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ON PSEUDOMONOTONE EQUILIBRIUM AND FIXED POINT PROBLEMS IN HADAMARD SPACES WITH APPLICATIONS

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Abstract. In this paper, we introduce a hybrid extragradient method for approximating a common solution of a finite family of equilibrium problems associated with pseudomonotone bifunctions and a fixed point problem of a generalized demimetric mapping. Under suitable conditions, we prove a strong convergence theorem in Hadamard spaces. Finally, in order to further support our main results, we provide a numerical experiment and an example of a pseudomonotone bifunction in Hadamard spaces.

Keywords. CAT(0) space; Δ-convergence; Extragradient method; Generalized Demimetric mapping; Pseudomonotone equilibrium problem.

1. Introduction

In the sense of Blum and Oetti [6], the equilibrium problem (EP) studied in this paper read as follows: find $p^* \in \mathscr{S}$ such that $\mathscr{L}(p^*,q) \geq 0$ for all $q \in \mathscr{S}$, where $\mathscr{S}$ is a nonempty, convex, and closed subset of a real Hilbert space $\mathscr{H}$ and $\mathscr{L} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is a given bifunction with $\mathscr{L}(p,p) = 0$ for all $p \in \mathscr{S}$. The point p^* in $\mathscr{S}$ is called an equilibrium point. The set of equilibrium points of the problem is denoted by $EP(\mathscr{L},\mathscr{S})$ in this paper.

The equilibrium problem (EP), which covers complementary problems, variational inequality problems, fixed point problems, and Nash equilibrium problems, has become a very effective model for various real-world problems; see, e.g, [1, 17, 32, 37, 40] and the references therein.

If $\mathscr{L}$ is monotone, to solve the EP, one of majority techniques is to consider the following regularization equilibrium problem (REP), i.e., at the k^{th} iteration step, known as p_k, determine the next approximation p_{k+1} as the solution of the equilibrium problem: find $p \in \mathscr{S}$ such that $\mathscr{L}(p,q) + \frac{1}{t_k}\langle q-p, p-p_k\rangle \geq 0$ for all $q \in \mathscr{S}$, where $t_k \geq d > 0$. Note that the regularization

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equilibrium problem is strongly monotone if bifunction $\mathscr{L}$ is monotone. Thus its solution exists and is unique if $\mathscr{L}$ is continuous. Unfortunately, in general, if $\mathscr{L}$ is pseudomonotone, then the regularization equilibrium problem is not strongly monotone, so the unique solvability can not be guaranteed. In this case, the authors in [2, 34] considered two strongly convex programs

$$\begin{cases} q_n = \arg\min\{\rho\mathscr{L}(p_k,q)+\frac{1}{2}\|p_k-q\|^2 : q\in\mathscr{S}\}, \\ p_{k+1} = \arg\min\{\rho\mathscr{L}(q_k,q)+\frac{1}{2}\|p_k-q\|^2 : q\in\mathscr{S}\} \end{cases}$$

where $\rho > 0$ satisfies some suitable conditions.

In addition to the methods in [20, 29], the proximal-like method presented in [16] is one of popular methods to study the EP. This method was further extended and investigated by the authors in [34] under different assumptions that the cost bifunctions are pseudomonotone and satisfy the Lipschitz-type condition in [28]. The method in [16, 34] is also called extragradient methods (or two-step proximal-like methods) due to the results obtained by Korpelevich in [23] for saddle problems. The advantages of the extragradient methods in [16, 34] are that they are used for the class of pseudomonotone bifunctions and can be easier to numerically solve than the proximal point method in [20, 29]. Further, Khatibzadeh and Mohebbi [21] recently introduced extragradient and Halpern's regularization methods in a Hadamard space as follows: Given a Hadanard space $\mathscr{X}$, let $\mathscr{S}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. Let $\mathscr{L} : \mathscr{S}\times\mathscr{S}\to\mathbb{R}$ be a pseudomonotone bifunction that also meets certain additional requirements. To solve the equilibrium problem, Khatibzadeh and Mohebbi [21] proposed the extragradient approach as follows:

$$q_k \in \mathrm{Argmin}_{q\in\mathscr{S}}\left\{\mathscr{L}(p_k,q)+\frac{1}{2\lambda_k}d^2(p_k,q)\right\} \tag{1.1}$$

$$p_{k+1} \in \mathrm{Argmin}_{q\in\mathscr{S}}\left\{\mathscr{L}(q_k,q)+\frac{1}{2\lambda_k}d^2(p_k,q)\right\} \tag{1.2}$$

where $\{\lambda_k\}$ is a positive sequence. They proved that the sequence generated by (1.1)-(1.2) Δ-converges to a solution of the equilibrium problem. They also proposed the following Halpern's regularization of the extragradient method (see [21]):

$$q_k \in \mathrm{Argmin}_{q\in\mathscr{S}}\left\{\mathscr{L}(p_k,q)+\frac{1}{2\lambda_k}d^2(p_k,q)\right\} \tag{1.3}$$

$$r_k \in \mathrm{Argmin}_{q\in\mathscr{S}}\left\{\mathscr{L}(q_k,q)+\frac{1}{2\lambda_k}d^2(p_k,q)\right\} \tag{1.4}$$

$$p_{k+1} = \alpha_k u\oplus(1-\alpha_k)r_k \tag{1.5}$$

where u is a fixed vector in $\mathscr{S}$, $\{\alpha_k\}$ is a sequence in $(0,1)$, and $\{\lambda_k\}$ is a positive sequence. They proved that the sequence defined by (1.3)-(1.5) converges strongly to a solution of the equilibrium problem. However, in the setting of Hilbert spaces, Banach spaces, and Hahamard spaces, numerous authors studied the problem of approximating common solutions of the equilibrium problem, monotone inclusion problems, variational inequality problems, and fixed point problems; see, e.g., [13, 14, 26, 33, 36]. To our knowledge, there are no results on the common solutions of the fixed point problem of generalized demimetric mappings and the equilibrium problem with a family of pseudomonotone bifunctions in complete CAT(0) spaces.

Let p in $\mathscr{S}$ be a fixed point of a nonlinear mapping $\mathscr{G}:\mathscr{S}\to\mathscr{S}$, that is, $\mathscr{G}p = p$. Next, we borrow $F(\mathscr{G}) := \{p\in\mathscr{S} : \mathscr{G}p = p\}$ to denote the fixed point set of $\mathscr{G}$ on $\mathscr{S}$. Recently, the notion of quasilinearization was investigated by Aremu et al. [4] and Ugwunnadi et al. [38] to define demimetric and $\mu-$generalized demimetric mappings, respectively, in CAT(0) spaces.

Definition 1.1. Let $\mathscr{X}$ be a CAT(0) space, and let $\mathscr{S}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. The mapping $\mathscr{G}$ from $\mathscr{S}$ into $\mathscr{X}$ is said to be:

(i) τ-demimetric (see [4]) if $F(\mathscr{G}) \neq \emptyset$ and there exists $\tau \in (-\infty, 1)$ such that

$$\langle \overrightarrow{pq}, \overrightarrow{p\mathscr{G}p} \rangle \geq \frac{1-\tau}{2} d^2(p, \mathscr{G}p), \quad \text{for all} \quad p \in \mathscr{S} \text{ and } q \in F(\mathscr{G}).$$

(ii) τ-generalized demimetric (see [38]) if $F(\mathscr{G}) \neq \emptyset$ and there exists $\tau \in \mathbb{R}$ such that

$$d^2(p, \mathscr{G}p) \leq \theta \langle \overrightarrow{pq}, \overrightarrow{p\mathscr{G}p} \rangle, \ \forall \ p \in \mathscr{S} \text{ and } q \in F(\mathscr{G}).$$

Remark 1.2. For each $\tau \in (-\infty, 1)$, it is obvious from Definition 1.1 that a τ-demimetric mapping is $\frac{2}{1-\tau}$-generalized demimetric. $\left(1 - \frac{2}{\tau}\right)$ is a τ-generalized demimetric, likewise for $\tau > 0$.

Inspired by the studies mentioned above, we, in Hadamard spaces, present a hybrid extragardient method for fixed points of a generalized demimetric mapping and a common solution of a finite family of pseudomonotone bifunctions. We study the convergence analysis of our new iterative method and also provide an example and an application of a pseudomonotone bifunction that is not monotone but fulfills the lipschitz-type requirement in Hadamard spaces.

2. Preliminaries

Let $(\mathscr{X}, d)$ be a metric space and $p, q \in \mathscr{X}$. Recall that a map $\gamma : [x, y] \subseteq \mathbb{R} \to \mathscr{X}$ such that $\gamma(x) = p$, $\gamma(y) = q$, and $d(\gamma(s), \gamma(s')) = |s - s'|$ for all $s, s' \in [x, y]$ constitute a geodesic path connecting points p and q. In particular, $d(p, q) = y - x$ and γ is an isometry. The image $\gamma([x, y])$ of a geodesic path in $\mathscr{X}$ connecting p and q is a geodesic segment in $\mathscr{X}$. Recall that a metric space $\mathscr{X}$ is considered uniquely geodesic if there exists precisely one geodesic joining points p and q for each $p, q \in \mathscr{X}$. Similarly, $\mathscr{X}$ is uniquely geodesic if there exists precisely one geodesic joining p and q for each $p, q \in \mathscr{X}$. If a geodesic segment is unique, it is represented by $[p, q]$. Hadamard spaces are generally considered to be complete CAT(0) spaces; see [32] for more details. Consider $\lambda \in [0, 1]$ and $p, q \in \mathscr{X}$. The notation $\lambda p \oplus (1 - \lambda) q$ for the unique point $r \in [p, q]$ such that $d(r, p) = (1 - \lambda) d(p, q)$ and $d(r, q) = \lambda d(p, q)$ is used throughout this paper. Any subset $\mathscr{S}$ of a CAT(0) space $\mathscr{X}$ that contains every geodesic segment connecting any two points in $\mathscr{S}$ is convex.

