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ALMOST ψ-CONTRACTIONS UNDER BINARY RELATIONS WITH APPLICATIONS TO NONLINEAR ELASTIC BEAM EQUATIONS

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Abstract. In this paper, we establish a fixed point theorem for strict almost ψ-contractions under binary relations in Kaleva-Seikkala's type fuzzy b-metric spaces. By introducing two novel lemmas, we provide new proofs that differ from the original ones in metric spaces. The results obtained herein not only strengthen but also extend the scope of pre-existing fixed point theorems in the relevant literature. Additionally, As a practical application, the existence and uniqueness of solutions to nonlinear elastic beam equations are effectively demonstrated thereby bridging the theoretical and applied aspects of this study.

Keywords. Almost ψ-contraction; Binary relation; Fixed point; Kaleva-Seikkala's type fuzzy b-metric spaces.
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1. INTRODUCTION

In 1984, Kaleva and Seikkala [14] initiated the fuzzy metric spaces (in the sense of Kaleva and Seikkala) (briefly, $\mathscr{KS}$-FMS) by defining the distance between two points as a nonnegative fuzzy real number and utilizing a novel triangle inequality that is similar to the conventional triangle inequality. Subsequent research expanded upon these findings, with key outcomes for mappings in fuzzy metric spaces—including variational principles, coincidence theorems, and diverse fixed point theorems—documented in works such as [13, 17, 19, 21]. In 2022, Li et al. [15] introduced the concept of the Kaleva-Seikkala's type fuzzy b-metric space (briefly, $\mathscr{KS}$-FbMS), which generalizes the notions of $\mathscr{KS}$-FMS and b-metric spaces (briefly, b-MS). Recently, we obtained a new fixed point theorem for generalized Ćirić-type contraction in $\mathscr{KS}$-FbMS [20].

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The Banach contraction principle is a widely used method for addressing uniqueness and existence issues in various areas of mathematical analysis. The contractions of Boyd-Wong type [11] generalized the Banach contraction principle to the case of nonlinear contractions. The concept of almost contraction is another noted variant of contraction mappings, initially introduced by Berinde [9] and further expanded upon by researchers, such as those in [4, 7, 8, 10]. An almost contraction is still a weakly Picard operator, which means it may not have a unique fixed point; however, the sequence generated by the Picard iteration converges to a fixed point of the underlying mapping. To establish a uniqueness theorem, Babu et al. [7] introduced a strict almost contraction. It is widely recognized that a strict almost contraction is an almost contraction, but not conversely [7].

Recently, there is a compelling area of research in metric fixed point theory to establish fixed point results within relational metric spaces. The structure of relational metric spaces was initiated by Alam and Imdad [1]. As of now, usual contractions remain stronger than relational contractions. The fixed point results in relational metric spaces require that the contraction must be satisfied for just comparative elements (with respect to the underlying binary relation). After the appearance of relational metric spaces, various fundamental fixed point results were established in the context of different relational metric spaces by modifying the underlying contraction conditions; see, e.g., [2, 3, 6].

In 2023, Alharbi and Khan [5] introduced a new generalization of Boyd-Wong type contractions, known as strict almost ϕ-contractions under binary relations, and derived the corresponding fixed point theorem. In light of the above, converting fixed point theorems from metric spaces to $\mathscr{K}\mathscr{S}$-FbMS is quite a challenging endeavor. In this paper, we extend the above results to $\mathscr{K}\mathscr{S}$-FbMS and present results on the existence and uniqueness of fixed points for strict almost ψ-contractions under binary relations in $\mathscr{K}\mathscr{S}$-FbMS. As an application, we demonstrate a unique solution to the nonlinear elastic beam equation.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the sets of positive integers and nonnegative real numbers, respectively.

We now review some definitions related to $\mathscr{K}\mathscr{S}$-FbMS as follows.

Definition 2.1 ([12], [13])**.** Let $\eta : \mathbb{R} \to [0,1]$ be a mapping, whose α-level set is denoted by $[\eta]_\alpha = \{\delta \in \mathbb{R} : \eta(\delta) \geq \alpha\}$. η is called a fuzzy real number or fuzzy interval if the following two conditions are satisfied:

(1) There exists $\delta_0 \in \mathbb{R}$ such that $\eta(\delta_0) = 1$.

(2) $[\eta]_\alpha = [\lambda_\alpha, \rho_\alpha]$ is a closed interval of $\mathbb{R}$ for each $\alpha \in (0,1]$, where $-\infty < \lambda_\alpha \leq \rho_\alpha < +\infty$.

The set of all such fuzzy real numbers is denote by $\mathscr{F}$. If $\eta \in \mathscr{F}$ and $\eta(\delta) = 0$ whenever $\delta < 0$, then η is called a non-negative fuzzy real number, and $\mathscr{F}^+$ denotes the set of all non-negative fuzzy real numbers.

The notation $\overline{0}$ stands for the fuzzy number satisfying $\overline{0}(0) = 1$ and $\overline{0}(\delta) = 0$ if $\delta \neq 0$. Clearly, $\overline{0} \in \mathscr{F}^+$. $\mathbb{R}$ can be embedded in $\mathscr{F}$: if $a \in \mathbb{R}$, then $\overline{a} \in \mathscr{F}$ satisfies $\overline{a}(\delta) = \overline{0}(\delta - a)$.

Definition 2.2 ([15])**.** Suppose that $\mathscr{M}$ is a non-empty set, $b \geq 1$ and that $\mathfrak{D}$ is a mapping from $\mathscr{M} \times \mathscr{M}$ into $\mathscr{F}^+$. Let $\mathscr{L}, \mathfrak{R} : [0,1] \times [0,1] \to [0,1]$ be two symmetric and nondecreasing functions such that $\mathscr{L}(0,0) = 0$ and $\mathfrak{R}(1,1) = 1$. For $\alpha \in (0,1]$ and $x, y \in \mathscr{M}$, define the

mapping

$$[\mathfrak{D}(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)].$$

Then, $\mathfrak{D}$ is called a fuzzy b-metric, and the quintuple $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is called a fuzzy b-metric space (briefly, $\mathscr{K}\mathscr{S}$-FbMS) with the coefficient b if

(BM1) $\mathfrak{D}(x,y) = \bar{0}$ if and only if $x = y$;

(BM2) $\mathfrak{D}(x,y) = \mathfrak{D}(y,x)$ for all $x, y \in \mathscr{M}$;

(BM3) for all $x, y, z \in \mathscr{M}$:

(BM3$\mathscr{L}$) $\mathfrak{D}(x,y)(b(\theta+\delta)) \geq \mathscr{L}(\mathfrak{D}(x,z)(\theta), \mathfrak{D}(z,y)(\delta))$, whenever $\theta \leq \lambda_1(x,z)$, $\delta \leq \lambda_1(z,y)$ and $(b(\theta+\delta)) \leq \lambda_1(x,y)$;

(BM3$\mathfrak{R}$) $\mathfrak{D}(x,y)(b(\theta+\delta)) \leq \mathfrak{R}(\mathfrak{D}(x,z)(\theta), \mathfrak{D}(z,y)(\delta))$, whenever $\theta \geq \lambda_1(x,z)$, $\delta \geq \lambda_1(z,y)$ and $(b(\theta+\delta)) \geq \lambda_1(x,y)$.

