,dt \\ \leq&\varphi(\int_0^{E(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})}\phi(t)\,dt)\int_0^{E(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})}\phi(t)\,dt,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}E(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) =&\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) + |\varpi(\mathfrak{p}_{m_k-1},\mathbb{T}\mathfrak{p}_{m_k-1}) - \varpi(\mathfrak{p}_{n_k-1},\mathbb{T}\mathfrak{p}_{n_k-1})| \\ =&\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) + |\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{m_k}) - \varpi(\mathfrak{p}_{n_k-1},\mathfrak{p}_{n_k})|.\end{aligned} \tag{3.11}$$

Taking the inferior limit as $k \to \infty$ in (3.11), we arrive at

$$\liminf_{k\to\infty}E(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) = \liminf_{k\to\infty}\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \leq \limsup_{k\to\infty}\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1}) \leq s^3\varepsilon,$$

Taking the inferior limit as $k \to \infty$ in (3.10), we see that

$$\begin{aligned}\int_0^{s^3\varepsilon}\phi(t)\,dt \leq& \limsup_{k\to\infty}\int_0^{\alpha(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})\varpi(\mathbb{T}\mathfrak{p}_{m_k-1},\mathbb{T}\mathfrak{p}_{n_k-1})}\phi(t)\,dt \\ \leq& \limsup_{k\to\infty}\varphi(\int_0^{\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})}\phi(t)\,dt)\int_0^{\varpi(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})}\phi(t)\,dt \\ \leq&\frac{1}{s}\int_0^{s^3\varepsilon}\phi(t)\,dt,\end{aligned} \tag{3.12}$$

which is a contradiction. Thus $\{\mathfrak{p}_n\}$ is a Cauchy sequence. As Ξ is complete, there exists $\mathfrak{p}^* \in \Xi$ such that $\lim_{n\to\infty}\mathfrak{p}_n = \mathfrak{p}^*$.

Now, we prove that $\mathfrak{p}^*$ is a fixed point of $\mathbb{T}$. By (iii), we have $\alpha(\mathfrak{p}^*,\mathfrak{p}_{n-1}) \geq s^p$. Letting $\mathfrak{p} = \mathfrak{p}^*, \mathfrak{q} = \mathfrak{p}_{n-1}$, we obtain

$$\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{p}_{n-1})\varpi(\mathbb{T}\mathfrak{p}^*,\mathbb{T}\mathfrak{p}_{n-1})}\phi(t)\,dt \leq\varphi(\int_0^{E(\mathfrak{p}^*,\mathfrak{p}_{n-1})}\phi(t)\,dt)\int_0^{E(\mathfrak{p}^*,\mathfrak{p}_{n-1})}\phi(t)\,dt, \tag{3.13}$$

where

$$\begin{aligned}E(\mathfrak{p}^*,\mathfrak{p}_{n-1}) =&\varpi(\mathfrak{p}^*,\mathfrak{p}_{n-1}) + |\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*) - \varpi(\mathfrak{p}_{n-1},\mathbb{T}\mathfrak{p}_{n-1})| \\ =&\varpi(\mathfrak{p}^*,\mathfrak{p}_{n-1}) + |\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*) - \varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)|.\end{aligned} \tag{3.14}$$

Letting $k\to\infty$ in (3.14), we obtain $\lim_{k\to\infty}E(\mathfrak{p}^*,\mathfrak{p}_{n-1})=\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*)$. Taking the superior limit as $k\to\infty$ in (3.13), we obtain that

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*)}\phi(t)\,dt &\le \limsup_{n\to\infty}\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{p}_{n-1})\varpi(\mathbb{T}\mathfrak{p}^*,\mathbb{T}\mathfrak{p}_{n-1})}\phi(t)\,dt\\ &\le \limsup_{n\to\infty}\varphi(\int_0^{E(\mathfrak{p}^*,\mathfrak{p}_{n-1})}\phi(t)\,dt)\int_0^{E(\mathfrak{p}^*,\mathfrak{p}_{n-1})}\phi(t)\,dt\\ &\le \frac{1}{s}\int_0^{\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*)}\phi(t)\,dt,\end{aligned}$$

which is impossible. So $\mathfrak{p}^*$ is a fixed point of $\mathbb{T}$. We assume that there exists another point $\mathfrak{q}^*\in\Xi$, with $\mathfrak{p}^*\neq\mathfrak{q}^*$ such that $\mathbb{T}\mathfrak{q}^*=\mathfrak{q}^*$. Taking $\mathfrak{p}=\mathfrak{p}^*,\mathfrak{q}=\mathfrak{q}^*$ in (3.3), we deduce

