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SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS VIA TRIPLE COMPOSED RECTANGULAR METRIC SPACES

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Abstract. In this paper, we prove some fixed point theorems by introducing the notion of triple composed rectangular metric space. Some examples and an application are presented to support our main results, which expand and generalize some recent known results obtained in the literature.

Keywords. Fractional differential equations; Fixed point; Rectangular metric space; Triple composed rectangular metric space.

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1. INTRODUCTION

Metrics play an important role in fixed point theory. To guarantee and seek a fixed point of nonlinear operators, various metrics were considered; see, e.g., [2, 5, 8, 12, 13] and the references therein. In 2000, Branciari [3] investigated the concept of metric by replacing the triangle inequality with a quadrilateral inequality to obtain a new distance function. Some authors referred to this modified metric function as a general metric, while others termed it rectangular metric. For this new function, we employ the Branciari metric in this paper. In [3], Branciari effectively defined an open ball and a topology for the Brianciari metric. However, the Branciari metric's topology significantly differs from that of the typical metric; see, e.g., [4, 10, 16] for further information on the Branciari metric. In addition to the intriguing topological characteristics brought about by the Branciari metric, Branciari [3] also described

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the renowned Banach contraction mapping principle, which was generalized and extended in a number of ways; see, e.g., [4, 6, 7, 10, 11, 14, 15, 17] and the references therein. In [1], Ayoob, Chuan and Mlaiki proved some fixed point theorems on double-composed metric spaces, whose the triangle inequality has a special form with two control functions; see also Mlaiki [12] for double-controlled metric-like spaces as a further generalization of double-controlled metric-type spaces. In addition, Ayoob, Chuan and Mlaiki [1] also provided some examples and applications.

In this paper, we prove fixed point theorems on triple composed rectangular metric spaces. We also present an example and an appropriate application for our fixed point theorems.

2. Preliminaries

In this section, we present some necessary known definitions for our main results; see [3].

Let Γ be a non-void set. Recall that a function $\varpi : \Gamma \times \Gamma \to \mathbb{R}^+$ is said to be rectangular metric if it satisfies:

(a) $\varpi(\check{g},\check{s}) = 0$ iff $\check{g} = \check{s}$, for all $\check{g},\check{s} \in \Gamma$.
(b) $\varpi(\check{g},\check{s}) = \varpi(\check{s},\check{g})$, for all $\check{g},\check{s} \in \Gamma$.
(c) $\varpi(\check{g},\check{s}) \leq \varpi(\check{g},\check{a}) + \varpi(\check{a},\check{b}) + \varpi(\check{b},\check{s})$, for all $\check{g},\check{s} \in \Gamma$ and for all different points $\check{a},\check{b} \in \Gamma - \{\check{g},\check{s}\}$.

The pair (Γ,ϖ) is named as rectangular metric space.

Let Γ be a non-void set and $\Omega,\beta,\vartheta : [0,\infty) \to [0,\infty)$. Recall that a function $\varpi : \Gamma \times \Gamma \to \mathbb{R}^+$ is said to be a triple composed rectangular metric if it satisfies:

(a) $\varpi(\check{g},\check{s}) = 0$ iff $\check{g} = \check{s}$, for all $\check{g},\check{s} \in \Gamma$.
(b) $\varpi(\check{g},\check{s}) = \varpi(\check{s},\check{g})$, for all $\check{g},\check{s} \in \Gamma$.
(c) $\varpi(\check{g},\check{s}) \leq \check{a}(\varpi(\check{g},\check{a})) + \beta(\varpi(\check{a},\check{b})) + \vartheta(\varpi(\check{b},\check{s}))$, for all $\check{g},\check{s} \in \Gamma$ and for all different points $\check{a},\check{b} \in \Gamma - \{\check{g},\check{s}\}$.

The pair (Γ,ϖ) is called a triple composed rectangular metric space (TCRMS).

Let (Γ,ϖ) be a TCRMS, $\{\check{g}_\psi\}$ be a sequence in Γ, and $\check{g} \in \Gamma$. We say that $\{\check{g}_\psi\}$ is convergent to $\check{g}$ if and only $\varpi(\check{g}_\psi,\check{g}) \to 0$ as $\psi \to \infty$. This is indicated by $\check{g}_\psi \to \check{g}$ in this paper.

Let (Γ,ϖ) be a TCRMS, and $\{\check{g}_\psi\}$ be a sequence in Γ. We say that $\{\check{g}_\psi\}$ is a Cauchy sequence if and only if $\varpi(\check{g}_\psi,\check{g}_\mathfrak{m}) \to 0$ as $\psi,\mathfrak{m} \to \infty$.