Next, we give a simple example of $\mathscr{K}\mathscr{S}$-FbMS.

Example 2.3. Let $(\mathscr{M}, d, b)$ be a b-MS with $b \geq 1$, and let $\mathfrak{D}(x,y) : \mathbb{R} \to \mathbb{R}$ be a mapping. If $x = y \in \mathscr{M}$, we define $\mathfrak{D}(x,y)(\xi) = \bar{0}(\xi)$ for any $\xi \in \mathbb{R}$. If, $x, y \in \mathscr{M}$ with $x \neq y, \mathfrak{D}(x,y)$ is defined by

$$\mathfrak{D}(x,y)(\xi) = \begin{cases} \dfrac{1}{1+\frac{\xi}{d(x,y)}}, & \xi \geq 0, \\ 0, & \xi < 0, \end{cases}$$

$\mathscr{L}(\alpha,\beta) = \min\{\alpha,\beta\}$ and $\mathfrak{R}(\alpha,\beta) = \max\{\alpha,\beta\}$, then $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is a $\mathscr{K}\mathscr{S}$-FbMS with ($\mathfrak{R}2$) and the coefficient $b \geq 1$. (For more details see [15, 20]).

This example effectively illustrates the concept of a $\mathscr{K}\mathscr{S}$-FbMS. It starts with a b-MS and constructs a fuzzy b-metric $\mathfrak{D}$. The specific form of $\mathfrak{D}$ for equal and unequal points in $\mathscr{M}$ is clearly defined. It shows how a traditional metric space can be transformed into a fuzzy b-metric space, providing a practical and tangible way to understand the abstract definition. This process aids in visualizing and applying the theoretical framework of $\mathscr{K}\mathscr{S}$-FbMS.

Proposition 2.4 ([15]). *Let $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ be a $\mathscr{K}\mathscr{S}$-FbMS. Then ($\mathfrak{R}2$) $\Rightarrow$ for each $\mathfrak{t} \in (0,1]$, there exists $\mathfrak{s} = \mathfrak{s}(\mathfrak{t}) \in (0,\mathfrak{t}]$ such that $\rho_t(x,y) \leq b[\rho_\mathfrak{s}(x,z) + \rho_\mathfrak{t}(z,y)]$ for all $x, y, z \in \mathscr{M}$.*

The following definition and lemma were introduced by Li et al. [15].

Definition 2.5 ([15]). Let $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ be a $\mathscr{K}\mathscr{S}$-FbMS.

(1) A sequence $\{x_n\}$ in $\mathscr{M}$ is said to converge to $x \in \mathscr{M}$ (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} \mathfrak{D}(x_n, x) = \bar{0}$, i.e., $\lim_{n\to\infty} \rho_\mathfrak{t}(x_n, x) = 0$ for each $\mathfrak{t} \in (0,1]$;

(2) A sequence $\{x_n\}$ in $\mathscr{M}$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \mathfrak{D}(x_n, x_m) = \bar{0}$, equivalently, for any given $\varepsilon > 0$ and $\mathfrak{t} \in (0,1]$, there exists $N = N(\varepsilon, \mathfrak{t}) \in \mathbb{Z}^+$ such that $\rho_\mathfrak{t}(x_n, x_m) < \varepsilon$, whenever $n, m \geq N$;

(3) $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is called complete if every Cauchy sequence in $\mathscr{M}$ converges.

This definition offers a lucid and comprehensive framework for understanding convergence, Cauchy sequences, and completeness within the context of $\mathscr{K}\mathscr{S}$-FbMS. By stipulating that convergence occurs when $\lim_{n\to\infty} \mathfrak{D}(x_n, x) = \bar{0}$, it presents a distinctively fuzzy metric-oriented perspective. The formulation of the Cauchy sequence criterion, which is intricately tied to the

$\rho_{\mathfrak{t}}$ values, is well-organized. The concept of completeness, which rests upon the convergence of all Cauchy sequences, is thereby precisely defined. Overall, this framework empowers a meticulous examination of the topological characteristics within this specific type of fuzzy b-metric space. Under such a definition, the limit of the sequence in $\mathscr{K}\mathscr{S}$-FbMS is unique, further enhancing the analytical rigor and predictability of the space's behavior.

Under $(\mathscr{R}2)$, the limit of the sequence in $\mathscr{K}\mathscr{S}$-FbMS is unique.

Lemma 2.6 ([15]). *Let $(\mathscr{M},\mathfrak{D},\mathscr{L},\mathfrak{R},b)$ be a $\mathscr{K}\mathscr{S}$-FbMS satisfying condition $(\mathfrak{R}2)$, let and $\{x_n\}\subseteq\mathscr{M}$ be a sequence. If there exist $x,y\in\mathscr{M}$ such that $\lim_{n\to\infty}\rho_{\mathfrak{t}}(x_n,x)=\lim_{n\to\infty}\rho_{\mathfrak{t}}(x_n,y)=0$, then $x=y$.*

This lemma is a crucial result in the context of $\mathscr{K}\mathscr{S}$-FbMS. It states that if a sequence has the same limit behavior with respect to two different points in terms of the $\rho_{\mathfrak{t}}$ values approaching zero, then those two points must be equal. This is helpful for establishing the uniqueness of limits in the fuzzy b-metric space. It provides an important tool for proving other properties related to convergence and the structure of sequences within the $\mathscr{K}\mathscr{S}$-FbMS, strengthening the theoretical foundation and allowing for more precise analysis of the space's characteristics.

Next, we firstly recall that a subset of $\mathscr{M}\times\mathscr{M}$ is referred to as a binary relation (or, simply a relation) on $\mathscr{M}$, as follows:

Definition 2.7 ([1]). A pair $x,y\in\mathscr{M}$ is $\mathscr{R}$-comparative if $(x,y)\in\mathscr{R}$ or $(y,x)\in\mathscr{R}$. We denote such a pair by $[x,y]\in\mathscr{R}$.