In a metric space $\mathscr{X}$, Berg and Nikolaev [17] introduced the following concept known as the quasilinearization. Let the pair $(w, x) \in \mathscr{X} \times \mathscr{X}$ be denoted by $\overrightarrow{wx}$, which is referred to as a vector. The map $\langle ., . \rangle : (\mathscr{X} \times \mathscr{X}) \times (\mathscr{X} \times \mathscr{X}) \to \mathbb{R}$ is called a quasilinearization if, for all $w, x, y, z \in \mathscr{X}$,

$$\langle \overrightarrow{wx}, \overrightarrow{yz} \rangle = \frac{1}{2} \Big(d^2(w, z) + d^2(x, y) - d^2(w, y) - d^2(x, z) \Big). \tag{2.1}$$

It is clear to see that $\langle \overrightarrow{wx}, \overrightarrow{yz} \rangle = \langle \overrightarrow{yz}, \overrightarrow{wx} \rangle$, $\langle \overrightarrow{wx}, \overrightarrow{yz} \rangle = -\langle \overrightarrow{xw}, \overrightarrow{yz} \rangle$ and

$$\langle \overrightarrow{wb}, \overrightarrow{yz} \rangle + \langle \overrightarrow{bx}, \overrightarrow{yz} \rangle = \langle \overrightarrow{wx}, \overrightarrow{yz} \rangle \ \text{ for all } \ w, x, y, z, b \in \mathscr{X}. \tag{2.2}$$

The Cauchy-Schwarz inequality is said to be satisfied by $\mathscr{X}$ if, for all $w, x, y, z \in \mathscr{X}$,

$$\langle \overrightarrow{wx}, \overrightarrow{yz} \rangle \leq d(w, x) d(y, z). \tag{2.3}$$

Geodesically connected metric spaces are known to be CAT(0) spaces if and only if they satisfy the Cauchy-Schwarz inequality [17]. We outline some relevant lemmas and significant properties in CAT(0) space that are essential to our finding. For complete CAT(0) space $\mathscr{X}$, let $\{p_k\}$ be a bounded sequence. If $p \in \mathscr{X}$, we let $s(p,\{p_k\}) = \limsup_{k\to\infty} d(p,p_k)$. Given $\{p_k\}$, the asymptotic radius $s(\{p_k\})$ is given by $s(\{p_k\}) = \inf\{s(p,\{p_k\}) : p \in \mathscr{X}\}$, and $A(\{p_k\})$, the asymptotic center of $\{p_k\}$, is the set $A(\{p_k\}) = \{p \in \mathscr{X} : s(p,\{p_k\}) = s(\{p_k\})\}$. It is widely known that in a CAT(0) space, $A(\{p_k\})$ contains exactly one point ([13]). In $\mathscr{X}$, a sequence $\{p_k\}$ is considered $\triangle-$convergent to $p \in \mathscr{X}$, indicated by $\triangle - \lim_k p_k = p$, if p represents the distinct asymptotic center of $\{v_k\}$, for each subsequence $\{v_k\}$ of $\{p_k\}$. From [22], we see that there exists a $\triangle-$convergent subsequence for every bounded sequence in a Hadamard space. For every closed and convex subset $\mathscr{S}$ of a Hadamard space, if $\{p_k\}$ is a bounded sequence in $\mathscr{S}$, then $\mathscr{S}$ contains the asymptotic center of $\{p_k\}$ [40]. Let $\mathscr{S}$ be a closed and convex subset of $\mathscr{X}$ that contains $\{p_k\}$. Let $\{p_k\}$ be a bounded sequence in a Hadamard space $\mathscr{X}$. We use the notation: $\{p_k\} \rightharpoonup z \Leftrightarrow \limsup_{k\to\infty} d(p_k,z) = \inf_{p\in\mathscr{S}}(\limsup_{k\to\infty} d(p_k,p))$. It is known that $\{p_k\} \rightharpoonup z$ if and only if $A(\{p_k\}) = \{z\}$ (see [31]). In a Hadamard space $\mathscr{X}$, $\triangle - \lim\limits_{k\to\infty} p_k = u^*$ means ([31]) that $\{p_k\} \rightharpoonup u^*$ if $\{p_k\}$ is a bounded sequence in a closed and convex subset $\mathscr{S}$ of $\mathscr{X}$. Consider a nonempty, closed, and convex subset of the Hadamard space $\mathscr{X}$, denoted by $\mathscr{S}$. Recall that the metric projection $P_{\mathscr{S}} : \mathscr{X} \to \mathscr{S}$ is given by, for all $p \in \mathscr{X}$,

$$u = P_{\mathscr{S}}(p) \quad \Leftrightarrow \quad d(u,p) = \inf\{d(q,p) : q \in \mathscr{S}\}.$$

Consider the Hadamard space $\mathscr{X}$. Then, the following inequality holds for every $u,p,q \in \mathscr{X}$ [41]: $d^2(p,u) \le d^2(q,u) + 2\langle\overrightarrow{pq},\overrightarrow{pu}\rangle$. For every $r \in [0,1]$ and every $v,p,q \in \mathscr{X}$, the following inequality holds [24] if $a_1 = rp \oplus (1-r)v$ and $a_2 = rq \oplus (1-r)v$, $\langle\overrightarrow{a_1a_2},\overrightarrow{pa_2}\rangle \le r\langle\overrightarrow{pq},\overrightarrow{xu}\rangle$.

Let $\mathscr{S}$ be a nonempty, closed, and convex subset of a CAT(0) space $\mathscr{X}$.

(1) Let $\mathscr{L} : \mathscr{X} \to \mathbb{R} \cup \{\infty\}$ be a function. The domain of $\mathscr{L}$ is the set defined by $D(\mathscr{L}) := \{p \in \mathscr{X} : \mathscr{L}(p) < \infty\}$. $\mathscr{L}$ is called proper if $D(\mathscr{L}) \neq \emptyset$. There is at least one point $a \in D(\mathscr{L})$ such that $\mathscr{L}(a) \in \mathbb{R}$. $\mathscr{L}$ is said to be:
   (a) convex if and only if there exists $r \in (0,1)$ such that
   $$\mathscr{L}(rp \oplus (1-r)q) \le r\mathscr{L}(p) + (1-r)\mathscr{L}(q), \ \ \forall p,q \in \mathscr{X};$$
   (b) strictly convex if and only if there exists $r \in (0,1)$ such that
   $$\mathscr{L}(rp \oplus (1-r)q) \le r\mathscr{L}(p) + (1-r)\mathscr{L}(q), \ \ \forall p,q \in \mathscr{X}, p \neq q.$$

(2) $\mathscr{L} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ is said to be:
   (a) monotone if $\mathscr{L}(p,q) + \mathscr{L}(q,p) \le 0$ for all $p,q \in \mathscr{S}$.
   (b) pseudomonotone on $\mathscr{S}$ if $\mathscr{L}(p,q) \ge 0 \Rightarrow \mathscr{L}(q,p) \le 0, \ \ \forall p,q \in \mathscr{S}$;
   (c) continuous Lipschitz-type on $\mathscr{S}$ if there exists two positive constants t_1,t_2 such that, for all $p,q,z \in \mathscr{S}$, $\mathscr{L}(p,q) + \mathscr{L}(q,z) \ge \mathscr{L}(p,z) - t_1d^2(p,q) - t_2d^2(q,z)$.

We use the following crucial conditions on the bifunction $\mathscr{L}$ in the sequel:

(A1) $\mathscr{L}(p,.) : \mathscr{X} \to \mathbb{R}$ is lower semicontinuous and convex (shortly, lsc) for all $p \in \mathscr{X}$.
(A2) $\mathscr{L}$ is pseudomonotone.
(A3) $\mathscr{L}$ is continuous Lipschitz-type.
(A4) $\mathscr{L}(.,q)$ is $\Delta-$upper semicontinuous for all $q \in \mathscr{X}$.

Lemma 2.1. [26] *In a Hadamard space $\mathscr{X}$, let $\{p_k\}$ be a sequence and $p \in \mathscr{X}$. $\{p_k\}$ is $\triangle-$convergent to p if and only if, for every $q \in \mathscr{X}$, $\limsup_{k\to\infty}\langle \overrightarrow{pp_k}, \overrightarrow{pq}\rangle \leq 0$.*

Lemma 2.2. [41] *Let $\mathscr{X}$ be a CAT(0) space, and let $a_r = ra \oplus (1-r)b$ for any $a,b,\in X$ and $r \in (0,1)$. Then, for every $p,q \in \mathscr{X}$,*

(i) $\langle \overrightarrow{a_r}, \overrightarrow{a_r q}\rangle \leq r\langle \overrightarrow{ap}, \overrightarrow{a_r q}\rangle + (1-r)\langle \overrightarrow{bp}, \overrightarrow{a_r q}\rangle$;
(ii) $\langle \overrightarrow{a_r p}, \overrightarrow{aq}\rangle \leq r\langle \overrightarrow{ap}, \overrightarrow{aq}\rangle + (1-r)\langle \overrightarrow{bp}, \overrightarrow{aq}\rangle$
and $\langle \overrightarrow{a_r p}, \overrightarrow{bq}\rangle \leq r\langle \overrightarrow{ap}, \overrightarrow{bq}\rangle + (1-r)\langle \overrightarrow{bp}, \overrightarrow{vy}\rangle$.

Lemma 2.3. [21] *Consider a nonempty, closed, convex subset of the Hadamard space $\mathscr{X}$, denoted as $\mathscr{S}$ and a bifunction $\mathscr{L} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ that has properties (A1)-(A4). Let $\{p_k\}$ be any arbitrary sequence in $\mathscr{S}$ and $\{\lambda_k\}$ in $(0,+\infty)$. Let $\{w_k\}$ and $\{z_k\}$ be sequences defined by*

$$\begin{cases} w_k = argmin\{\mathscr{L}(p_k,q) + \frac{1}{2\lambda_k}d^2(p_k,q) : y \in \mathscr{S}\} \\ z_n = argmin\{\mathscr{L}(w_k,q) + \frac{1}{2\lambda_k}d^2(p_k,q) : q \in \mathscr{S}\}. \end{cases}$$

For every $p^* \in EP(\mathscr{L},\mathscr{S})$, $d^2(z_k,p^*) \leq d^2(p_k,p^*) - (1-2t_1\lambda_k)d^2(p_k,w_k) - (1-2t_2\lambda_k)d^2(w_k,z_k)$.