Let (Γ,ϖ) be a TCRMS. We say that (Γ,ϖ) is complete if and only if every Cauchy sequence in Γ converges to some element in Γ.

3. Main Results

We now demonstrate fixed point theorems on TCRMS in this section.

Theorem 3.1. *Let (Γ,ϖ) be a complete TCRMS. Let $\Upsilon : \Gamma \to \Gamma$ such that $\varpi(\Upsilon\check{g},\Upsilon\check{s}) \leq \lambda\varpi(\check{g},\check{s})$ for all $\check{g},\check{s} \in \Gamma$, where $0 < \lambda < 1$. Assume the following assumptions hold*

(H1) ϑ is sub additive;

(H2)

$$\lim_{\psi,\mathfrak{m}\to\infty}\Bigg[\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))+\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1))$$
$$+\vartheta^{\mathfrak{m}}(\lambda^{\psi+2\mathfrak{m}}\,\varpi(\check{g}_0,\check{g}_1))\Bigg]=0,$$

where $\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$, *and* $\vartheta^{\mathfrak{m}}(\lambda^{\psi+2\mathfrak{m}}\,\varpi(\check{g}_0,\check{g}_1))$ *are the composite functions,*

(H3)

$$\lim_{\psi,\mathfrak{m}\to\infty}\Bigg[\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))+\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1))$$
$$+\vartheta^{\mathfrak{m}-1}(\lambda^{\psi+2\mathfrak{m}-2}\,\varpi(\check{g}_0,\check{g}_1))\Bigg]=0,$$

where $\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$ *and* $\vartheta^{\mathfrak{m}-1}(\lambda^{\psi+2\mathfrak{m}-2}\,\varpi(\check{g}_0,\check{g}_1))$ *are the composite functions.*

Then Υ *has a UFP(unique fixed point).*

Proof. Let $\check{g}_0\in\Gamma$. For each $\psi\in\mathbb{N}$, define $\Upsilon(\check{g}_\psi)=\check{g}_{\psi+1}$. Then $\{\check{g}_\psi\}$ is a sequence on (Γ,ϖ) and

$$\varpi(\check{g}_\psi,\check{g}_{\psi+1})=\varpi(\Upsilon\check{g}_{\psi-1},\Upsilon\check{g}_\psi)\leq\lambda\varpi(\check{g}_{\psi-1},\check{g}_\psi)\vdots\leq\lambda^{\psi}\varpi(\check{g}_0,\check{g}_1).$$

Suppose that $\check{g}_0$ is not a periodic point of Υ. In fact, if $\check{g}_0=\check{g}_\psi$ then, for any $\psi\geq 2$, $\varpi(\check{g}_0,\Upsilon\check{g}_0)=\varpi(\check{g}_\psi,\Upsilon\check{g}_\psi)$ and

$$\varpi(\check{g}_0,\check{g}_1)=\varpi(\check{g}_\psi,\check{g}_{\psi+1})\leq\cdots\leq\lambda^{\psi}\varpi(\check{g}_0,\check{g}_1),$$

which is a contradiction. Therefore, $\varpi(\check{g}_0,\check{g}_1)=0$, i.e., $\check{g}_0=\check{g}_1$ and so $\check{g}_0$ is a fixed point of Υ. Thus we assume that $\check{g}_\psi\neq\check{g}_{\mathfrak{m}}$ for all distinct $\psi,\mathfrak{m}\in\mathbb{N}$. Again, for any $\psi\in\mathbb{N}$, we obtain

$$\varpi(\check{g}_\psi,\check{g}_{\psi+2})=\varpi(\Upsilon\check{g}_{\psi-1},\Upsilon\check{g}_{\psi+1})\leq\lambda\varpi(\check{g}_{\psi-1},\check{g}_{\psi+1})\leq\cdots\leq\lambda^{\psi}\varpi(\check{g}_0,\check{g}_2).$$