Definition 2.8 ([16]). The relation $\mathscr{R}^{-1}:=\{(x,y)\in\mathscr{M}\times\mathscr{M}:(y,x)\in\mathscr{R}\}$ is called an inverse of $\mathscr{R}$. Also, $\mathscr{R}^s:=\mathscr{R}\cup\mathscr{R}^{-1}$defines a symmetric relation on $\mathscr{M}$, often called symmetric closure of $\mathscr{R}$.

Remark 2.9 ([1]). $[x,y]\in\mathscr{R}\iff(x,y)\in\mathscr{R}^s$.

Definition 2.10 ([5]). For a subset $\mathscr{A}\subseteq\mathscr{M}$, the set defined by $\mathscr{R}\big|_{\mathscr{A}}:=\mathscr{R}\cap(\mathscr{A}\times\mathscr{A})$ is referred to as the restriction of $\mathscr{R}$ to $\mathscr{A}$, and it constitutes a relation on $\mathscr{A}$.

Definition 2.11 ([1]). $\mathscr{R}$ is T-closed if it satisfies $(Tx,Ty)\in\mathscr{R}$, $\forall x,y\in\mathscr{M}$ verifying $(x,y)\in\mathscr{R}$.

Definition 2.12 ([1]). A sequence $x_n\subset\mathscr{M}$ satisfying $(x_n,x_{n+1})\in\mathscr{M}$, for all $n\in\mathbb{N}$, is termed as $\mathscr{R}$-preserving.

Definition 2.13. The $(\mathscr{M},\mathfrak{D},\mathscr{L},\mathfrak{R},b)$ is called $\mathscr{R}$-complete if each $\mathscr{R}$-preserving Cauchy sequence in $\mathscr{M}$ converges.

Definition 2.14 ([18]). A subset $\mathscr{A}\subseteq\mathscr{M}$ is $\mathscr{R}$-directed if, for every pair $x,y\in\mathscr{R}$, there exists $z\in\mathscr{M}$ satisfying $(x,z)\in\mathscr{R}$ and $(z,y)\in\mathscr{R}$.

In [5], the definition of $\mathscr{R}$-continuous in metric spaces is given. We next present a similar definition in the context of $\mathscr{K}\mathscr{S}$-FbMS.

Definition 2.15. The mapping T is referred to as $\mathscr{R}$-continuous at $x\in\mathscr{M}$ if, for every $\mathscr{R}$-preserving sequence $x_n\subseteq\mathscr{M}$ with $\rho_{\mathfrak{t}}(x_n,x)\to0$ *as* $n\to\infty$,

$$\rho_{\mathfrak{t}}(Tx_n,Tx)\to0 \text{ as } n\to\infty,$$

for any $\mathfrak{t}\in(0,1]$. A mapping, which remains $\mathscr{R}$-continuous function at every point, is called $\mathscr{R}$-continuous.

In [5], the definition of d-self-closed in metric spaces is given. We will present a similar definition in the context of $\mathscr{K}\mathscr{S}$-FbMS.

Definition 2.16. $\mathscr{R}$ is termed as $\mathfrak{D}$-self-closed if every $\mathscr{R}$-preserving $\{x_n\} \subseteq \mathscr{M}$ verifying $\rho_t(x_n, x) \to 0$ ($x \in \mathscr{M}$) has a subsequence $\{x_{n_k}\}$ satisfying $[x_{n_k}, x] \in \mathscr{R}$.

In 2004, Berinde [9] proposed a new generalization of the Banach contraction principle, often called "almost contraction".

Definition 2.17. ([9]) A self-function T on a metric space $(\mathscr{M}, d)$ is referred to as an almost contraction if there exist $L \in [0, \infty)$ and $k \in (0, 1)$ such that, for all $x, y \in \mathscr{M}$, $d(Tx, Ty) \leq kd(x, y) + Ld(x, Tx)$.

In 2008, Babu et al. [7] defined slightly stronger class of almost contraction condition, called "strict almost contraction".

Definition 2.18. ([7]) A self-function T on a metric space $(\mathscr{M}, d)$ is referred to as strict almost contraction if there exist $k \in (0, 1)$ and $L \in [0, \infty)$ such that

$$d(Tx, Ty) \leq kd(x, y) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in \mathscr{M}$.

Theorem 2.19. *([7]) An strict almost contraction self-function on a complete metric space admits a unique fixed point.*

In 2023, Alharbi and Khan [5] introduced a new type of strict almost ϕ-contractions under binary relations in metric spaces, and derived the corresponding fixed point theorem.

$$\Phi = \left\{ \phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi^{-1}(\{0\}) = \{0\}, \phi(r) < r \text{ and } \limsup_{q \to r} \phi(q) < r, \text{ for all } r > 0 \right\}.$$

Theorem 2.20. *([5, Theorem 4]). Suppose that $(\mathscr{M}, d)$ is a metric space, and $\mathscr{R}$ is a relation on $\mathscr{M}$ while $T : \mathscr{M} \to \mathscr{M}$ remains a mapping. Also, suppose that the following assumptions hold*

(i) $(\mathscr{M}, d)$ *is* $\mathscr{R}$*-complete,*
(ii) there exists $x_0 \in \mathscr{M}$ *satisfying* $(x_0, Tx_0) \in \mathscr{R}$*,*
(iii) $\mathscr{R}$ *is locally* T*-transitive and* T*-closed,*
(iv) T *is* $\mathscr{R}$*-continuous, or* $\mathscr{R}$ *is* d*-self-closed,*
(v) there exists $\phi \in \Phi$ *and* $L \geq 0$ *verifying*

$$d(Tx, Ty)) \leq \phi(d(x, y)) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in \mathscr{M}$ *with* $(x, y) \in \mathscr{R}$*.*
(vi) $T(\mathscr{M})$ *is* $\mathscr{R}$*-directed.*

Then, if conditions $(i) - (v)$ *are satisfied, then* T *admits a fixed point. Additionally, if condition* (vi) *is also satisfied, then* T *admits a unique fixed point.*

In light of the above, in this paper, we extend the above conclusion to $\mathscr{K}\mathscr{S}$-FbMS and present results on the existence and uniqueness of fixed point for strict almost ψ-contraction under binary relations in $\mathscr{K}\mathscr{S}$-FbMS.

$$\Psi=\left\{\psi:\mathbb{R}^+\to\mathbb{R}^+:\psi^{-1}(\{0\})=\{0\},\psi(r)>r \text{ and } \liminf_{q\to r}\psi(q)>r,\ \text{for all } r>0\right\} \quad (2.1)$$

Similarly, we present the definition of a strict almost ψ-contraction in $\mathscr{K}\mathscr{S}$-FbMS.

Definition 2.21. A self-function T on $\mathscr{K}\mathscr{S}$-FbMS is referred to as strict almost ψ-contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)+L\min\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\},$$

for all $\mathfrak{t}\in(0,1]$ and for all $x,y\in\mathscr{M}$ with $(x,y)\in\mathscr{R}$.