Lemma 2.4. [30] *Let $\mathscr{S}$ be a closed and convex subset of $\mathscr{X}$, a Hadamard space. The set of solutions to the equilibrium problem $EP(\mathscr{L},\mathscr{L})$ is closed and convex if a bifunction $\mathscr{L}$ on $\mathscr{S}$ meets requirements A1, A2, and A4.*

Dhompongsa et al. [15] proposed the following notation in CAT(0) spaces to write a finite convex combination of elements: Given a CAT(0) space $\mathscr{X}$ and $\alpha_1, \alpha_2, \ldots, \alpha_N \in (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$, let $\{p_i : i = 1,2,\ldots,N\}$ be points. Then

$$\begin{aligned} \bigoplus_{i=1}^N \alpha_i p_i &:= (1-\alpha_N)\Big(\frac{\alpha_1}{1-\alpha_N}p_1 \oplus \frac{\alpha_2}{1-\alpha_N}p_2 \oplus \cdots \oplus \frac{\alpha_{N-1}}{1-\alpha_N}p_{N-1}\Big) \oplus \alpha_N p_N \\ &= (1-\alpha_N)\bigoplus_{i=1}^{N-1}\frac{\alpha_i}{1-\alpha_N}p_i \oplus \alpha_N p_N. \end{aligned} \tag{2.4}$$

Lemma 2.5. [8] *Consider a nonempty, closed, convex subset of the CAT(0) space $\mathscr{X}$, denoted as $\mathscr{S}$. Let $\{p_i : i = 1,2,\ldots,N\}$ be in $\mathscr{S}$. Similarly, let $\alpha_1, \alpha_2, \ldots, \alpha_N \in (0,1)$ so that $\sum_{i=1}^N \alpha_i = 1$. For any $z \in \mathscr{S}$, then $d\Big(z, \bigoplus_{i=1}^N \alpha_i p_i\Big) \leq \sum_{i=1}^N \alpha_i d(z,p_i)$.*

Lemma 2.6. [38] *Given a CAT(0) space $\mathscr{X}$, let $\mathscr{S}$ be a nonempty, convex, and closed subset. Let $\mathscr{T} : \mathscr{S} \to \mathscr{X}$ be a θ-generalized demimetric mapping with $\theta \in \mathbb{R}$. Then $F(\mathscr{T})$ is then convex and closed and $(1-\tau)I \oplus \tau\mathscr{T}$ is then the $\theta\tau$-generalized demimetric from $\mathscr{S}$ into $\mathscr{X}$ for all $\theta \in [0,\infty)$ and $\tau \in (0,1]$.*

Lemma 2.7. [4] *Given a CAT(0) space $\mathscr{X}$, consider the following: $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is a τ-demimetric mapping, where $\tau \in (-\infty,\lambda)$, $\lambda \in (0,1)$, and $F(\mathscr{T}) \neq \emptyset$. Assume that $\mathscr{T}_\lambda p := (1-\lambda)p \oplus \lambda\mathscr{T}p$. Hence, $F(\mathscr{T}_\lambda) = F(\mathscr{T})$ and $\mathscr{T}_\lambda$ is a quasi-nonexpansive mapping.*

Lemma 2.8. [31] *Let $\mathscr{T}$ be a Hadamard space. If, for any bounded sequence $\{p_k\}$ in $\mathscr{X}$, $\mathscr{T}p^* = p^*$ and $\lim_{k\to\infty} d(p_k,\mathscr{T}p_k) = 0$, then $\mathscr{T}p^* = p^*$. In such a situation, $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be Δ-demiclosed at* 0.

Lemma 2.9. [14, 11] *Assume that $\mathscr{X}$ is a CAT(0) space with $r \in [0,1]$ and $p,q,z \in \mathscr{X}$. Then*

(i): $d(rp \oplus (1-r)q, z) \leq rd(p,z) + (1-r)d(q,z)$.

(ii): $d^2(rp \oplus (1-r)q, z) \leq rd^2(p,z) + (1-r)d^2(q,z) - r(1-r)d^2(p,q)$.

Lemma 2.10. [42] *If the real nonnegative real sequence $\{\mu_k\}$ satisfies $\mu_{k+1} \leq (1-\alpha_k)\mu_k + \alpha_k\sigma_k + \gamma_k$, $k \geq 0$, where (i) $\{\alpha_k\} \subset [0,1]$ and $\sum \alpha_k = \infty$; (ii) $\lim sup\, \sigma_k \leq 0$; and (iii) $\gamma_k \geq 0$; $(k \geq 0)$ and $\sum \gamma_k < \infty$. Then, $\mu_k \to 0$ as $k \to \infty$.*

Lemma 2.11. [27] *If $\{\mu_k\}$ is a real sequence and there exists a subsequence $\{k_i\}$ of $\{k\}$ such that, for all $i \in \mathbb{N}$, $\mu_{k_i} < \mu_{k_i+1}$, then there exists a nondecreasing sequence $\{s_j\} \subset \mathbb{N}$ such that $s_j \to \infty$ and certain properties are met: $\mu_{s_j} \leq \mu_{s_j+1}$ and $\mu_j \leq \mu_{s_j+1}$. for all $j \in \mathbb{N}$ reasonably big numbers. Thus $s_j = \max\{l \leq j : \mu_l < \mu_{l+1}\}$.*

3. Main Results

This section examines a hybrid extragardient approach to locate a common solution of the problems under investigation. We demonstrate a convergence theorem in a Hadamard space.

Theorem 3.1. *Given a Hadamard space $\mathscr{X}$, let $\mathscr{S}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. For every $i = 1, \cdots, N$, let $\mathscr{L}_i : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Consider that $\mathscr{T} : \mathscr{S} \to \mathscr{S}$ is a $\theta-$generalized demimetric mapping, which is $\Delta-$demiclosed at 0, and $\theta \in (0,\infty)$. Let $\tau \in (0,\sigma)$, $\sigma \in (0,1)$ and $\theta\tau > 0$ such that $\Gamma := F(\mathscr{T}) \bigcap (\cap_{i=1}^N EP(\mathscr{L}_i))$ is not empty. In $(0,1)$, let $\{\alpha_k\}, \{\beta_k\}$, and $\{\gamma_k\}$ be sequences such that $\alpha_k + \beta_k + \gamma_k = 1$ for any fixed vector $u \in \mathscr{S}$. Let $\{p_k\}$ be a sequence generated by $p_1 \in \mathscr{S}$ and*

$$\begin{cases} w_{ki} = \arg\min\{\mathscr{L}_i(p_k, y) + \frac{1}{2\lambda_k} d^2(p_k, y) : y \in \mathscr{S}\},\ i = 1, \cdots, N \\ z_{ki} = \arg\min\{\mathscr{L}_i(w_{ki}, y) + \frac{1}{2\lambda_k} d^2(p_k, y) : y \in \mathscr{S}\},\ i = 1, \cdots, N \\ \bar{z}_k = \arg\max\{d(z_{ki}, p_k) :\ i = 1, \cdots, N\} \\ q_k = (1-\sigma)\bar{z}_k \oplus \sigma[(1-\tau) \oplus \tau\mathscr{T}]\bar{z}_k \\ p_{k+1} = \alpha_k u \oplus \beta_k p_k \oplus \gamma_k q_k, \end{cases} \tag{3.1}$$

Assume that the following restrictions are satisfied

(C1) $\lim\limits_{k\to\infty} \alpha_k = 0$ *and* $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$;

(C2) $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$;

(C3) *For every $i = 1, \cdots, N$, $c_1 = \max_{1\leq i\leq N} c_{i,1}$, $c_2 = \max_{1\leq i\leq N} c_{i,2}$, and $c_{i,1}, c_{i,2}$ are the Lipschitz-type coefficients of $\mathscr{L}_i$. $\{\lambda_k\} \subset [a,b] \subset (0,d)$.*

Then sequence $\{p_k\}$ converges to a point $P_\Gamma u$.

Proof. Based on Lemma 2.6, $F(\mathscr{T})$ is both convex and closed. Additionally, by Lemma 2.4, $EP(\mathscr{L}_i, \mathscr{S})$ is convex and closed for each $i = 1, \cdots, N$. Consequently, $\cap_{i=1}^N EP(\mathscr{L}_i, \mathscr{S})$ is convex and closed, and $\Gamma := F(\mathscr{T}) \bigcap (\cap_{i=1}^N EP(\mathscr{L}_i, \mathscr{S})$ is as well convex and closed. As a result, P_Γ is well-defined.

Furthermore, since $\mathscr{T}$ is $\theta-$generalized demimetric with $\theta \in (0,\infty)$, then, for any $\tau \in (0,\sigma)$ with $\sigma \in (0,1)$, it follows from Lemma 2.6 that $(1-\tau) \oplus \tau\mathscr{T}$ is $\theta\tau-$generalized demimetric. By Remark 1.2, we see that $(1-\tau) \oplus \tau\mathscr{T}$ is $\left(1 - \frac{2}{\theta\tau}\right)-$demimetric. Since $\theta\tau > 0$, by Lemma

2.7, we have that $B := (1-\sigma)\oplus\sigma[(1-\tau)\oplus\tau\mathscr{T}]$ is quasi-nonexpansive and $F(B)=F(\mathscr{T})$. Let $p^*\in\Gamma$ and $i^k\in\{1,\cdots,N\}$ such that $\bar{z}_k = z_{ki^k}$. It follows from (3.1) and Lemma 2.3 that

$$\begin{aligned}
d^2(q_k,p^*) &= d^2(B\bar{z}_k,p^*)\le d^2(\bar{z}_k,p^*)\\
&\le d^2(p_k,p^*)-(1-2c_1\lambda_k)d^2(p_k,w_{ki^k})-(1-2c_2\lambda_k)d^2(\bar{z}_k,w_{ki^k})\\
&\le d^2(p_k,p^*).
\end{aligned}\tag{3.2}$$

In view of Lemma 2.5, we obtain

$$\begin{aligned}
d(p_{k+1},p^*) &\le \alpha_k d(u,p^*)+\beta_k d(p_k,p^*)+\gamma_k d(p_k,p^*)\\
&= \alpha_k d(u,p^*)+(1-\alpha_k)d(p_k,p^*)\\
&\le \max\{d(u,p^*),d(p_k,p^*)\}\\
&\vdots \quad \vdots\\
&\le \max\{d(u,p^*),d(p_1,p^*)\}.
\end{aligned}$$

Therefore, $\{p_k\}$ is bounded. From Lemma 2.9 (ii) and (2.4), we arrive at

$$\begin{aligned}
d^2(p_{k+1},p^*) &= d^2\Big(\alpha_k u\oplus(1-\alpha_k)\Big(\frac{\beta_k}{1-\alpha_k}p_k\oplus\frac{\gamma_k}{1-\alpha_k}q_k\Big),p^*\Big)\\
&\le \alpha_k d^2(u,p^*)+(1-\alpha_k)d^2\Big(\frac{\beta_k}{1-\alpha_k}p_k\oplus\frac{\gamma_k}{1-\alpha_k}q_k,p^*\Big)\\
&\le \alpha_k d^2(u,p^*)+\beta_k d^2(p_k,p^*)+\gamma_k d^2(q_k,p^*)-\frac{\beta_k\gamma_k}{1-\alpha_k}d^2(p_k,q_k)\\
&\le \alpha_k d^2(u,p^*)+(1-\alpha_k)d^2(p_k,p^*)-\beta_k\gamma_k d^2(p_k,q_k).
\end{aligned}$$

Thus

$$\beta_k\gamma_k d^2(p_k,q_k)\le\alpha_k(d^2(u,p^*)-d^2(p_k,p^*))+d^2(p_k,p^*)-d^2(p_{k+1},p^*).\tag{3.3}$$

We next split the remaining proof into two cases.