For $\{\check{g}_\psi\}$, we consider $\varpi(\check{g}_\psi,\check{s}_{\psi+\mathfrak{p}})$ in two cases. If $\mathfrak{p}$ is odd say $2\mathfrak{m}+1$, then

$$\begin{aligned}\varpi(\check{g}_\psi,\check{g}_{\psi+2\mathfrak{m}+1})&\leq\Omega(\varpi(\check{g}_\psi,\check{g}_{\psi+1}))+\beta(\varpi(\check{g}_{\psi+1},\check{g}_{\psi+2}))+\vartheta(\varpi(\check{g}_{\psi+2},\check{g}_{\psi+2\mathfrak{m}+1}))\\&\leq\Omega(\varpi(\check{g}_\psi,\check{g}_{\psi+1}))+\beta(\varpi(\check{g}_{\psi+1},\check{g}_{\psi+2}))+\vartheta\Omega(\varpi(\check{g}_{\psi+2},\check{g}_{\psi+3}))\\&\quad+\vartheta\beta(\varpi(\check{g}_{\psi+3},\check{g}_{\psi+4}))+\vartheta^2(\varpi(\check{g}_{\psi+4},\check{g}_{\psi+2\mathfrak{m}+1}))\\&\cdots\\&\leq\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_{\mathfrak{l}},\check{g}_{\mathfrak{l}+1}))+\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1},\check{g}_{\mathfrak{l}+2}))\\&\quad+\vartheta^{\mathfrak{m}}(\,\varpi(\check{g}_{\psi+2\mathfrak{m}},\check{g}_{\psi+2\mathfrak{m}+1})).\end{aligned}$$

Since Ω, β, and ϑ are non-decreasing functions, then compositions $\vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_{\mathfrak{l}},\ \check{g}_{\mathfrak{l}+1}))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1},\check{g}_{\mathfrak{l}+2}))$, and $\vartheta^{\mathfrak{m}}(\ \varpi(\check{g}_{\psi+2\mathfrak{m}},\check{g}_{\psi+2\mathfrak{m}+1}))$ are also non-decreasing. Thus

$$\begin{aligned}\varpi(\check{g}_{\psi},\check{g}_{\psi+2\mathfrak{m}+1}) \leq & \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1)) \\ & + \vartheta^{\mathfrak{m}}(\lambda^{\psi+2\mathfrak{m}}\ \varpi(\check{g}_0,\check{g}_1)).\end{aligned}$$

As $\psi,\mathfrak{m}\to\infty$, we see that $\lim_{\psi,\mathfrak{m}\to\infty}\varpi(\check{g}_{\psi},\check{g}_{\psi+2\mathfrak{m}+1})=0$. If $\mathfrak{p}$ is even, say $2\mathfrak{m}$, then

$$\begin{aligned}\varpi(\check{g}_{\psi},\check{g}_{\psi+2\mathfrak{m}}) &\leq \Omega(\varpi(\check{g}_{\psi},\check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1},\check{g}_{\psi+2})) + \vartheta(\varpi(\check{g}_{\psi+2},\check{g}_{\psi+2\mathfrak{m}})) \\ &\leq \Omega(\varpi(\check{g}_{\psi},\check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1},\check{g}_{\psi+2})) + \vartheta\Omega(\varpi(\check{g}_{\psi+2},\check{g}_{\psi+3})) \\ &+ \vartheta\beta(\varpi(\check{g}_{\psi+3},\check{g}_{\psi+4})) + \vartheta^2(\varpi(\check{g}_{\psi+4},\check{g}_{\psi+2\mathfrak{m}})) \\ &\cdots \\ &\leq \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_{\mathfrak{l}},\check{g}_{\mathfrak{l}+1})) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1},\check{g}_{\mathfrak{l}})) \\ &+ \vartheta^{\mathfrak{m}-1}(\ \varpi(\check{g}_{\psi+2\mathfrak{m}-2},\check{g}_{\psi+2\mathfrak{m}})).\end{aligned}$$

Since Ω, β, and ϑ are non-decreasing functions, then compositions $\vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_{\mathfrak{l}},\ \check{g}_{\mathfrak{l}+1}))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1},\check{g}_{\mathfrak{l}}))$, and $\vartheta^{2\mathfrak{m}-1}(\ \varpi(\check{g}_{\psi+2\mathfrak{m}-2},\check{g}_{\psi+2\mathfrak{m}}))$ are also non-decreasing. Thus

$$\begin{aligned}\varpi(\check{g}_{\psi},\check{g}_{\psi+2\mathfrak{m}}) \leq & \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1)) \\ & + \vartheta^{\mathfrak{m}-1}(\lambda^{\psi+2\mathfrak{m}-2}\ \varpi(\check{g}_0,\check{g}_1)).\end{aligned}$$