Since the symmetry of $\mathfrak{D}$ indicates that $\rho_{\mathfrak{t}}$ is symmetric, we can conclude the following.

Proposition 2.22. *Given any $\psi\in\Psi$ and $L\geq 0$, the following conditions are equivalent:*

(1) $\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)+L\min\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\}$, *for all $\mathfrak{t}\in(0,1]$, for all $x,y\in\mathscr{M}$ with $(x,y)\in\mathscr{R}$.*

(2) $\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)+L\min\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\}$, *for all $\mathfrak{t}\in(0,1]$, for all $x,y\in\mathscr{M}$ with $[x,y]\in\mathscr{R}$.*

Proof. The conclusion (2) $\Rightarrow$ (1) holds trivially. Conversely, let (1) holds. Assume that $x,y\in\mathscr{M}$ with $[x,y]\in\mathscr{R}$. Then, in case $(x,y)\in\mathscr{R}$, (1) $\Rightarrow$ (2). Otherwise, in case $(y,x)\in\mathscr{R}$, by symmetry of $\rho_{\mathfrak{t}}$ and (1), one obtains

$$\begin{aligned}\psi(\rho_{\mathfrak{t}}(Tx,Ty))&=\psi(\rho_{\mathfrak{t}}(Ty,Tx))\\&\leq\rho_{\mathfrak{t}}(y,x)+L\min\{\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Tx),\rho_{\mathfrak{t}}(x,Ty)\},\\&=\rho_{\mathfrak{t}}(x,y)+L\min\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\}.\end{aligned}$$

It follows that (1) $\Rightarrow$ (2). □

3. Main Results

The following two lemmas discuss the relevant properties of Ψ in (2.1). They are new and useful and are needed to prove our theorems.

Lemma 3.1. *Let $[\alpha,\beta]$ be a closed interval with $\alpha>0$, If $\psi\in\Psi$, then $\inf_{\alpha\leq r\leq\beta}[\psi(r)-r]>0$.*

Proof. If not, then $\inf_{\alpha\leq r\leq\beta}[\psi(r)-r]=0$. It follows that there exists a sequence $\{r_k\}\subset[\alpha,\beta]$ such that $\psi(r_k)-r_k\to 0$ as $k\to\infty$. Since $\{r_k\}$ is bounded, there exists a subsequence $\{r_{k_i}\}$ of $\{r_k\}$ satisfying $r_{k_i}\to r_0$ as $i\to\infty$, $r_0\in[\alpha,\beta]$. In view of $\psi(r_{k_i})=[\psi(r_{k_i})-r_{k_i}]+r_{k_i}\to r_0$ as $i\to\infty$, we have that $r_0<\liminf_{r\to r_0}\psi(r)\leq\lim_{i\to\infty}\psi(r_{k_i})=r_0$, which is a contradiction. □

Lemma 3.2. *Let $(\mathscr{M},\mathfrak{D},\mathscr{L},\Re,b)$ be a $\mathscr{K}\mathscr{S}$-FbMS with ($\Re$2) and the coefficient $b\geq 1$. Let $[\alpha,\beta]$ be a closed interval with $0<\alpha<\beta<+\infty$. Suppose that $T:\mathscr{M}\to\mathscr{M}$ is a mapping, and let $\{x_n\}_{n=0}^{\infty}\subseteq\mathscr{M}$ be a sequence by $x_n=Tx_{n-1}=T^nx_0$, for all $n\in\mathbb{N}$, (where it is understood that $T^0x_0=x_0$). If there exists $\psi\in\Psi$ and $L\geq 0$ with*

$$\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)+L\min\left\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\right\}, \quad (3.1)$$

for all $\mathfrak{t}\in(0,1]$ and $x,y\in\mathscr{M}$, then following conclusions hold:

(i) $\lim_{n\to\infty}\rho_{\mathfrak{t}}(x_n,x_{n-1})=0$, $\forall \mathfrak{t}\in(0,1]$,

(ii) *then for all* $\mathfrak{t}\in(0,1]$, *there exist positive integers* N *and* K, *for all* $x\in\mathscr{M}$ *satisfying* $\rho_{\mathfrak{t}}(x,T^nx_0)<\beta$, *we have* $\rho_{\mathfrak{t}}(T^kx,T^{k+n}x_0)<\alpha$ $(k>K,n>N)$.

Proof. (i) By $\psi\in\Psi$, we have $\psi^{-1}(\{0\})=\{0\}$ and $\psi(r)>r$, $\forall r>0$, Moreover, $\psi(r)\geq r$, $\forall r\in\mathbb{R}^+$. Suppose that $\mathfrak{t}\in(0,1]$ and let $u_n(\mathfrak{t})=\rho_{\mathfrak{t}}(x_n,x_{n-1})$. By (3.1), we have

$$\begin{aligned}u_{n+1}(\mathfrak{t})&\leq\psi(u_{n+1}(\mathfrak{t}))\\&=\psi(\rho_{\mathfrak{t}}(Tx_n,Tx_{n-1}))\\&\leq\rho_{\mathfrak{t}}(x_n,x_{n-1})+L\min\{\rho_{\mathfrak{t}}(x_n,x_{n+1}),\rho_{\mathfrak{t}}(x_{n-1},x_n),0,\rho_{\mathfrak{t}}(x_{n-1},x_{n+1})\}\\&=u_n(\mathfrak{t}).\end{aligned}$$

Moreover, $\{u_n(\mathfrak{t})\}$ is a nonnegative non-increasing sequence; and hence $\lim_{n\to\infty}u_n(\mathfrak{t})=u(\mathfrak{t})\geq 0$. In fact, if $u(\mathfrak{t})>0$, then

$$u(\mathfrak{t})<\liminf_{q\to u(\mathfrak{t})}\psi(q)\leq\liminf_{n\to\infty}\psi(u_{n+1}(\mathfrak{t}))\leq\lim_{n\to\infty}u_n(\mathfrak{t})=u(\mathfrak{t}),$$

which is a contradiction. Hence, $\lim_{n\to\infty}u_n(\mathfrak{t})=0$ for all $\mathfrak{t}\in(0,1]$.