Case 1. Consider a non-increasing sequence of real numbers, $\{d(p_k,p^*)\}_{k=1}^\infty$. Due to its boundedness, we have that $\{d(p_k,p^*)\}_{k=1}^\infty$ has limits. Thus, from (3.3) and the fact that $\alpha_k\to 0$ as $k\to\infty$ and $\beta_k\gamma_k>0$ for all $k\ge 1$, we obtain that $\lim_{k\to\infty}d(p_k,q_k)=0$. Furthermore, from (3.2) and (3.3), we obtain

$$\begin{aligned}
d^2(p_{k+1},p^*) \le\ & \alpha_k d^2(u,p^*)+\beta_k d^2(p_k,p^*)+\gamma_k\Big[d^2(p_k,p^*)\\
&-(1-2c_1\lambda_k)d^2(p_k,w_{ki^k})-(1-2c_2\lambda_k)d^2(\bar{z}_k,w_{ki^k})\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&\gamma_k(1-2c_1\lambda_k)d^2(p_k,w_{ki^k})+\gamma_k(1-2c_2\lambda_k)d^2(\bar{z}_k,w_{ki^k})\\
&\quad\le \alpha_k(d^2(u,p^*)-d^2(p_k,p^*))+d^2(p_k,p^*)-d^2(p_{k+1},p^*).
\end{aligned}\tag{3.4}$$

Using C1-C3, we obtain $\lim\limits_{k\to\infty}d(p_k,w_{ki^k})=0=\lim\limits_{k\to\infty}d(\bar{z}_k,w_{ki^k})$. Observe $d(\bar{z}_k,p_k)\le d(\bar{z}_k,w_{ki^k})+d(w_{ki^k},p_k)\to 0$ as $k\to\infty$. From the definition of $\bar{z}_k$ in (3.1), for each $i\in\{1,\cdots,N\}$, we have

$\lim_{k\to\infty} d(z_{ki}, p_k) = 0$. It follows from (3.1) and Lemma 2.3 that

$$\begin{aligned}(1-2c_2\lambda_k)d^2(p_k, w_{ki}) &\leq d^2(p_k, p^*) - d^2(z_{ki}, p^*) - (1-2c_1\lambda_k)d^2(w_{ki}, z_{ki})\\ &\leq \Big(d(p_k, p^*) - d(z_{ki}, p^*)\Big)[d(p_k, p^*) + d(z_{ki}, p^*)]\\ &\leq d(p_k, z_{ki})[d(p_k, p^*) + d(z_{ki}, p^*)].\end{aligned}$$

For each $i \in \{1, \cdots, N\}$, we obtain $\lim_{k\to\infty} d(p_k, w_{ki}) = 0$, which further implies that $d(z_{ki}, w_{ki}) \leq d(z_{ki}, p_k) + d(p_k, w_{ki}) \to 0$ as $k \to \infty$. We also have $d(q_k, \bar{z}_k) \leq d(q_k, p_k) + d(p_k, \bar{z}_k) \to 0$ as $k \to \infty$.

Next, we demonstrate that $\lim_{k\to\infty} d(\bar{z}_k, \mathscr{T}\bar{z}_k) = 0$. Given that $\mathscr{T}$ is the $\theta-$generalized demimetric for $\theta \in (0, \infty)$, we obtain the following by using (3.1) and Lemma 2.2(ii):

$$\begin{aligned}\langle \overrightarrow{\bar{z}_k q_k}, \overrightarrow{\bar{z}_k p^*}\rangle &= -\langle \overrightarrow{[(1-\gamma)\bar{z}_k \oplus \gamma B\bar{z}_k]\bar{z}_k}, \overrightarrow{\bar{z}_k p^*}\rangle\\ &\geq -\gamma\langle \overrightarrow{B\bar{z}_k\bar{z}_n}, \overrightarrow{\bar{z}_n p^*}\rangle\\ &= -\gamma\langle \overrightarrow{[(1-\tau)\bar{z}_k \oplus \tau\mathscr{T}\bar{z}_k]\bar{z}_k}, \bar{z}_k p^*\rangle\\ &\geq -\gamma\tau\langle \overrightarrow{\mathscr{T}\bar{z}_k\bar{z}_k}, \overrightarrow{\bar{z}_k p^*}\rangle\\ &\geq \frac{\gamma\tau}{\theta} d^2(\bar{z}_k, \mathscr{T}\bar{z}_k).\end{aligned}$$

Consequently, it follows from (2.3) that

$$\frac{\gamma\tau}{\theta} d^2(\bar{z}_k, \mathscr{T}\bar{z}_k) \leq \langle \overrightarrow{\bar{z}_k q_k}, \overrightarrow{\bar{z}_k p^*}\rangle \leq d(\bar{z}_k q_k) d(\bar{z}_k, p^*).$$

Since $\frac{\gamma\tau}{\theta} > 0$, we obtain $\lim k \to \infty d(\bar{z}_k, \mathscr{T}\bar{z}_k) = 0$. For every $i \in \{1, \cdots, N\}$, we obtain by using the definition of $\bar{z}_k$ in (3.1) that $\lim k \to \infty d(z_{ki}, \mathscr{T}z_{ki}) = 0$. Moreover, since $\{p_k\}$ is bounded and $\mathscr{X}$ is complete, then the $\{p_{k_j}\}$ of $\{p_k\}$ is a subsequence such that $\Delta - \lim p_{k_j} = p$ is in $\mathscr{S}$. Also, for every $i \in \{1, \cdots, N\}$, we obtain $\Delta - \lim z_{k_j} = p$. Since $\mathscr{T}$ is $\Delta-$demiclosed at 0, we conclude that $p \in F(\mathscr{T})$ due to $\lim k \to \infty d(z_{ki}, \mathscr{T}z_{ki}) = 0$.

Next, we show that $p \in \cap_{i=1}^{N} EP(\mathscr{L}_i, \mathscr{S})$. Since $\{p_k\}$ is in $\mathscr{S}$, which is convex and closed, then $p \in \mathscr{S}$. Now, from (3.1), for every $i \in \{1, \cdots, N\}$, the minimization problem is solved by z_{nki}. If $y = tz_{ki} \oplus (1-t)p^*$ for $t \in (0,1)$, then

$$\begin{aligned}&\mathscr{L}_i(w_{ki}, z_{ki}) + \frac{1}{2\lambda_k} d^2(p_k, z_{ki})\\ &\leq \mathscr{L}_i(p_k, y) + \frac{1}{2\lambda_k} d^2(p_k, y)\\ &\leq \mathscr{L}_i(w_{ki}, tz_{ki} \oplus (1-t)p^*) + \frac{1}{2\lambda_k} d^2(p_k, tz_{ki} \oplus (1-t)p^*)\\ &\leq \mathscr{L}_i(w_{ki}, z_{ki}) + (1-t)\mathscr{L}_i(w_{ki}, p^*) + \frac{1}{2\lambda_k}[td^2(p_k, z_{ki}) + (1-t)d^2(p_k, p^*) - t(1-t)d^2(z_{ki}, p^*)].\end{aligned}$$

Since $p^* \in \cap_{i=1}^N EP(\mathscr{L}_i, \mathscr{S})$, then $\mathscr{L}_i(p^*, w_{ki}) \geq 0$ for $i \in \{1, \cdots, N\}$. From the pseudomonotonicity of $\mathscr{L}_i$ for each $i \in \{1, \cdots, N\}$, we have $\mathscr{L}_i(w_{ki}, p^*) \leq 0$. Letting $t \to 1^-$, we obtain

$$\begin{aligned}
\mathscr{L}_i(w_{ki}, z_{ki}) &\leq \frac{1}{2\lambda_k}[d^2(p_k, p^*) - d^2(p_k, z_{ki}) - d^2(z_{ki}, p^*)] \\
&\leq \frac{1}{2\lambda_k}\Big([d(p_k, p^*) + d(z_{ki}, p^*)]d(p_k, z_{ki}) - d^2(p_k, z_{ki})\Big) \\
&\leq \frac{1}{2\lambda_k}[d(p_k, p^*) + d(p_k, p^*)]d(p_k, z_{ki}) \\
&= \frac{1}{\lambda_k} d(p_k, p^*) d(p_k, z_{ki}). \qquad (3.5)
\end{aligned}$$

Using (w_{ki}) in (3.1) and letting $y = tw_{ki} \oplus (1-t)z_{ki}$ for any $t \in [0, 1)$, we have

$$\begin{aligned}
&\mathscr{L}_i(p_k, w_{ki}) + \frac{1}{2\lambda_n} d^2(p_k, w_{ki}) \\
&\leq \mathscr{L}_i(p_k, y) + \frac{1}{2\lambda_k} d^2(p_k, y) \\
&\leq t\mathscr{L}_i(p_k, w_{ki}) + (1-t)\mathscr{L}_i(p_k, z_{ki}) + \frac{1}{2\lambda_k}[td^2(p_k, w_{ki}) + (1-t)d^2(p_k, z_{ki}) - t(1-t)d^2(w_{ki}, z_{ki})].
\end{aligned}$$

Letting $t \to 1^-$, we obtain

$$\mathscr{L}_i(p_k, w_{ki}) - \mathscr{L}_i(p_k, z_{ki}) \leq \frac{1}{2\lambda_n}[d^(p_k, z_{ki}) - d^2(p_k, w_{ki}) - d^2(w_{ki}, z_{ki})]. \qquad (3.6)$$