Letting $\psi,\mathfrak{m}\to\infty$, we see that $\lim_{\psi,\mathfrak{m}\to\infty}\varpi(\check{g}_{\psi},\check{g}_{\psi+2\mathfrak{m}})=0$. Thus $\{\check{g}_{\psi}\}$ is a Cauchy sequence. Since (Γ,ϖ) is complete, then $\{\check{g}_{\psi}\}$ converges, to a point π, and $\{\Upsilon(\check{g}_{\psi})\}=\{\check{g}_{\psi+1}\}\to\pi\in\Gamma$ guarantees that $\{\Upsilon(\check{g}_{\psi})\}$ has a unique limit. Since Υ is continuous $\Upsilon(\check{g}_{\psi})\to\Upsilon(\pi)$, then $\Upsilon(\pi)=\pi$. Hence π is a fixed point of Υ. If ν is any fixed point of Υ, then

$$\varpi(\pi,\nu)=\varpi(\Upsilon(\pi),\Upsilon(\nu)\leq\lambda\varpi(\pi,\nu)$$

where $0<\lambda<1$, which implies $\varpi(\pi,\nu)=0$, so $\pi=\nu$. This completes the proof. □

Remark 3.2. Letting $\Omega(\check{g})=\beta(\check{g})=\vartheta(\check{g})=\check{g}$ in Theorem 3.1, we obtain the classical Banach fixed point theorem.

Example 3.3. Let $\Gamma=\{\mathscr{M}_{\psi}(\mathbb{R}):\mathscr{U}_{\psi}(\mathbb{R})$ is matrices over $\mathbb{R}\}$ and $\varpi:\Gamma\times\mathscr{Y}\to\mathbb{R}^{+}$ be defined by

$$\varpi(\check{W},\check{U})=\sum_{\aleph,\wp=1}^{\psi}|\chi_{\aleph\wp}-\mathfrak{q}_{\aleph\wp}|$$

for all $\check{W}=(\chi_{\aleph\wp})_{\psi\times\psi}\in\Gamma$, $\check{U}=(\mathfrak{q}_{\aleph\wp})_{\psi\times\psi}\in\mathscr{Y}$ and $\Omega(\check{g})=\beta(\check{g})=\vartheta(\check{g})=e^{\check{g}}$. Then (Γ,ϖ) is a complete TCRMS. Define $\Upsilon:\Gamma\to\Gamma$ by

$$\Upsilon(\check{W})=\left(\frac{\chi_{\aleph\wp}}{5}\right)_{\psi\times\psi}$$

for all $\check{W} = (\chi_{\aleph\wp})_{\psi\times\psi} \in \mathscr{M}_{\psi}(\mathbb{R})$. Now, we have

$$\begin{aligned}\varpi(\Upsilon(\check{W}),\Upsilon(\check{U})) &= \frac{1}{5}\sum_{\aleph,\wp=1}^{\psi}|\chi_{\aleph\wp}-\mathfrak{q}_{\aleph\wp}|\\ &\leq \frac{1}{2}\sum_{\aleph,\wp=1}^{\psi}|\chi_{\aleph\wp}-\mathfrak{q}_{\aleph\wp}|\\ &= \lambda\varpi(\check{W},\check{U})\end{aligned}$$

for all $\check{W} = (\chi_{\aleph\wp})_{\psi\times\psi}, \check{U} = (\mathfrak{q}_{\aleph\wp})_{\psi\times\psi} \in \Gamma$. Thus all the assumptions of Theorem 3.1 are fulfilled with $\lambda = \frac{1}{2}$ and Υ has a UFP $(0_{\psi\times\psi}, 0_{\psi\times\psi}) \in \mathscr{M}_{\psi}(\mathbb{R})$, where $0_{\psi\times\psi}$ is zero matrix.

Example 3.4. Let $\Gamma = [0,1]$ be equipped with $\varpi(\check{g},\check{s}) = |\check{g}-\check{s}|$ for all $\check{g} \in \Gamma$, $\check{s} \in \mathscr{Y}$, and $\Omega(\check{g}) = \beta(\check{g}) = \vartheta(\check{g}) = e^{\check{g}}$. Then, (Γ,ϖ) is a complete TCRMS. Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon(\check{g}) = \frac{(\sqrt{3}+1)-\check{g}}{\sqrt{3}}$$

for all $\check{g} \in \Gamma$. Let $\check{g},\check{s} \in \Gamma$. Then

$$\varpi(\Upsilon\check{g},\Upsilon\check{s}) = |\frac{\check{g}}{\sqrt{3}} - \frac{\check{s}}{\sqrt{3}}| \leq \frac{1}{2}|\check{g}-\check{s}|.$$

Hence all the assumptions of Theorem 3.1 are fulfilled and Υ has a UFP $\check{g} = 0$.

We present a theorem based on Kannan's fixed point result [9].