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt &\le \int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{T}\mathfrak{p}^*,\mathbb{T}\mathfrak{q}^*)}\phi(t)\,dt\\ &\le \varphi(\int_0^{E(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt)\int_0^{E(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt,\end{aligned}\tag{3.15}$$

where $E(\mathfrak{p}^*,\mathfrak{q}^*)=\varpi(\mathfrak{p}^*,\mathfrak{q}^*)+|\varpi(\mathfrak{p}^*,\mathbb{T}\mathfrak{p}^*)-\varpi(\mathfrak{q}^*,\mathbb{T}\mathfrak{q}^*)|=\varpi(\mathfrak{p}^*,\mathfrak{q}^*)$. According to (3.15), we have

$$\begin{aligned}\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt &\le \int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{T}\mathfrak{p}^*,\mathbb{T}\mathfrak{q}^*)}\phi(t)\,dt\\ &\le \varphi(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt)\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt\\ &< \frac{1}{s}\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\phi(t)\,dt,\end{aligned}$$

which is a contradiction. Hence, $\mathfrak{p}^*=\mathfrak{q}^*$. That is, $\mathfrak{p}^*$ is a unique fixed point of $\mathbb{T}$. □

Example 3.5. Let $\Xi=\{0,\frac{1}{10},\frac{1}{20}\}$ and $\varpi:\Xi\times\Xi\to[0,+\infty)$ be defined by $\varpi(\mathfrak{p},\mathfrak{p})=0,\mathfrak{p}\in\Xi$, $\varpi(0,\frac{1}{10})=\varpi(\frac{1}{10},0)=\frac{1}{600},\varpi(0,\frac{1}{20})=\varpi(\frac{1}{20},0)=\frac{1}{700},\varpi(\frac{1}{10},\frac{1}{20})=\varpi(\frac{1}{20},\frac{1}{10})=\frac{1}{800}$. Thus (Ξ,ϖ) is a $b-$metric space with $s=\frac{11}{10}$. Let $\mathbb{T}:\Xi\to\Xi$ be defined as $\mathbb{T}(0)=0,\mathbb{T}(\frac{1}{10})=\frac{1}{20},\mathbb{T}(\frac{1}{20})=\frac{1}{10}$. Define $\varphi(t)=\frac{9}{11}$ and $\phi(t)=2t$, $\alpha:\Xi\times\Xi\to[0,+\infty)$ by

$$\alpha(\mathfrak{p},\mathfrak{q})=\begin{cases}0, & \mathfrak{p}=\mathfrak{q}\ or\ \mathfrak{p}=\dfrac{1}{10},\mathfrak{q}=\dfrac{1}{20}\ or\ \mathfrak{p}=\dfrac{1}{20},\mathfrak{q}=\dfrac{1}{10},\\ s^3, & \text{otherwise}.\end{cases}$$

Consider the following cases:

Case 1. If $\mathfrak{p}=0,\mathfrak{q}=\frac{1}{10}$, then

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}\mathfrak{p},\mathbb{T}\mathfrak{q})}\phi(t)\,dt=\int_0^{(\frac{11}{10})^3\cdot\frac{1}{700}}2t\,dt=(\frac{1331}{700000})^2,$$

$$E(0,\frac{1}{10})=\frac{1}{600}+\frac{1}{800}=\frac{7}{2400},$$

and

$$\varphi(\int_0^{\frac{7}{2400}}2t\,dt)\int_0^{\frac{7}{2400}}2t\,dt=\frac{9}{11}\int_0^{\frac{7}{2400}}2t\,dt=\frac{9}{11}\cdot(\frac{7}{2400})^2.$$

So

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}\mathfrak{p},\mathbb{T}\mathfrak{q})} \phi(t)\,dt \le \varphi\Big(\int_0^{E(\mathfrak{p},\mathfrak{q})} \phi(t)\,dt\Big)\int_0^{E(\mathfrak{p},\mathfrak{q})} \phi(t)\,dt.$$

Case 2. If $\mathfrak{p}=0, \mathfrak{q}=\frac{1}{20}$, then

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}\mathfrak{p},\mathbb{T}\mathfrak{q})} \phi(t)\,dt = \int_0^{(\frac{11}{10})^3\cdot\frac{1}{600}} 2t\,dt = (\frac{1331}{600000})^2,$$

$$E(0,\frac{1}{20}) = \frac{1}{700}+\frac{1}{800} = \frac{15}{5600},$$

and

$$\varphi\Big(\int_0^{\frac{15}{5600}} 2t\,dt\Big)\int_0^{\frac{15}{5600}} 2t\,dt = \frac{9}{11}\int_0^{\frac{15}{5600}} 2t\,dt = \frac{9}{11}\cdot(\frac{15}{5600})^2.$$

That is,

$$\int_0^{\alpha(\mathfrak{p},\mathfrak{q})\varpi(\mathbb{T}\mathfrak{p},\mathbb{T}\mathfrak{q})} \phi(t)\,dt \le \varphi\Big(\int_0^{E(\mathfrak{p},\mathfrak{q})} \phi(t)\,dt\Big)\int_0^{E(\mathfrak{p},\mathfrak{q})} \phi(t)\,dt.$$

Hence, all the conditions of Theorem 3.4 are fulfilled with $p=3$, and $\mathbb{T}$ has a unique fixed point 0.

4. Conclusions

In this paper, we first introduced two new classes of contractive mappings of integral type in b-metric spaces. Then we studied the existence and uniqueness of fixed points of these contractions of integral type. Finally, two examples were given to prove the practicability of our theorems.

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