(ii) By using Lemma 3.1, we denote

$$\sigma_1=\inf_{\frac{\alpha}{2}\leq r\leq\beta}[\psi(r)-r]>0$$

and

$$\sigma_\varepsilon=\inf_{\varepsilon\leq r\leq\beta}[\psi(r)-r]>0,(0<\varepsilon<\frac{\alpha}{2}).$$

It is obvious that $\sigma_\varepsilon\leq\sigma_1$. From (i), it can be seen that $\lim_{n\to\infty}\rho_{\mathfrak{t}}(T^nx_0,T^{n+1}x_0)=0$ $(\mathfrak{t}\in(0,1])$. Furthermore, there exists a positive integer N such that

$$\rho_{\mathfrak{t}}(T^nx_0,T^{n+1}x_0)<\min\left\{\frac{\sigma_1}{2L},\frac{\alpha}{2L}\right\},$$

for all $n\geq N$. We take $K=\left[\frac{2\beta-\alpha}{\sigma_1}\right]+1$ and denote $u_k=\rho_{\mathfrak{t}}(x,T^nx_k)$, for all $x\in\mathscr{M}$ satisfying $u_0=\rho_{\mathfrak{t}}(x,T^nx_0)<\beta$. From (3.1), one sees that

$$\frac{\sigma_1}{2}+\rho_{\mathfrak{t}}(x,T^nx_0)-\rho_{\mathfrak{t}}(Tx,TT^nx_0)\geq\psi(\rho_{\mathfrak{t}}(Tx,TT^nx_0))-\rho_{\mathfrak{t}}(Tx,TT^nx_0)\geq\sigma_1.$$

It follows that

$$u_1=\rho_{\mathfrak{t}}(Tx,T^{n+1}x_0)\leq u_0-\sigma_1+\frac{\sigma_1}{2}<\beta-\frac{\sigma_1}{2}$$

and

$$u_2=\rho_{\mathfrak{t}}(T^2x,T^{n+2}x_0)\leq u_1-\sigma_1+\frac{\sigma_1}{2}<\beta-2\cdot\frac{\sigma_1}{2}.$$

Continue this process for $u_K=\rho_{\mathfrak{t}}(T^Kx,T^{n+K}x_0)<\beta-K\cdot\frac{\sigma_1}{2}<\frac{\alpha}{2}<\alpha$. Without loss of generality, we assume u_K is the first term that is less than $\frac{\alpha}{2}$, and there exists $\varepsilon_0<u_K$ such that

$$u_{K+1}=\rho_{\mathfrak{t}}(T^{K+1}x,T^{n+K+1}x_0)\leq u_K-\sigma_{\varepsilon_0}+\min\left\{\frac{\sigma_1}{2},\frac{\alpha}{2}\right\}<\frac{\alpha}{2}-\sigma_{\varepsilon_0}+\frac{\alpha}{2}<\alpha.$$

If $u_{K+1} < \frac{\alpha}{2}$, there exists $\varepsilon_1 < u_{K+1}$ such that

$$u_{K+2} \leq u_{K+1} - \sigma_{\varepsilon_1} + \min\left\{\frac{\sigma_1}{2}, \frac{\alpha}{2}\right\} < \frac{\alpha}{2} - \sigma_{\varepsilon_1} + \frac{\alpha}{2} < \alpha.$$

If $\frac{\alpha}{2} < u_{K+1} < \alpha$, then

$$u_{K+2} \leq u_{K+1} - \sigma_1 + \min\left\{\frac{\sigma_1}{2}, \frac{\alpha}{2}\right\} < \alpha - \sigma_1 + \frac{\sigma_1}{2} < \alpha.$$

Therefore, when $k \geq K$, it holds that $u_k = \rho_{\mathfrak{t}}(T^k x, T^{n+k} x_0) < \alpha$. □

We initially characterized the degree of nonlinear compression by examining locally interval-compressed decreasing quantities. Our lemma establishes a lower bound for the degree of ψ-compression, which guarantees that the sequence produced by the compressed mapping is a Cauchy sequence. This sets the stage for the introduction of the following technical lemma, which plays a crucial role in the proof of the fixed point theorem.

Now, we present the following result on the existence and uniqueness of fixed point for strict almost ψ-contraction under binary relations in $\mathscr{K}\mathscr{S}$-FbMS.

Theorem 3.3. *Let $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ be a $\mathscr{K}\mathscr{S}$-FbMS with ($\mathfrak{R}2$) and the coefficient $b \geq 1$. Let $\mathscr{R}$ be a relation on $\mathscr{M}$ and $T : \mathscr{M} \to \mathscr{M}$ be a mapping. Also, suppose the following assumptions*

- (i) *$(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is $\mathscr{R}$-complete,*
- (ii) *there exists $x_0 \in \mathscr{M}$ satisfying $(x_0, Tx_0) \in \mathscr{R}$,*
- (iii) *$\mathscr{R}$ is T-closed,*
- (iv) *T is $\mathscr{R}$-continuous, or $\mathscr{R}$ is $\mathfrak{D}$-self-closed,*
- (v) *there exists $\psi \in \Psi$ and $L \geq 0$ verifying*

$$\psi(\rho_{\mathfrak{t}}(Tx, Ty)) \leq \rho_{\mathfrak{t}}(x, y) + L\min\{\rho_{\mathfrak{t}}(x, Tx), \rho_{\mathfrak{t}}(y, Ty), \rho_{\mathfrak{t}}(x, Ty), \rho_{\mathfrak{t}}(y, Tx)\}, \tag{3.2}$$

  for all $\mathfrak{t} \in (0, 1]$, for all $x, y \in \mathscr{M}$ with $(x, y) \in \mathscr{R}$,

- (vi) *$T(\mathscr{M})$ is $\mathscr{R}$-directed.*

If conditions $(i) - (v)$ are satisfied, T admits a fixed point. Additionally, if condition (vi) is also satisfied, then T admits a unique fixed point.

Proof. By (ii), we take $x_0 \in \mathscr{M}$ satisfying $(x_0, Tx_0) \in \mathscr{R}$, and let $\{x_n\}_{n=0}^{\infty} \subseteq \mathscr{M}$ be a sequence by $x_n = Tx_{n-1} = T^n x_0$, for all $n \in \mathbb{N}$, (where it is understood that $T^0 x_0 = x_0$).

We next prove in several steps:

1. We claim that $\{x_n\}$ is an $\mathscr{R}$-preserving sequence.

By (ii) and $\mathscr{R}$ is T-closed, we obtain $(T^n x_0, T^{n+1} x_0) \in \mathscr{R}$. Hence, $(x_n, x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$.