Theorem 3.5. *Let (Γ,ϖ) be a complete TCRMS. Let $\Upsilon : \Gamma \to \Gamma$ be a mapping such that $\varpi(\Upsilon\check{g},\Upsilon\check{s}) \leq \check{b}(\varpi(\check{g},\Upsilon\check{g}) + \varpi(\check{s},\Upsilon\check{s}))$ for all $\check{g},\check{s} \in \Gamma$, where $0 < \check{b} < \frac{1}{2}$. Assume*

(H1) ϑ is sub additive;

(H2)

$$\lim_{\psi,\mathfrak{m}\to\infty}\Bigg[\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1)) + \vartheta^{\mathfrak{m}-1}(\lambda^{\psi+2\mathfrak{m}-3}\,\varpi(\check{g}_0,\check{g}_1))\Bigg] = 0,$$

where $\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1))$, and $\vartheta^{\mathfrak{m}-1}(\varpi(\lambda^{\psi+2\mathfrak{m}-3}\varpi(\check{g}_0,\check{g}_1))$ denote the composite functions;

(H3)

$$\lim_{\psi,\mathfrak{m}\to\infty}\Bigg[\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2}\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1)) + \vartheta^{\mathfrak{m}}(\lambda^{\psi+2\mathfrak{m}}\,\varpi(\check{g}_0,\check{g}_1))\Bigg] = 0,$$

where $\vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^{\mathfrak{l}}\varpi(\check{g}_0,\check{g}_1))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0,\check{g}_1))$, and $\vartheta^{\mathfrak{m}}(\lambda^{\psi+2\mathfrak{m}}\,\varpi(\check{g}_0,\check{g}_1))$ denote the composite functions.

Then Υ has a UFP.

Proof. Let $\check{g}_0 \in \Gamma$. For each non-negative integer ψ, we define $\check{g}_{\psi+1} = \Upsilon\check{g}_\psi$. Then,

$$\begin{aligned}\varpi(\check{g}_\psi, \check{g}_{\psi+1}) &\leq \check{b}(\varpi(\check{g}_{\psi-1}, \Upsilon\check{g}_{\psi-1}) + \varpi(\check{g}_\psi, \Upsilon\check{g}_\psi,)) \\ &= \check{b}(\varpi(\check{g}_{\psi-1}, \check{g}_\psi) + \varpi(\check{g}_\psi, \check{s}_{\psi+1}))\end{aligned}$$

for all integers $\psi \geq 1$.

Now, one has $\varpi(\check{g}_\psi, \check{g}_{\psi+1}) \leq \frac{\check{b}}{1-\check{b}}\varpi(\check{g}_{\psi-1}, \check{g}_\psi)$. Assume $\lambda := \frac{\check{b}}{1-\check{b}}$, where $\lambda \in (0,1)$, since $\check{b} \in (0, \frac{1}{2})$. Now, one has $\varpi(\check{g}_\psi, \check{g}_{\psi+1}) \leq \lambda^\psi \varpi(\check{g}_0, \check{g}_1)$. Assume that $\check{g}_0$ is not a periodic point of Υ. In fact, if $\check{g}_0 = \check{g}_\psi$ then, for any $\psi \geq 2$, $\varpi(\check{g}_0, \Upsilon\check{g}_0) = \varpi(\check{g}_\psi, \Upsilon\check{g}_\psi)$, and

$$\varpi(\check{g}_0, \check{g}_1) = \varpi(\check{g}_\psi, \check{g}_{\psi+1}) \leq \cdots \leq \lambda^\psi \varpi(\check{g}_0, \check{g}_1),$$

which is a contradiction. Therefore, $\varpi(\check{g}_0, \check{g}_1) = 0$, i.e., $\check{g}_0 = \check{g}_1$ and so $\check{g}_0$ is a fixed point of Υ. Thus we assume that $\check{g}_\psi \neq \check{g}_\mathfrak{m}$ for all distinct $\psi, \mathfrak{m} \in \mathbb{N}$. Again, for any $\psi \in \mathbb{N}$, we obtain

$$\begin{aligned}\varpi(\check{g}_\psi, \check{g}_{\psi+2}) &\leq \check{b}(\varpi(\check{g}_{\psi-1}, \Upsilon\check{g}_{\psi-1}) + \varpi(\check{g}_{\psi+1}, \Upsilon\check{g}_{\psi+1})) \\ &\leq \check{b}(\varpi(\check{g}_{\psi-1}, \check{g}_\psi) + \varpi(\check{g}_{\psi+1}, \check{g}_{\psi+2})) \\ &\leq \check{b}\lambda^{\psi-1}\varpi(\check{g}_0, \check{g}_1) + \lambda^{\psi+1}\varpi(\check{g}_0, \check{g}_1) \\ &\leq \check{b}\lambda^{\psi-1}(1+\lambda^2)\varpi(\check{g}_0, \check{g}_1).\end{aligned}$$

Therefore, $\varpi(\check{g}_\psi, \check{g}_{\psi+2}) \leq \kappa\lambda^{\psi-1}\varpi(\check{g}_0, \check{g}_1)$, where $\kappa = \check{b}(1+\lambda^2) > 0$. For $\{\check{g}_\psi\}$, we consider $\varpi(\check{g}_\psi, \check{s}_{\psi+\mathfrak{p}})$ in two cases.