By A3, for each $i \in \{1, \cdots, N\}$, $\mathscr{L}_i$ is continuous Lipschitz-type with coefficient c_{1i} and c_{2i} for $i \in \{1, \cdots, N\}$. Using $c_1 = \max_{1 \leq i \leq N} c_{1i}$ and $c_2 = \max_{1 \leq i \leq N} c_{2i}$, we obtain

$$\mathscr{L}_i(p_k, w_{ki}) + \mathscr{L}_i(w_{ki}, z_{ki}) \geq \mathscr{L}_i(p_k, z_{ki}) - c_1 d^2(p_k, w_{ki}) - c_2 d^2(w_{ki}, z_{ki}).$$

It follows from (3.6) that

$$\begin{aligned}
&-\mathscr{L}_i(w_{ki}, z_{ki}) - c_1 d^2(p_k, w_{ki}) - c_2 d^2(w_{ki}, z_{ki}) \\
&\leq \frac{1}{2\lambda_k}[d^2(p_k, z_{ki}) - d^2(p_k, w_{ki}) - d^2(w_{ki}, z_{ki})].
\end{aligned}$$

Hence

$$(\frac{1}{2\lambda_k} - c_1)d^2(p_k, w_{ki}) + (\frac{1}{2\lambda_k} - c_2)d^2(w_{ki}, z_{ki}) - \frac{1}{2\lambda_k}d^2(p_k, z_{ki}) \leq \mathscr{L}_i(w_{ki}, z_{ki}).$$

In light of this and (3.5), we have

$$\begin{aligned}
(\frac{1}{2\lambda_k} - c_1)d^2(p_k, w_{ki}) + (\frac{1}{2\lambda_k} - c_2)d^2(w_{ki}, z_{ki}) - \frac{1}{2\lambda_k}d^2(p_k, z_{ki}) &\leq \mathscr{L}_i(w_{ki}, z_{ki}) \\
&\leq \frac{1}{\lambda_k} d(p_k, p^*) d(p_k, z_{ki}).
\end{aligned}$$

For each $i \in \{1, \cdots, N\}$, we have $\lim_{k \to \infty} \mathscr{L}_i(w_{ki}, z_{ki}) = 0$. Furthermore, from (3.1), for each $i \in \{1, \cdots, N\}$, z_{ki} is a solution to the minimization problem. Letting $z = tz_{ki} \oplus (1-t)y$ for

$t \in [0,1)$, we obtain

$$\begin{aligned}
&\mathscr{L}_i(w_{ki}, z_{ki}) + \frac{1}{2\lambda_k} d^2(p_k, z_{ki}) \leq \mathscr{L}_i(w_{ki}, z) + \frac{1}{2\lambda_k} d^(p_k, z) \\
= \;& \mathscr{L}_i(w_{ki}, t z_{ki} \oplus (1-t)y) + \frac{1}{2\lambda_k} d^2(p_k, t z_{ki} \oplus (1-t)y) \\
\leq \;& t\mathscr{L}_i(w_{ki}, z_{ki}) + (1-t)\mathscr{L}_i(w_{ki}, y) + \frac{1}{2\lambda_k}\Big(t d^2(p_k, z_{ki}) \\
& + (1-t) d^2(p_k, y) - t(1-t) d^2(z_{ki}, y)\Big)
\end{aligned}$$

Letting $t \to 1^-$, we further have

$$\frac{1}{2\lambda_k}[d^2(p_k, z_{ki}) + d^2(z_{ki}, y) - d^2(p_k, y)] \leq \mathscr{L}_i(w_{ki}, y) - \mathscr{L}_i(w_{ki}, z_{ki}). \tag{3.7}$$

However, we obtain from the left side of (3.7) that

$$\begin{aligned}
& d^2(p_k, z_{ki}) + d^2(z_{ki}, y) - d^2(p_k, y) \\
& \geq \; d^2(p_k, z_{ki}) + d(z_{ki}, p_k)[d(z_{ki}, y) - d(p_k, y)] \\
& = \; d(p_k, z_{ki})\Big[d(z_{ki}, p_k) + d(z_{ki}, y) - d(p_k, y)\Big] \\
& \geq \; d(p_k, z_{ki})[d(p_k, y) - d(p_k, y)] = 0,
\end{aligned}$$

which together with (3.7) and $\frac{1}{2\lambda_k} > 0$ for $k \geq 1$ yields that

$$0 \leq \mathscr{L}_i(w_{ki}, y) - \mathscr{L}_i(w_{ki}, z_{ki}). \tag{3.8}$$

Since, for every $i \in \{1, \cdots, N\}$, $\Delta - \lim p_{k_j i} = p$, we obtain $\Delta - \lim w_{k_j i} = p = \Delta - \lim z_{k_j i}$ for all $i \in \{1, \cdots, N\}$. For $i \in \{1, \cdots, N\}$, replacing k with k_j in (3.8) and taking the limsup, it follows from the fact that $\mathscr{L}_i$ is $\Delta-$upper semicontinuous that $0 \leq \limsup_{j\to\infty} \mathscr{L}_i(w_{k_j i}, y) \leq \mathscr{L}_i(p, y)$ for all $y \in c$, $i = 1, \cdots, N$. Hence, $p \in \cap_{i=1}^N EP(\mathscr{L}_i, \mathscr{S})$, and then $p \in \Gamma := F(T) \bigcap (\cap_{i=1}^N EP(\mathscr{L}_i, \mathscr{S}))$. Lemma 2.1 further yields that $\limsup_{k\to\infty} \langle \overrightarrow{up}, \overrightarrow{p_k p} \rangle \leq 0$. Letting $v_k := \frac{\beta_k}{1-\alpha_k} p_k \oplus \frac{\gamma_k}{1-\alpha_k} q_k$ and using (2.2) and (2.3), we obtain $\langle \overrightarrow{up}, \overrightarrow{v_k p} \rangle \leq d(u,p) d(v_k, p_k) + \langle \overrightarrow{up}, \overrightarrow{p_k p} \rangle$, which yields that $\limsup_{k\to\infty} \langle \overrightarrow{up}, \overrightarrow{v_k p} \rangle \leq 0$. Thus we demonstrate that $p_k \to p$ as $k \to \infty$. Let $r_k := \alpha_k p \oplus \beta_k p_k \oplus \gamma_k q_k$. Using the idea in (2.4), we let $v_k := \frac{\beta_k}{1-\alpha_k} p_k \oplus \frac{\gamma_k}{1-\alpha_k} q_k$. From (3.1), $p_{k+1} = \alpha_k u \oplus (1-\alpha_k) v_k$. Thus, using Lemma 2.9 (ii) yields

$$\begin{aligned}
d^2(p_{k+1}, p) \;&\leq\; d^2(r_k, p) + 2\langle \overrightarrow{p_{k+1} r_k}, \overrightarrow{p_{k+1} p} \rangle \\
&=\; d^2\Big(\alpha_k p \oplus (1-\alpha_k)\Big(\frac{\beta_k}{1-\alpha_k} p_k \oplus \frac{\gamma_k}{1-\alpha_k} q_k\Big), p\Big) + 2\langle \overrightarrow{r_k p_{k+1}}, \overrightarrow{p p_{k+1}} \rangle \\
&\leq\; (1-\alpha_k) d^2\Big(\frac{\beta_k}{1-\alpha_k} p_k \oplus \frac{\gamma_k}{1-\alpha_k} q_k, p\Big) + 2\langle \overrightarrow{r_k p_{k+1}}, \overrightarrow{p p_{k+1}} \rangle \\
&\leq\; \beta_k d^2(p_k, p) + \gamma_n d^2(q_k, p) + 2\langle \overrightarrow{r_k p_{k+1}}, \overrightarrow{p p_{k+1}} \rangle \\
&\leq\; \beta_k d^2(p_k, p) + \gamma_k d^2(p_k, p) + 2\alpha_k \langle \overrightarrow{up}, \overrightarrow{v_k p} \rangle \\
&\leq\; (1-\alpha_k) d^2(p_k, p) + 2\alpha_k \langle \overrightarrow{up}, \overrightarrow{v_k p} \rangle.
\end{aligned}$$

Therefore

$$d^2(p_{k+1}, p) \leq (1-\alpha_k) d^2(p_k, p) + 2\alpha_k \langle \overrightarrow{up}, \overrightarrow{v_k p} \rangle. \tag{3.9}$$

Consequently, we obtain $d(p_k, p) \to 0$ as $k \to \infty$, that is, $p_k \to p$ as $k \to \infty$, by (3.9) and Lemma 2.10.

Case 2. Assume that $\{d(p_k, p^*)\}_{k=1}^{\infty}$ is monotone nondecreasing real sequence. If $\Upsilon_k := d(p_k, p^*)$ for every $k \geq 1$, then there exists a subsequence Υ_{k_s} of Υ_k such that $\Upsilon_{k_s} < \Upsilon_{k_s+1}$ for all $s \geq 1$. Define $\xi : \mathbb{N} \to \mathbb{N}$ $\xi(k) = \max\{s \leq k : \Upsilon_k < \Upsilon_{k+1}\}$. Lemma 2.11 implies that $\Upsilon_{\xi(k)} \leq \Upsilon_{\xi(k)+1}$. Then by (3.3) and $\beta_{\xi(k)}\gamma_{\xi(k)} > 0$, we have

$$
\begin{aligned}
0 < \beta_{\xi(k)}\gamma_{\xi(k)} d^2(p_{\xi(k)}, q_{\xi(k)}) \leq\ & \alpha_{\xi(k)}(d^2(u, p^*) - d^2(p_{\xi(k)}, p^*)) \\
& + d^2(p_{\xi(k)}, p^*) - d^2(p_{\xi(k)+1}, p^*).
\end{aligned}
$$

Since $\alpha_{\xi(k)} \to 0$ as $k \to \infty$, then $\lim_{k\to\infty} d(p_{\xi(k)}, q_{\xi(k)}) = 0$. It follows from (3.4) that

$$
\begin{aligned}
& \gamma_{\xi(k)}(1 - 2c_1\lambda_{\xi(k)})d^2(p_{\xi(k)}, w_{\xi(k)i^k}) + \gamma_{\xi(k)}(1 - 2c_2\lambda_{\xi(k)})d^2(\bar{z}_{\xi(k)}, w_{\xi(k)i^k}) \\
\leq\ & \alpha_{\xi(k)}(d^2(u, p^*) - d^2(p_{\xi(k)}, p^*)) + d^2(p_{\xi(k)}, p^*) - d^2(p_{\xi(k)+1}, p^*).
\end{aligned}
$$