2. We claim that $\{x_n\}$ is a Cauchy sequence in $\mathscr{M}$.

If not, then there exists $\varepsilon_0 > 0$, $\mathfrak{t} \in (0, 1]$ and two sequences $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k \geq k$ and

$$\rho_{\mathfrak{t}}(x_{m_k}, x_{n_k}) \geq \varepsilon_0 \text{ and } \rho_{\mathfrak{t}}(x_{m_k - 1}, x_{n_k}) < \varepsilon_0, \text{ for all } k \in \mathbb{Z}^+.$$

Without loss of generality, we can assume that $\varepsilon_0 < \frac{\varepsilon}{2b^2}$. For the interval $[\varepsilon_0, b^2\varepsilon_0 + \frac{\varepsilon}{3}]$, according to Lemma 3.2, there exists a finite positive integer i such that $\rho_{\mathfrak{t}}(T^i x, T^i y) < \varepsilon_0$ for all

$\mathfrak{t} \in (0,1]$ and $x, y \in \mathscr{M}$ with $\rho_{\mathfrak{t}}(x,y) < b^2\varepsilon_0 + \dfrac{\varepsilon}{3}$. It follows that Proposition 2.4 that there exists $\mathfrak{s} = \mathfrak{s}(\mathfrak{t}) \in (0,\mathfrak{t}]$ such that

$$\begin{aligned}\rho_{\mathfrak{t}}(x_{m_k-i}, x_{n_k-i}) &\leq b[\rho_{\mathfrak{t}}(x_{m_k-i}, x_{n_k}) + \rho_{\mathfrak{s}}(x_{n_k}, x_{n_k-i})] \\ &\leq b^2\rho_{\mathfrak{s}}(x_{m_k-i}, x_{m_k-1}) + b^2\rho_{\mathfrak{t}}(x_{m_k-1}, x_{n_k}) + b\rho_{\mathfrak{s}}(x_{n_k}, x_{n_k-i}) \\ &< b^2\rho_{\mathfrak{s}}(x_{m_k-i}, x_{m_k-1}) + b^2\varepsilon_0 + b\rho_{\mathfrak{s}}(x_{n_k}, x_{n_k-i}).\end{aligned}$$

Since $\{\rho_{\mathfrak{s}}(x_{m_k-i}, x_{m_k-1})\}$ and $\{\rho_{\mathfrak{s}}(x_{n_k}, x_{n_k-i})\}$ converge to 0 as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $\rho_{\mathfrak{t}}(x_{m_k-i}, x_{n_k-i}) < b^2\varepsilon_0 + \dfrac{\varepsilon}{3} < \varepsilon$ for all $k > K$. Hence, by Lemma 3.2, we obtain

$$\varepsilon_0 \leq \rho_{\mathfrak{t}}(x_{m_k}, x_{n_k}) = \rho_{\mathfrak{t}}(T^i x_{m_k-i}, T^i x_{n_k-i}) < \varepsilon_0,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ is an $\mathscr{R}$-preserving and Cauchy, one sees by assumption (i) that there exists $x^* \in \mathscr{M}$ such that $\rho_{\mathfrak{t}}(x_n, x^*) \to 0$, as $n \to \infty$, for any $\mathfrak{t} \in (0,1]$.

3. We claim that x^* is the fixed point of T.

Since $\{x_n\}$ is a $\mathscr{R}$-preserving sequence. If T is $\mathscr{R}$-continuous with $\rho_{\mathfrak{t}}(x_n, x^*) \to 0$ as $n \to \infty$, then $\rho_{\mathfrak{t}}(Tx_n, Tx^*) \to 0$ as $n \to \infty$, for any $\mathfrak{t} \in (0,1]$. This shows that $\rho_{\mathfrak{t}}(x_{n+1}, Tx^*) \to 0$ as $n \to \infty$. By Lemma 2.6, we have $Tx^* = x^*$. If $\mathscr{R}$ is $\mathfrak{D}$-self-closed, then $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ with $[x_{n_k}, x^*] \in \mathscr{R}$, for all $k \in \mathbb{N}$. From the hypotheses on ψ, (3.2), Proposition 2.22, and $[x_{n_k}, x^*] \in \mathscr{R}$, we obtain

$$\begin{aligned}\rho_{\mathfrak{t}}(x_{n_{k+1}}, Tx^*) &\leq \psi(\rho_{\mathfrak{t}}(Tx_{n_k}, Tx^*)) \\ &\leq \rho_{\mathfrak{t}}(x_{n_k}, x^*) + \omega\left(\rho_{\mathfrak{t}}(x_{n_k}, x_{n_{k+1}}), \rho_{\mathfrak{t}}(x^*, Tx^*), \rho_{\mathfrak{t}}(x_{n_k}, Tx^*), \rho_{\mathfrak{t}}(x^*, x_{n_{k+1}})\right).\end{aligned}$$

Computing the limit of the above and using $\rho_{\mathfrak{t}}(x_{n_k}, x^*) \to 0$, we derive $\rho_{\mathfrak{t}}(x_{n_{k+1}}, Tx^*) \to 0$. From Lemma 2.6, we find $Tx^* = x^*$.

4. We claim that T has a unique fixed point $x^* \in \mathscr{M}$.

If $y^* \in \mathscr{M}$ with $Ty^* = y^*$ and $x^* \neq y^*$, then there exists $\mathfrak{t} \in (0,1]$ such that $\rho_{\mathfrak{t}}(x^*, y^*) > 0$. As $x^*, y^* \in T(\mathscr{M})$, by (vi), there exists $z \in \mathscr{M}$ satisfying $(x^*, z) \in \mathscr{R}$ and $(y^*, z) \in \mathscr{R}$. Denote $v_n(\mathfrak{t}) = \rho_{\mathfrak{t}}(x^*, T^n z)$. Since $\mathscr{R}$ is T-closed, $\psi(r) \geq r$, and (3.2), one obtains

$$\begin{aligned}\rho_{\mathfrak{t}}(x^*, T^n z) &= \rho_{\mathfrak{t}}(T(T^{n-1}x^*), T(T^{n-1}z)) \\ &\leq \psi(\rho_{\mathfrak{t}}(Tx^*, T(T^{n-1}z))) \\ &\leq \rho_{\mathfrak{t}}(x^*, T^{n-1}z)) + \omega\left(0, \rho_{\mathfrak{t}}(T^{n-1}z, T^n z), \rho_{\mathfrak{t}}(x^*, T^n z), \rho_{\mathfrak{t}}(T^{n-1}z, Tx^*)\right) \\ &\leq \rho_{\mathfrak{t}}(x^*, T^{n-1}z).\end{aligned}$$

Thus $v_n(\mathfrak{t}) \leq v_{n-1}(\mathfrak{t})$. Moreover, $\{v_n(\mathfrak{t})\}$ is a nonnegative non-increasing sequence. Thus $\lim\limits_{n\to\infty} v_n(\mathfrak{t}) = v(\mathfrak{t}) \geq 0$.