If $\mathfrak{p}$ is odd, say $2\mathfrak{m}+1$, then

$$\begin{aligned}\varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}+1}) &\leq \Omega(\varpi(\check{g}_\psi, \check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1}, \check{g}_{\psi+2})) + \vartheta(\varpi(\check{g}_{\psi+2}, \check{g}_{\psi+2\mathfrak{m}+1})) \\ &\leq \Omega(\varpi(\check{g}_\psi, \check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1}, \check{g}_{\psi+2})) + \vartheta\Omega(\varpi(\check{g}_{\psi+2}, \check{g}_{\psi+3})) \\ &\quad + \vartheta\beta(\varpi(\check{g}_{\psi+3}, \check{g}_{\psi+4})) + \vartheta^2(\varpi(\check{g}_{\psi+4}, \check{g}_{\psi+2\mathfrak{m}+1})) \\ &\cdots \\ &\leq \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_\mathfrak{l}, \check{g}_{\mathfrak{l}+1})) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1}, \check{g}_{\mathfrak{l}+2})) \\ &\quad + \vartheta^\mathfrak{m}(\varpi(\check{g}_{\psi+2\mathfrak{m}}, \check{g}_{\psi+2\mathfrak{m}+1})).\end{aligned}$$

Since Ω, β, and ϑ are non-decreasing, then $\vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_\mathfrak{l}, \check{g}_{\mathfrak{l}+1}))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1}, \check{g}_{\mathfrak{l}+2}))$, and $\vartheta^\mathfrak{m}(\varpi(\check{g}_{\psi+2\mathfrak{m}}, \check{g}_{\psi+2\mathfrak{m}+1}))$ are also non-decreasing. Therefore, we obtain

$$\begin{aligned}\varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}+1}) \leq &\sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^\mathfrak{l}\varpi(\check{g}_0, \check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0, \check{g}_1)) \\ &+ \vartheta^\mathfrak{m}(\lambda^{\psi+2\mathfrak{m}}\varpi(\check{g}_0, \check{g}_1)).\end{aligned}$$

As $\psi, \mathfrak{m} \to \infty$, we obtain $\lim_{\psi,\mathfrak{m}\to\infty} \varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}+1}) = 0$. If $\mathfrak{p}$ is even, say $2\mathfrak{m}$, then

$$\begin{aligned}
\varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}}) &\leq \Omega(\varpi(\check{g}_\psi, \check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1}, \check{g}_{\psi+2})) + \vartheta(\varpi(\check{g}_{\psi+2}, \check{g}_{\psi+2\mathfrak{m}})) \\
&\leq \Omega(\varpi(\check{g}_\psi, \check{g}_{\psi+1})) + \beta(\varpi(\check{g}_{\psi+1}, \check{g}_{\psi+2})) + \vartheta\Omega(\varpi(\check{g}_{\psi+2}, \check{g}_{\psi+3})) \\
&\quad + \vartheta\beta(\varpi(\check{g}_{\psi+3}, \check{g}_{\psi+4})) + \vartheta^2(\varpi(\check{g}_{\psi+4}, \check{g}_{\psi+2\mathfrak{m}})) \\
&\cdots \\
&\leq \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_\mathfrak{l}, \check{g}_{\mathfrak{l}+1})) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1}, \check{g}_\mathfrak{l})) \\
&\quad + \vartheta^{\mathfrak{m}-1}(\varpi(\check{g}_{\psi+2\mathfrak{m}-2}, \check{g}_{\psi+2\mathfrak{m}})).
\end{aligned}$$