Since $\gamma_{\xi(k)}(1 - 2c_j\lambda_{\xi(k)}) > 0$ for each $j = 1, 2$ and $\alpha_{\xi(k)} \to 0$ as $k \to \infty$, then $\lim_{k\to\infty} d(p_{\xi(k)}, w_{\xi(k)i^k}) = 0 = \lim_{k\to\infty} d(z_{\xi(k)}, w_{\xi(k)i^k})$. With similar procedure in Case 1, we obtain $\lim_{k\to\infty} d(p_{\xi(k)}, z_{\xi(k)i}) = 0$, $\lim_{k\to\infty} d(p_{\xi(k)}, w_{\xi(k)i})$, and $\lim_{k\to\infty} d(z_{\xi(k)i}, w_{\xi(k)i}) = 0$, and $\lim_{k\to\infty} d(z_{\xi(k)i}, \mathscr{T} z_{\xi(k)i})$. Furthermore, we have from (3.1) that

$$
d(p_{\xi(k)+1}, p_{\xi(k)}) \leq \alpha_{\xi(k)} d(u, p_{\xi(k)}) + \gamma_{\xi(k)} d(q_{\xi(k)}, p_{\xi(k)}),
$$

which implies $\lim_{k\to\infty} d(p_{\xi(k)+1}, p_{\xi(k)}) = 0$. Note that $\limsup_{k\to\infty}\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle \leq 0$. Furthermore, we have

$$
d^2(p_{\xi(k)+1}, p) \leq (1 - \alpha_{\xi(k)}) d(p_{\xi(k)}, p) + 2\alpha_{\xi(k)}\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle.
$$

Since $d^2(p_{\xi(k)}, p) < d^2(p_{\xi(k)+1}, p)$, then

$$
\begin{aligned}
\alpha_{\xi(k)} d^2(p_{\xi(k)}, p) & \leq d^2(p_{\xi(k)}, p) - d^2(p_{\xi(k)+1}, p) + 2\alpha_{\xi(k)}\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle \\
& < 2\alpha_{\xi(k)}\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle
\end{aligned}
$$

In view of $\alpha_{\xi(k)} > 0$, we can obtain $d^2(p_{\xi(k)}, p) < 2\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle$. With $\limsup_{k\to\infty}\langle \overrightarrow{up}, \overrightarrow{v_{\xi(k)}p}\rangle \leq 0$, we obtain $\limsup_{k\to\infty} d^2(p_{\xi(k)}, p) \leq 0$. Therefore $\lim_{k\to\infty} d(p_{\xi(k)}, p) = 0$ and

$$
\lim_{k\to\infty} d(p_{\xi(k)}, p) = \lim_{k\to\infty} d(p_{\xi(k)+1}, p) = 0.
$$

Thus we obtain $d(p_k, p) \leq d(p_{\xi(k)+1}, p) \to 0$ as $k \to \infty$ according to Lemma 2.11. This proves $p_k \to p$ as $k \to \infty$. □

The following result is due to Theorem 3.1 since every generalized hybrid mapping in the sense of Lin et al. [25] with nonempty fixed points is a 4-generalized demimetric mapping (see [38]).

Corollary 3.2. *Given a Hadamard space $\mathscr{X}$, let $\mathscr{S}1$ be a nonempty, closed, and convex subset. For every $i = 1, \cdots, N$, let $\mathscr{L}_i : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ be a bifunction satisfying A1-A4. Let $\Gamma := F(\mathscr{T}) \bigcap (\cap_{i=1}^N EP(\mathscr{L}_i))$ be nonempty, and let $\mathscr{T} : \mathscr{S} \to \mathscr{S}$ be a generalized demimetric mapping and $\Delta-$demiclosed at* 0*. Assume that $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, $\{\lambda_k\}$, u, and $\{p_k\}$ are assigned values as in Theorem 3.1 with $\sigma = 1$. Then, $\{p_k\}$ converges to $P_\Gamma u$.*

4. Applications

We prove a new strong convergence theorem in this section, which is related to the bifunction-associated with a finite family of pseudomonotones and α-inverse strongly monotone mappings in the sense of [3]. Assume that $\mathscr{S}$ is a nonempty, closed, and convex subset of a Hadamard space $\mathscr{X}$. Recall that mapping $\mathscr{T}:\mathscr{S}\to\mathscr{X}$ is α-inverse strongly monotone if

$$d^2(p,q)-\langle\overrightarrow{\mathscr{T}p\mathscr{T}q},\overrightarrow{pq}\rangle\geq\alpha\Psi_{\mathscr{T}}(p,q),\quad\forall p,q\in\mathscr{S},\tag{4.1}$$

where α is a positive real number, $\Psi_{\mathscr{T}}(p,q):=d^2(p,q)+d^2(\mathscr{T}p,\mathscr{T}q)-2\langle\overrightarrow{\mathscr{T}p\mathscr{T}q},\overrightarrow{pq}\rangle$, and $\Psi_{\mathscr{T}}(p,q)$ is nonnegative.

Lemma 4.1. [3] *Assume that $\mathscr{S}$ is a nonempty, closed, and convex subset of a Hadamard space $\mathscr{X}$. Let $\mathscr{T}:\mathscr{S}\to\mathscr{X}$ be an α-inverse strongly monotone mapping and define $\mathscr{T}_\mu:\mathscr{S}\to\mathscr{X}$ by $\mathscr{T}_\mu=(1-\mu)I\oplus\mu\mathscr{T}$. If $0<\mu<2\alpha$, then $F(\mathscr{T}_\mu)=F(\mathscr{T})$ is a nonexpansive mapping.*

Lemma 4.2. [22] *Assume that $\mathscr{S}$ is a nonempty, closed, and convex subset of a Hadamard space $\mathscr{X}$. Let $\mathscr{T}:\mathscr{S}\to\mathscr{X}$ be a nonexpansive mapping. In $\mathscr{S}$, let $\{p_n\}$ be a bounded sequence such that $\Delta-\lim_{k\to\infty}p_k=p$ and $\lim_{k\to\infty}d(\mathscr{T}x_k,x_k)=0$. Then $p=\mathscr{T}p$.*

Lemma 4.3. *Let $\mathscr{T}$ be a mapping from $\mathscr{S}$ into $\mathscr{X}$, where $\mathscr{S}$ is a nonempty, closed, and convex subset of a CAT(0) space. $\mathscr{T}$ is a $\frac{1}{\alpha}$-generalized demimetric mapping if $\mathscr{T}$ is a α-inverse strongly monotone, as in (4.1) with $F(\mathscr{T})\neq\emptyset$.*

Proof. Let $a\in F(\mathscr{T})$ and $p\in\mathscr{S}$. By (4.1), we obtain $d^2(p,a)-\langle\overrightarrow{\mathscr{T}pa},\overrightarrow{pa}\rangle\geq\alpha\Psi_{\mathscr{T}}(p,a)$,where

$$\Psi_{\mathscr{T}}(p,a)=d^2(p,a)+d^2(\mathscr{T}p,a)-2\langle\overrightarrow{\mathscr{T}pa},\overrightarrow{pa}\rangle.$$

Using the definition of quasilinerization, we have

$$\begin{aligned}
&d^2(p,a)-\langle\overrightarrow{\mathscr{T}pp},\overrightarrow{py}\rangle-\langle\overrightarrow{pa},\overrightarrow{pa}\rangle\\
\geq\;&\alpha\left[d^2(p,a)+d^2(\mathscr{T}p,a)-2\left[\langle\overrightarrow{\mathscr{T}pp},\overrightarrow{pa}\rangle+\langle\overrightarrow{pa},\overrightarrow{pa}\rangle\right]\right]\\
=\;&\left[d^2(p,a)+d^2(\mathscr{T}p,a)-2\langle\overrightarrow{\mathscr{T}pp},\overrightarrow{pa}\rangle-2d^2(p,a)\right]\\
=\;&\alpha\left[d^2(\mathscr{T}p,a)-d^2(p,a)-\left(d^2(\mathscr{T}p,a)-d^2(\mathscr{T}p,p)\ -d^2(p,a)\right)\right]\\
=\;&\alpha d^2(\mathscr{T}p,p),
\end{aligned}$$

Hence $\langle\overrightarrow{p\mathscr{T}p},\overrightarrow{pa}\rangle\geq\alpha d^2(\mathscr{T}p,p)$ and then $d^2(\mathscr{T}p,p)\leq\frac{1}{\alpha}\langle\overrightarrow{pa},\overrightarrow{p\mathscr{T}p}\rangle$. If $\mathscr{T}$ is α-inverse strongly monotone in the sense of [3] with $F(\mathscr{T})\neq\emptyset$, then $\mathscr{T}$ is $\frac{1}{\alpha}$-generalized demimetric. □

Theorem 4.4. *Given the Hadamard space $\mathscr{X}$, let $\mathscr{S}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. For every $i=1,\cdots,N$, let $\mathscr{L}_i:\mathscr{S}\times\mathscr{S}\to\mathbb{R}$ be a bifunction satisfying A1-A4. Assume that $\Gamma:=F(\mathscr{T})\bigcap(\cap_{i=1}^N EP(\mathscr{L}_i))$ is nonempty, where $\mathscr{T}:\mathscr{S}\to\mathscr{X}$ is an α-inverse strongly monotone. In $(0,1)$, let $\{\alpha_k\},\{\beta_k\}$, and $\{\gamma_k\}$ be sequences such that $\alpha_k+\beta_k+\gamma_k=1$. For a fixed vector $u\in\mathscr{S}$, we define a sequence $\{p_k\}$, with an initial point p_1, in $\mathscr{S}$ and*

$$\begin{cases}
w_{ki}=\arg\min\{\mathscr{L}_i(p_k,y)+\frac{1}{2\lambda_k}d^2(p_k,y):y\in\mathscr{S}\},\ i=1,\cdots,N\\
z_{ki}=\arg\min\{\mathscr{L}_i(w_{ki},y)+\frac{1}{2\lambda_k}d^2(p_k,y):y\in\mathscr{S}\},\ i=1,\cdots,N\\
\bar{z}_k=\arg\max\{d(z_{ki},p_k):\ i=1,\cdots,N\}\\
p_{k+1}=\alpha_k u\oplus\beta_k p_k\oplus\gamma_n\mathscr{T}_\mu\bar{z}_k,
\end{cases}$$

where $\mathscr{T}_\mu = (1-\mu)I \oplus \mu\mathscr{T}$ and $\mu \in [0,1]$. Assume that the following conditions are satisfied:

(c1) $\lim\limits_{k\to\infty} \alpha_k = 0$ *and* $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$;

(c2) $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$;

(c3) *For every* $i = 1, \cdots, N$, $c_1 = \max_{1\leq i\leq N} c_{i,1}$, $c_2 = \max_{1\leq i\leq N} c_{i,2}$, *and* $c_{i,1}, c_{i,2}$ *are the Lipschitz-type coefficients of* $\mathscr{L}_i$. $\{\lambda_k\} \subset [a,b] \subset (0,d)$.