In fact, if $v(\mathfrak{t}) > 0$, then $v(\mathfrak{t}) < \liminf\limits_{q \to v(\mathfrak{t})} \psi(q) \leq \liminf\limits_{n\to\infty} \psi(v_{n+1}(\mathfrak{t})) \leq \lim\limits_{n\to\infty} v_n(\mathfrak{t}) = v(\mathfrak{t})$, which is a contradiction. Hence, $\lim\limits_{n\to\infty} v_n(\mathfrak{t}) = \rho_{\mathfrak{t}}(x^*, T^n z) = 0$ for all $\mathfrak{t} \in (0,1]$. Similarly, one can find $\rho_{\mathfrak{t}}(y^*, T^n z) = 0$ for all $\mathfrak{t} \in (0,1]$

$$\rho_{\mathfrak{t}}(x^*, y^*) \leq b\rho_{\mathfrak{s}}(x^*, T^n z) + b\rho_{\mathfrak{t}}(y^*, T^n z) \to 0, \text{ as } n \to \infty.$$

Hence, $x^* = y^*$, i.e., the uniqueness of x^* is true. This completes the proof. □

Using our result, we can obtain some known fixed point theorems based on a review of recent research. It is known that $\mathscr{K}\mathscr{S}$-FbMS is a generalization of $\mathscr{K}\mathscr{S}$-FMS and b-MS.

Using Theorem 3.3, we can obtain the following fixed point theorem of strict almost ψ-contractions under binary relations in $\mathscr{K}\mathscr{S}$-FMS.

Corollary 3.4. *Let $(\mathscr{M},\mathfrak{D},\mathscr{L},\mathfrak{R})$be a $\mathscr{K}\mathscr{S}$-FMS with $(\mathfrak{R}2)$. Let $\mathscr{R}$ be a relation on $\mathscr{M}$ and let $T:\mathscr{M}\to\mathscr{M}$ be a mapping. Also, suppose that the following assumptions hold*

(i) *$(\mathscr{M},\mathfrak{D},\mathscr{L},\mathfrak{R})$ is $\mathscr{R}$-complete,*
(ii) *there exists $x_0\in\mathscr{M}$ satisfying $(x_0,Tx_0)\in\mathscr{R}$,*
(iii) *$\mathscr{R}$ is T-closed,*
(iv) *T is $\mathscr{R}$-continuous, or $\mathscr{R}$ is $\mathfrak{D}$-self-closed,*
(v) *there exist $\psi\in\Psi$ and $L\geq 0$ verifying*

$$\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)+L\min\{\rho_{\mathfrak{t}}(x,Tx),\rho_{\mathfrak{t}}(y,Ty),\rho_{\mathfrak{t}}(x,Ty),\rho_{\mathfrak{t}}(y,Tx)\},$$

for all $\mathfrak{t}\in(0,1]$, for all $x,y\in\mathscr{M}$ with $(x,y)\in\mathscr{R}$,
(vi) *$T(\mathscr{M})$ is $\mathscr{R}$-directed.*

Iff conditions (i)-(v) are satisfied, then T admits a fixed point. Additionally, if condition (vi) is also satisfied, then T admits a unique fixed point.

Remark 3.5. Note that the conclusion is up to date in $\mathscr{K}\mathscr{S}$-FMS.

Under the restriction $\mathscr{R}=\mathscr{M}\times\mathscr{M}$, the universal relation, Corollary 3.4 deduces the following fixed point theorem.

Corollary 3.6. *(*[21, Theorem 3.2]*). Let $(\mathscr{M},\mathfrak{D},\mathscr{L},\mathfrak{R})$ be a complete $\mathscr{K}\mathscr{S}$-FMS with $(\mathfrak{R}2)$. Suppose that $T:\mathscr{M}\to\mathscr{M}$ is a mapping satisfies $\psi(\rho_{\mathfrak{t}}(Tx,Ty))\leq\rho_{\mathfrak{t}}(x,y)$ for all$\mathfrak{t}\in(0,1]$ and $x,y\in\mathscr{M}$, where $\psi\in\Psi$. Then T has a unique fixed point $x^*\in\mathscr{M}$ such that $Tx^*=x^*$.*

Remark 3.7. This type of contractivity condition in $\mathscr{K}\mathscr{S}$-FMS is referred to as a Boyd-Wong contraction or a ψ-contraction.

Using Theorem 3.3, we can obtain the following fixed point theorem of strict almost ψ-contractions under binary relations in b-MS. Notice that the following result is up to date in b-MS.

Corollary 3.8. *Let $(\mathscr{M},d,b)$ be a b-MS with $b\geq 1$. Let $\mathscr{R}$ be a relation on $\mathscr{M}$ and let $T:\mathscr{M}\to\mathscr{M}$ be a mapping. Also, suppose that the following assumptions hold*

(i) *$(\mathscr{M},d,b)$ is $\mathscr{R}$-complete,*
(ii) *there exists $x_0\in\mathscr{M}$ satisfying $(x_0,Tx_0)\in\mathscr{R}$,*
(iii) *$\mathscr{R}$ is T-closed,*
(iv) *T is $\mathscr{R}$-continuous, or $\mathscr{R}$ is d-self-closed,*
(v) *there exist $\psi\in\Psi$ and $L\geq 0$ verifying*

$$\psi(d(Tx,Ty))\leq d(x,y)+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},$$

for all $x,y\in\mathscr{M}$ with $(x,y)\in\mathscr{R}$,
(vi) *$T(\mathscr{M})$ is $\mathscr{R}$-directed.*

If conditions $(i)-(v)$ are satisfied, then T admits a fixed point. Additionally, if condition (vi) is also satisfied, then T admits a unique fixed point.

Under the restriction $\mathscr{R} = \mathscr{M} \times \mathscr{M}$, the universal relation, Corollary 3.8 deduces the following fixed point theorem.

Corollary 3.9. *Let $(\mathscr{M}, d, b)$ be a complete b-MS with $b \geq 1$. Suppose that $T : \mathscr{M} \to \mathscr{M}$ is a mapping satisfies $\psi(d(Tx,Ty)) \leq d(x,y)$ for all $x, y \in \mathscr{M}$, where $\psi \in \Psi$. Then T has a unique fixed point $x^* \in \mathscr{M}$ such that $Tx^* = x^*$.*

Remark 3.10. Note that the conclusion is up to date in b-MS.

4. The Application to Nonlinear Elastic Beam Equation

Fourth-order two-point Boundary Value Problems are significant in engineering and science, as they model the deflection of an elastic beam in equilibrium. These problems have applications across various fields, including materials mechanics, physics, micro-electromechanical systems, chemical sensors, aircraft design and medical diagnostics.

In this paper, one considers a special case of the Boundary Value Problems associated with nonlinear elastic beam equation:

$$\begin{cases} x''''(s) = h(s, x(s)), s \in [0,1], \\ x(0) = x'(0) = x''(1) = x'''(1) = 0. \end{cases} \tag{4.1}$$

where $h : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. Equation (4.1) characterizes the bending equilibria of an elastic beam model of length 1, whose both ends are rigidly fixed. Such a beam is called cantilever beam in material mechanics.