Since Ω, β, and ϑ are non-decreasing, then $\vartheta^{\mathfrak{l}-\psi}\Omega(\varpi(\check{g}_\mathfrak{l}, \check{g}_{\mathfrak{l}+1}))$, $\vartheta^{\mathfrak{l}-\psi}\beta(\varpi(\check{g}_{\mathfrak{l}+1}, \check{g}_{\mathfrak{l}+2}))$ and $\vartheta^{\mathfrak{m}-1}(\varpi(\check{g}_{\psi+2\mathfrak{m}-2}, \check{g}_{\psi+2\mathfrak{m}}))$ are also non-decreasing. Therefore, we obtain

$$\begin{aligned}
\varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}}) \leq & \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\Omega(\lambda^\mathfrak{l}\varpi(\check{g}_0, \check{g}_1)) + \sum_{\mathfrak{l}=\psi}^{2\mathfrak{m}+\psi-2} \vartheta^{\mathfrak{l}-\psi}\beta(\lambda^{\mathfrak{l}+1}\varpi(\check{g}_0, \check{g}_1)) \\
& + \vartheta^{\mathfrak{m}-1}(\lambda^{\psi+2\mathfrak{m}-3}\varpi(\check{g}_0, \check{g}_1)).
\end{aligned}$$

As $\psi, \mathfrak{m} \to \infty$, we see that $\lim_{\psi,\mathfrak{m}\to\infty} \varpi(\check{g}_\psi, \check{g}_{\psi+2\mathfrak{m}}) = 0$. Hence, $\{\check{g}_\psi\}$ is a Cauchy sequence. Since (Γ, ϖ) is complete, then $\{\check{g}_\psi\}$ converges, to a point π, and $\{\Upsilon(\check{g}_\psi)\} = \{\check{g}_{\psi+1}\} \to \pi \in \Gamma$. Conversely,

$$\varpi(\Upsilon\pi, \Upsilon\check{g}_\psi) \leq \check{b}(\varpi(\check{g}_\psi, \Upsilon\check{g}_\psi) + \varpi(\Upsilon\pi, \pi)) = \check{b}(\varpi(\check{g}_\psi, \check{g}_{\psi+1}) + \varpi(\Upsilon\pi, \pi));$$

which in turn implies that $\varpi(\Upsilon\pi, \pi) \leq \check{b}\varpi(\Upsilon\pi, \pi)$. Hence π is a fixed point of Υ. If ν is any fixed point of Υ, then $\varpi(\pi, \nu) = \varpi(\Upsilon\pi, \Upsilon\nu) \leq \check{b}(\varpi(\pi, \pi) + \varpi(\nu, \nu)) = 0$. Consequently, $\pi = \nu$. □

Example 3.6. Let $\Gamma = \{\mathscr{M}_\psi(\mathbb{R}) : \mathscr{U}_\psi(\mathbb{R}) \text{ is matrices over } \mathbb{R}\}$ and $\varpi : \Gamma \times \mathscr{Y} \to \mathbb{R}^+$ be defined by

$$\varpi(\check{W}, \check{U}) = \sum_{\aleph,\wp=1}^{\psi} \left(|\chi_{\aleph\wp} - \frac{\chi_{\aleph\wp}}{5} + \mathfrak{q}_{\aleph\wp} - \frac{\mathfrak{q}_{\aleph\wp}}{5}| \right)$$

for all $\check{W} = (\chi_{\aleph\wp})_{\psi\times\psi} \in \Gamma$, $\check{U} = (\mathfrak{q}_{\aleph\wp})_{\psi\times\psi} \in \mathscr{Y}$, and $\Omega(\check{g}) = \beta(\check{g}) = \vartheta(\check{g}) = e^{\check{g}}$. Then (Γ, ϖ) is a complete TCRMS. Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon(\check{W}) = \left(\frac{\chi_{\aleph\wp}}{5} \right)_{\psi\times\psi}$$

for all $\check{W} = (\chi_{\aleph\wp})_{\psi\times\psi} \in \mathscr{M}_\psi(\mathbb{R})$. Now,

$$\begin{aligned}
\varpi(\Upsilon(\check{W}), \Upsilon(\check{U})) &= \frac{1}{5} \sum_{\aleph,\wp=1}^{\psi} \left(|\chi_{\aleph\wp} - \frac{\chi_{\aleph\wp}}{5} + \mathfrak{q}_{\aleph\wp} - \frac{\mathfrak{q}_{\aleph\wp}}{5}| \right) \\
&\leq \frac{1}{2} \sum_{\aleph,\wp=1}^{\psi} |\chi_{\aleph\wp} - \frac{\chi_{\aleph\wp}}{5}| + \frac{1}{2} \sum_{\aleph,\wp=1}^{\psi} |\mathfrak{q}_{\aleph\wp} - \frac{\mathfrak{q}_{\aleph\wp}}{5}| \\
&= \lambda(\varpi(\check{W}, \Upsilon(\check{W})) + \varpi(\check{U}, \Upsilon(\check{U})))
\end{aligned}$$

for all $\check{W} = (\chi_{\aleph\wp})_{\psi\times\psi}, \check{U} = (\mathfrak{q}_{\aleph\wp})_{\psi\times\psi} \in \Gamma$. All the assumptions of Theorem 3.5 are verified with $\lambda = \frac{1}{2}$ and Υ has a UFP $(0_{\psi\times\psi}, 0_{\psi\times\psi}) \in \mathscr{M}_{\psi}(\mathbb{R})$, where $0_{\psi\times\psi}$ is zero matrix.