Thn $\{p_k\}$ *then converges to a point* $P_\Gamma u$.

Proof. Using $0 < \mu < 2\alpha$, we see Lemma 4.1 that $F(\mathscr{T}) = F(\mathscr{T}_\mu)$ and $\mathscr{T}_\mu$ is nonexpansive. Lemma 4.2 states that $\mathscr{T}_\mu$ is $\Delta-$demilosed at 0. Additionally, since $F(\mathscr{T}) \neq \emptyset$, $\mathscr{T}$ is $\frac{1}{\alpha}-$generalized demimetric mapping, according to Lemma 4.3. As a result, by using $k = 1$ and $\sigma = 1$ in Theorem 3.1, Theorem 3.1 yields the desired result immediately. □

Given a Hadamard $\mathscr{X}$, let $\mathscr{S}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. Recently, Kakavandi and Amini [19] defined the dual space in a Hadamard space $\mathscr{X}$ as follows by utilizing the notion of quasilinearization. Let $\Theta : \mathbb{R} \times \mathscr{X} \times \mathscr{X} \to C(\mathscr{X}, \mathbb{R})$ be the map defined by $\Theta(t,x,y)(p) = t\langle \overrightarrow{xy}, \overrightarrow{xp}\rangle$, $t \in \mathbb{R}, x, y, p \in \mathscr{X}$, in which the space of all continuous real-valued functions on $\mathscr{X}$ is denoted by $C(\mathscr{X}, \mathbb{R})$. Consequently, $\Theta(t,x,y)$ is a Lipschitz function with a Lipschitz semi-norm $L(\Theta(t,x,y)) = |t|d(x,y)$ $(t \in \mathbb{R},\ x, y \in \mathscr{X})$, based on the Cauchy-Schwartz inequality (2.3) where, for every function $\varphi : \mathscr{X} \to \mathbb{R}$

$$L(\Theta) = \sup\left\{ \frac{\varphi(p) - \varphi(q)}{d(p,q)} : p, q \in X, p \neq q \right\}$$

is the Lipschitz semi-norm. Define a pseudometric D on $\mathbb{R} \times \mathscr{X} \times \mathscr{X}$ as

$$D((t,x,y),(s,u,v)) = L(\Theta(t,x,y) - \Theta(s,u,v)), \quad (t,s \in \mathbb{R}, x,y,u,v \in \mathscr{X}).$$

The pseudometric space $(\mathbb{R} \times \mathscr{X} \times \mathscr{X}, D)$ can be viewed as a subset of the pseudometric space of all real-valued Lipschitz functions $(Lip(\mathscr{X}, \mathbb{R}), L)$ for a Hadamard space $(\mathscr{X}, d)$. Furthermore, it was examined in [19] that, for all $p, q \in \mathscr{X}$, $D((t,x,y),(s,u,v)) = 0$ if and only if $t\langle \overrightarrow{xy}, \overrightarrow{pq}\rangle = \langle \overrightarrow{uv}, \overrightarrow{pq}\rangle$. Consequently, D generates an equivalent relation on $\mathbb{R} \times \mathscr{X} \times \mathscr{X}$, in which (t,a,b) is the equivalence class $[t\overrightarrow{xy}] := \{s\overrightarrow{uv} : D((t,x,y),(s,u,v)) = 0\}$. The set $\mathscr{X}^* = \{[t\overrightarrow{xy}] : (t,x,y) \in \mathbb{R} \times \mathscr{X} \times \mathscr{X}\}$ is a metric space with the metric $D([t\overrightarrow{xy}],[s\overrightarrow{uv}]) := D((t,x,y),(s,u,v))$. The pair $(\mathscr{X}^*, D)$ is called the dual space of $(\mathscr{X}, d)$. The dual of a closed and convex subset of a Hilbert space $\mathscr{H}$ with a nonempty interior is $\mathscr{H}$ and $t(y-x) = [t\overrightarrow{xy}]$, as is demonstrated in [19], for every $t \in \mathbb{R}, x, y \in \mathscr{H}$. Moreover, $\mathscr{X}^*$ acts as follows on $\mathscr{X} \times \mathscr{X}$:

$$\langle p^*, \overrightarrow{pq}\rangle = t\langle \overrightarrow{xy}, \overrightarrow{pq}\rangle, \ \ (p^* = [t\overrightarrow{xy}] \in \mathscr{X}^*, p, q \in \mathscr{X}).$$

The concept of a monotone operator in a CAT(0) space was introduced by Ranjbar and Khatibzadeh [35] in 2016. They also examined some properties of a monotone operator and its resolvent operator. Consider the Hadamard space $\mathscr{X}$, and its dual space $\mathscr{X}^*$. With domain $\mathbb{D}(A) := \{p \in \mathscr{X} : Ap \neq \emptyset\}$, a multivalued operator $A : \mathscr{X} \to 2^{\mathscr{X}^*}$ is monotone if and only if $p^* \in Ap$ and $q^* \in Aq$ for every $p, q \in \mathbb{D}(A)$ $\langle p^* - q^*, \overrightarrow{qp}\rangle \geq 0$ holds. Let $\mathscr{X}$ be a Hadamard space, $\mathscr{X}^*$ its dual space, and $A : \mathscr{X} \to 2^{\mathscr{X}^*}$ be monotone. The set-valued mapping $J_\lambda^A : \mathscr{X} \to 2^{\mathscr{X}}$ is the resolvent of A of order $\lambda > 0$, defined by $J_\lambda^A(p) := \{z \in \mathscr{X} : [\frac{1}{\lambda}\overrightarrow{zp}] \in Ap\}$. The range requirement is satisfied by the monotone operator A if, for each $\lambda > 0$, $\mathbb{D}(J_\lambda^A) = \mathscr{X}$. We write J_λ for the resolvent of a monotone operator A in the sequel.

Lemma 4.5. [35] *Assume that $\mathscr{X}$ is a Hadamard space and that the resolvent of a monotone operator A of order $\lambda > 0$ is J_λ. Then,*

(i) *$F(J_\lambda) = A^{-1}(0)$, and $R(J_\lambda)$ denote the range of J_λ for any $\lambda > 0$, $R(J_\lambda) \subset \mathbb{D}(A)$;*
(ii) *J_λ is firmly nonexpansive and has a single value.*

Given a Hadamard space $\mathscr{X}$ and a monotone operator A of order $\lambda > 0$ with resolvent J_λ, we see from [39] that $d^2(a, J_\lambda p) + d^2(J_\lambda p, p) \leq d^2(a, p)$ for every $a \in A^{-1}(0)$, $p \in \mathbb{D}(J_\lambda)$ and $\lambda > 0$. If $a \in A^{-1}(0) = F(J_\lambda)$, then, for any $p \in D(J_\lambda)$, we obtain from (2.1) that

$$2\langle \overrightarrow{ap}, \overrightarrow{pJ_\lambda p}\rangle + d^2(a,p) + d^2(p, J_\lambda p) = d^2(a, J_\lambda p).$$

Note that $2\langle \overrightarrow{ap}, \overrightarrow{pJ_\lambda p}\rangle + d^2(p, J_\lambda p) \leq -d^2(p, J_\lambda p)$. Hence $d^2(J_\lambda p, p) \leq \langle \overrightarrow{pa}, \overrightarrow{pJ_\lambda p}\rangle$. As a result, J_λ is a $1-$generalized demimetric mapping in accordance with Definition 1.1(iv).

Theorem 4.6. *Suppose $\mathscr{X}$ is a Hadamard with dual $\mathscr{X}^*$, and $\mathscr{S}$ is a nonempty, closed, and convex subset of $\mathscr{X}$. Given multivalued monotone mappings $A : \mathscr{X} \to 2^{\mathscr{X}^*}$ with the range condition, let $A^{-1}(0) \neq \emptyset$ and J_μ be the resolvent of A for $\mu > 0$. Define a bifunction $\mathscr{L}_i : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ for each $i = 1, \cdots, N$ with A1-A4. Given an α-inverse strongly monotone $\mathscr{T} : \mathscr{S} \to \mathscr{X}$, let $\Gamma := A^{-1}(0) \bigcap (\cap_{i=1}^N EP(\mathscr{L}_i)) \neq \emptyset$. Consider the sequences $\{\alpha_k\}, \{\beta_k\}$, and $\{\gamma_k\}$ in $(0,1)$ such that $\alpha_k + \beta_k + \gamma_k = 1$. Let $\{p_k\}$ be a sequence produced by $p_1 \in \mathscr{S}$ and a fixed vector $u \in \mathscr{S}$*

$$\begin{cases} w_{ki} = \arg\min\{\mathscr{L}_i(p_k, y) + \frac{1}{2\lambda_n} d^2(p_k, y) : y \in \mathscr{S}\},\ i = 1, \cdots, N \\ z_{ki} = \arg\min\{\mathscr{L}_i(w_{ki}, y) + \frac{1}{2\lambda_n} d^2(p_k, y) : y \in \mathscr{S}\},\ i = 1, \cdots, N \\ \bar{z}_k = \arg\max\{d(z_{ki}, p_k) :\ i = 1, \cdots, N\} \\ p_{k+1} = \alpha_k u \oplus \beta_k p_k \oplus \gamma_k J_\mu \bar{z}_k, \end{cases}$$

where $\mathscr{T}_\mu = (1-\mu)I \oplus \mu \mathscr{T}$ and $\mu \in [0,1]$. If the following conditions are satisfied

(c1) *$\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$;*
(c2) *$0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$;*
(c3) *For every $i = 1, \cdots, N$, $c_1 = \max_{1\leq i\leq N} c_{i,1}$, $c_2 = \max_{1\leq i\leq N} c_{i,2}$, and $c_{i,1}, c_{i,2}$ are the Lipschitz-type coefficients of $\mathscr{L}_i$. $\{\lambda_k\} \subset [a,b] \subset (0,d)$,*

then $\{p_k\}$ then converges to a point $P_\Gamma u$.