Let $\mathscr{M} := C[0,1]$ be the set of all real continuous functions on a closed interval $[0,1]$, and define $d : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$, as follows:

$$d(x,y) = \begin{cases} \frac{1}{3} \max\limits_{s \in [0,1]} |x(s) - y(s)|, & x(s)y(s) \equiv 0, \forall s \in [0,1]; \\ \max\limits_{s \in [0,1]} |x(s) - y(s)|, & other; \end{cases}$$

where $x, y \in \mathscr{M}$.

Let $\mathfrak{D}(x,y) : \mathbb{R} \to \mathbb{R}$ be a mapping. If $x = y \in \mathscr{M}$, we define $\mathfrak{D}(x,y)(\xi) = \bar{0}(\xi)$, for any $\xi \in \mathbb{R}$. If $x, y \in \mathscr{M}$ with $x \neq y, \mathfrak{D}(x,y)$ is defined by

$$\mathfrak{D}(x,y)(\xi) = \begin{cases} 0, & \xi < 0, \\ \dfrac{1}{1 + \frac{\xi}{d(x,y)}}, & \xi \geq 0, \end{cases}$$

and $\mathscr{L}(\alpha,\beta) = \min\{\alpha,\beta\}, \mathscr{R}(\alpha,\beta) = \max\{\alpha,\beta\}$.

It is evident that $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is a complete $\mathscr{KS}$-FbMS with ($\mathfrak{R}$2) and the coefficient $b = 3$. In addition, there is the following property: $\rho_{\mathfrak{t}}(x,y) = \dfrac{1-\mathfrak{t}}{\mathfrak{t}} \cdot d(x,y)$, for all $\mathfrak{t} \in (0,1]$. (For more details, see [20]).

Let $\mathscr{R}$ be a relation on $\mathscr{M}$ as

$$(x,y) \in \mathscr{R} \iff x(s) \leq y(s), \forall x, y \in \mathscr{M}, \forall s \in [0,1].$$

It can be easily proved that $(\mathscr{M}, \mathfrak{D}, \mathscr{L}, \mathfrak{R}, b)$ is $\mathscr{R}$-complete and $\mathscr{R}$ is $\mathfrak{D}$-self-closed.

Proposition 4.1. *Consider problem (4.1) with ψ continuous and nondecreasing such that for any $x,y \in \mathscr{M}$ verifying $(x,y) \in \mathscr{R}$ and $\mathfrak{t} \in (0,1]$,*

$$\psi\left(\frac{1-\mathfrak{t}}{4\mathfrak{t}} \cdot [h(s,y(s)) - h(s,x(s))]\right) \leq \frac{1-\mathfrak{t}}{4\mathfrak{t}} \cdot [y(s) - x(s)], \tag{4.2}$$

where $\psi \in \Psi$ and $s \in [0,1]$. Assume that there exists $\alpha \in \mathscr{M}$ such that $\alpha(s) \leq \int_0^1 G(s,v)h(v,\alpha(v))dv$ for all $s \in [0,1]$, where Green function is

$$G(s,v) = \frac{1}{6}\begin{cases} v^2(3s-v), & 0 \leq v < s \leq 1, \\ s^2(3v-s), & 0 \leq s < v \leq 1. \end{cases}$$

Then there exists a unique solution for (4.1).

Proof. Problem (4.1) is equivalent to the integral equation $x(s) = \int_0^1 G(s,v)h(v,x(v))dv$. It is not difficult to obtain that

$$0 \leq G(s,v) \leq \frac{1}{2}s^2 v, \forall s,v \in [0,1].$$

Let $T : \mathscr{M} \to \mathscr{M}$ be defined by

$$(Tx)(s) = \int_0^1 G(s,v)h(v,x(v))dv. \tag{4.3}$$

With the help of (4.2), it can be verified that $\mathscr{R}$ is T-closed. By (4.2) and (4.3) for any $x,y \in \mathscr{M}$ verifying $(x,y) \in \mathscr{R}$ and $\mathfrak{t} \in (0,1]$,

$$\begin{aligned}
\psi(\rho_{\mathfrak{t}}(Tx,Ty)) &= \psi\left(\frac{1-\mathfrak{t}}{\mathfrak{t}} \cdot d(Tx,Ty)\right) \\
&\leq \psi\left(\frac{1-\mathfrak{t}}{\mathfrak{t}} \cdot \max_{s\in[0,1]} \int_0^1 G(s,v)\left[h(v,y(v)) - h(v,x(v))\right]dv\right) \\
&\leq \psi\left(\frac{1-\mathfrak{t}}{\mathfrak{t}} \cdot \frac{1}{4} \cdot \max_{s\in[0,1]} \int_0^1 \left[h(v,y(v)) - h(v,x(v))\right]dv\right) \\
&\leq \max_{s\in[0,1]} \psi\left(\frac{1-\mathfrak{t}}{4\mathfrak{t}} \cdot [h(s,y(s)) - h(s,x(s))]\right) \\
&\leq \frac{1-\mathfrak{t}}{4\mathfrak{t}} \cdot \max_{s\in[0,1]} |x(s) - y(s)| \\
&\leq \rho_{\mathfrak{t}}(x,y)
\end{aligned}$$

For any $x^*, y^* \in T(\mathscr{M})$, we take $z := \max\{Tx^*, Ty^*\}$, implying $Tx^* \leq z$ and $Ty^* \leq z$. This show that $T(\mathscr{M})$ is $\mathscr{R}$-directed. From Theorem 3.3, we see that T has a unique fixed point, i.e., equation (4.1) has a unique solution in $C[0,T]$. □

5. Conclusions

In this paper, we investigated the result on the existence and uniqueness of fixed points for strict almost ψ-contractions under binary relations in $\mathscr{KS}$-FbMS. The conditions (i)-(v) of Theorem 3.3 ensure the existence of fixed points, while condition (vi) guarantees their uniqueness. Moreover, condition (vi) can be replaced with the statement that "$\mathscr{R}|_{T(\mathscr{M})}$ remains complete", and it can be shown that the uniqueness still holds. Since $\mathscr{KS}$-FbMS is a generalization of both $\mathscr{KS}$-FMS and b-MS. Using our result, we can obtain some corollaries in $\mathscr{KS}$-FMS

or b-MS. Specifically, under the restriction $\mathscr{R} = \mathscr{M} \times \mathscr{M}$, which is the universal relation, our result implies [21, Theorem 3.2]. Obviously, the present investigation enriches the knowledge of fixed points in $\mathscr{K}\mathscr{S}$-FbMS. As an application of our results, we investigate the existence and uniqueness of solutions for the fourth-order two-point boundary value problem related to elastic beam equations.

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