Proof. Observe that J_{μ_k} is the resolvent of A and $A^{-1}(0) \neq \emptyset$. We have that J_{μ_k} is 1-generalized demimetric. Furthermore, it follows from Theorem 4.5 that $A^{-1}(0) = F(J_{\mu_n})$ and J_{μ_k} is firmly nonexpansive. By Lemma 4.2, J_{μ_k} is $\Delta-$demiclosed at zero. Putting $\sigma = 1 = k$ in Theorem 3.1, we obtain the desire result immediately. □

5. Numerical Example

In this section, we give an example to validate our main results.

Example 5.1. Let $(1, \mathbb{R}^+)$ be a Hadamard space with inner product $\langle p, q\rangle_u = \frac{pq}{u^2}$ for $u \in \mathbb{R}^+$ and $p, q \in \mathscr{T}_u\mathbb{R}^+ = \mathbb{R}$ the tangent space. Let $d : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, \infty)$ be defined by $d(u,v) = |\ln u - \ln v|$ for all $u, v \in \mathbb{R}^+$. Then $(\mathbb{R}^+, d)$ is a CAT(0) space (see [26] for details) with the geodesic from u to v defined as $\gamma(s) = u(\frac{v}{u})^s$ and $\ln\gamma(s) = \ln u(\frac{v}{u})^s = (1-s)\ln u + s\ln v$.

Now, given $\mathscr{X}=\mathbb{R}^+$, define $\mathscr{L}_i:\mathscr{X}\times\mathscr{X}\to\mathbb{R}$ as follows

$$\mathscr{L}_i(p,q)=(\ln(p))^{2^i}[\ln p-\ln q],\ \ i=1,2,3$$

$$\begin{aligned}\mathscr{L}_i(p,\gamma(s)) &= (\ln p)^{2^i}[\ln p-\ln\gamma(s)]\\ &= (\ln p)^{2^i}[s\ln p+(1-s)\ln p-[(1-s)\ln u+s\ln v]]\\ &= (1-s)(\ln p)^{2^i}[\ln p-\ln u]+s(\ln p)^{2^i}[\ln p-\ln v]\\ &= (1-s)\mathscr{L}_i(p,u)+s\mathscr{L}_i(p,v).\end{aligned}$$

Hence, $\mathscr{L}_i(p,\cdot)$ is convex for $p\in\mathscr{X}$. Since $\mathscr{L}_i$ is continuous on both argument, then $\mathscr{L}_i$ satisfy $A1$ and $A4$.

Next, we show that $\mathscr{L}_i$ is pseudomonotone but not monotone. Firstly, $\mathscr{L}_i$ is not monotone for all $p,q\in\mathscr{X}$

$$\mathscr{L}_i(p,q)+\mathscr{L}_i(q,p)=\left[(\ln p)^{2^i}-(\ln q)^{2^i}\right][\ln p-\ln q].$$

If $i=1$, then $\mathscr{L}_1(p,q)+\mathscr{L}_1(q,p)=(\ln p+\ln q)(\ln p-\ln q)^2\geq 0$. Thus $\mathscr{L}_1$ is not monotone.

For $i=2$

$$\mathscr{L}_2(p,q)+\mathscr{L}_2(q,p) = \left[(\ln p)^2+(\ln q)^2\right](\ln p+\ln q)(\ln p-\ln q)^2\geq 0.$$

Thus $\mathscr{L}_2$ is not monotone.

For $i=3$, $\mathscr{L}_3(p,q)+\mathscr{L}_3(q,p)=[\mathscr{L}_2(p,q)+\mathscr{L}_2(p,q)]\left[(\ln p)^4+(\ln q)^4\right]\geq 0$. $\mathscr{L}_3$ is not monotone.

Finally, we show that $\mathscr{L}_i$ is pseudomonotone. Let $p,q\in\mathscr{X}$. If $\mathscr{L}_i(p,q)\geq 0$, then

$$(\ln p)^{2^i}[\ln p-\ln q]\geq 0 \Longrightarrow\ (\ln p)^{2^i}[\ln q-\ln p]\leq 0.$$

Since $(\ln p)^{2^i}\geq 0$ for all $p\in\mathscr{X}$, then $(\ln q)^{2^i}[\ln q-\ln p]\leq 0$ for each $i=1,2,3$, which implies $\mathscr{L}_i(q,p):=(\ln q)^{2^i}[\ln q-\ln p]\leq 0$, that is $\mathscr{L}_i(q,p)\leq 0$ whenever $\mathscr{L}_i(p,q)\geq 0$. Hence $\mathscr{L}_i$ is pseudomonotone for each i.

Finally, we show that $\mathscr{L}_i$ is Lipschitz-type continuous for each i. Let $p,q,z\in\mathscr{X}$. It follows that

$$\begin{aligned}\mathscr{L}_i(p,q)+\mathscr{L}_i(q,z)-\mathscr{L}_i(p,z) &= (\ln p)^{2^i}[\ln z-\ln q]+(\ln q)^{2^i}[\ln q-\ln z]\\ &= \left[(\ln p)^{2^i}-(\ln q)^{2^i}\right][\ln z-\ln q]\\ &\geq -\frac{1}{2}\left[(\ln p)^{2^i}-(\ln q)^{2^i}\right]^2-\frac{1}{2}[\ln z-\ln q]^2.\end{aligned}$$

For $i=1$,

$$\begin{aligned}\mathscr{L}_1(p,q)+\mathscr{L}_1(q,z)-\mathscr{L}_1(p,z) &\geq -\frac{1}{2}[(\ln p-\ln q)(\ln p+\ln q)]^2-\frac{1}{2}|\ln z-\ln q|^2\\ &= -c_1d^2(p,q)-c_2d^2(z,q)\end{aligned}$$

where $c_{1,1}=\frac{(\ln p+\ln q)^2}{2}$ and $c_{2,1}=\frac{1}{2}$. Hence $\mathscr{L}_1$ is Lipschitz-type continuous with constant coefficients $c_{1,1}$ and $c_{2,2}$.

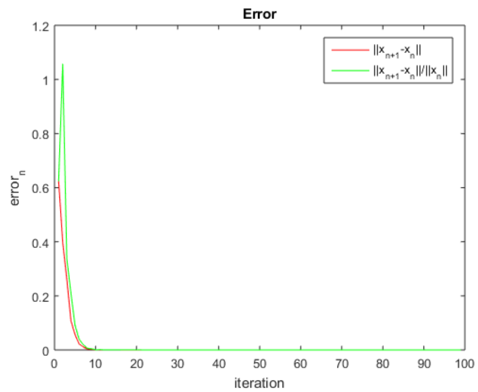

FIGURE 1. The convergence behaviour of the sequences generated by (5.1)

For $i=2$,

$$\begin{aligned}\mathscr{L}_2(p,q)+\mathscr{L}_2(q,z)-\mathscr{L}_2(p,z) &\geq -\frac{1}{2}\left[(\ln p)^4-(\ln q)^4\right]^2-\frac{1}{2}\left[\ln p-\ln q\right]^2\\ &= -c_{1,2}|\ln p-\ln q|^2-c_{2,2}|\ln z-\ln q|^2.\end{aligned}$$

where $c_{1,2}=\frac{1}{2}\left\{[\ln p-\ln q]\left[(\ln p)^2+(\ln q)^2\right]\right\}^2$ and $c_{2,2}=\frac{1}{2}$. Hence $\mathscr{L}_2$ is Lipschitz-type continuous.

For $i=3$,

$$\begin{aligned}\mathscr{L}_3(p,q)+\mathscr{L}_3(q,z)-\mathscr{L}_3(z,p) &\geq -\frac{1}{2}(\ln p-\ln q)^2\left[(\ln p+\ln q)\left((\ln p)^2+(\ln q)^2\right)\right.\\ &\qquad \times\left.\left((\ln p)^4+(\ln q)^4\right)\right]^2-\frac{1}{2}(\ln z-\ln q)^2\\ &= -c_{1,3}d^2(p,q)-c_{2,3}d^2(z,q)\end{aligned}$$

where $c_{1,3}=\frac{1}{2}[(\ln p+\ln q)((\ln p)^2+(\ln q)^2)((\ln p)^2+(\ln q)^4)]^2$ and $c_{2,3}=\frac{1}{2}$. Hence $\mathscr{L}_2$ is Lipschitz - type continuous.

Define $\mathscr{T}:\mathbb{R}^+\to\mathbb{R}$ via $\mathscr{T}p=\sqrt{p}$. Then, in the sense of d, $\mathscr{T}$ is a $\frac{1}{2}$-generalized demimetric mapping with $F(\mathscr{T})=\{1\}$. Let $\sigma=\frac{1}{5}$, $\tau=\frac{1}{4}$ and $\alpha_k=\frac{1}{12k}$, $\beta_k=\frac{2k-1}{4k}$ $\gamma_k=\frac{3k+1}{6k}$, $\lambda_k=\frac{1}{2}$ for $k\geq 1$ and $u=1.5$. Then all the conditions are satisfied and $\Omega=F(\mathscr{T})\cap\bigcap_{i=1}^3 F(\mathscr{L}_i)=\{1\}$. Computing (3.1), we obtain, for $y\in[1,2]$,

$$\begin{cases} w_{ki}=\arg\min\left\{(\ln p_k)^{2^i}(\ln p_k-\ln y)+|\ln p_k-\ln y|^2,\ \ i=1,2,3\right\},\\ z_{ki}=\arg\min\left\{(\ln w_{ki})^{2^i}(\ln w_{ki}-\ln y)+|\ln p_k-\ln y|^2,\ \ i=1,2,3\right\},\\ \bar{z}_k=\arg\min\{|\ln z_{ki}-\ln p_k|:\ \ i=1,2,3\},\\ p_{k+1}=\frac{1}{12k}u+\frac{2k-1}{4k}p_k+\frac{3k+1}{6k}\left(\frac{19\bar{z}_k+\sqrt{\bar{z}_k}}{20}\right)\end{cases}\tag{5.1}$$

6. CONCLUSION

We studied a modified version of the Halpern-type extragradient method to find a common solution for a set of equilibrium problems involving pseudomonotone bifunctions and fixed point problems of a generalized demimetric mapping in a Hadamard space. We also derived new convergence theorems and demonstrated our result with a numerical example by using a pseudomonotone bifunction that is not monotone in Hadamard spaces. Our work extends and improves many results obtained in this field.

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