4. Application in Production-Consumption Equilibrium

The existence and uniqueness of solutions to the integral equations of Theorem 3.1 are covered in this section. We apply our main results to develop a mathematical model and resolve an initial value issue that arises in the dynamic market equilibrium problem, a major economic issue. Daily pricing patterns and prices have a significant influence on markets, regardless of whether prices are growing or lowering for production $\check{b}_{\mathfrak{p}}$ and consumption $\check{b}_{\mathfrak{c}}$. The economist is therefore curious about the present price $\mathscr{Y}(\check{f})$. Now, consider

$$\begin{aligned}\check{b}_{\mathfrak{p}} =& \beta_1 + \gamma_1\mathscr{Y}(\check{f}) + \delta_1\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \rho_1\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2}\\ \check{b}_{\mathfrak{c}} =& \beta_2 + \gamma_2\mathscr{Y}(\check{f}) + \delta_2\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \rho_2\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2},\end{aligned}$$

initially $\mathscr{Y}(0) = 0, \frac{\mathrm{d}\mathscr{Y}}{\mathrm{d}\check{f}}(0) = 0$, where $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1$, and ρ_2 are constants. When market forces are in balance and the existing prices between production and consumption stabilize, this is known as dynamic economic equilibrium, that is, $\check{b}_{\mathfrak{p}} = \check{b}_{\mathfrak{c}}$. Thus,

$$\begin{aligned}&\beta_1 + \gamma_1\mathscr{Y}(\check{f}) + \delta_1\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \rho_1\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2} = \beta_2 + \gamma_1\mathscr{Y}(\check{f}) + \delta_2\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \rho_2\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2},\\ &(\beta_1 - \beta_2) + (\gamma_1 - \gamma_2)\mathscr{Y}(\check{f}) + (\delta_1 - \delta_2)\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + (\rho_1 - \rho_2)\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2} = 0,\\ &\rho\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2} + \delta\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \gamma\mathscr{Y}(\check{f}) = -\beta,\\ &\frac{\mathrm{d}^2\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}^2} + \frac{\delta}{\rho}\frac{\mathrm{d}\mathscr{Y}(\check{f})}{\mathrm{d}\check{f}} + \frac{\gamma}{\rho}\mathscr{Y}(\check{f}) = -\frac{\beta}{\rho},\end{aligned}$$

where $\beta = \beta_1 - \beta_2, \gamma = \gamma_1 - \gamma_2, \delta = \delta_1 - \delta_2$, and $\rho = \rho_1 - \rho_2$.

Our initial value problem is now listed as follows:

$$\mathscr{Y}''(\check{f}) + \frac{\delta}{\rho}\mathscr{Y}'(\check{f}) + \frac{\gamma}{\rho}\mathscr{Y}(\check{f}) = -\frac{\beta}{\rho}, \text{ with } \mathscr{Y}(0) = 0 \text{ and } \mathscr{Y}'(0) = 0. \tag{4.1}$$

Problem (4.1) is equivalent to if we examine the production and consumption duration time $\mathfrak{w}$

$$\mathscr{Y}(\check{f}) = \int_0^{\mathfrak{w}} \mathscr{G}(\check{f}, \check{f}^*)\mathscr{K}(\check{f}^*, \check{f}, \mathscr{Y}(\check{f}))\mathrm{d}\check{f}, \tag{4.2}$$

where Green function $\mathscr{G}(\check{f}, \check{f}^*)$ is

$$\mathscr{G}(\check{f}, \check{f}^*) = \begin{cases} \check{f}\check{h}^{\frac{\gamma}{2\delta}}(\check{f}^* - \check{f}), & 0 \le \check{f} \le \mathfrak{s} \le \mathfrak{w} \\ \mathfrak{s}\check{h}^{\frac{\gamma}{2\delta}}(\check{f} - \check{f}^*), & 0 \le \mathfrak{s} \le \check{f} \le \mathfrak{w} \end{cases}$$

and $\mathscr{K}:[0,\mathfrak{w}]\times\mathscr{U}^2\to\mathbb{R}$ is a continuous function. Define $\Upsilon:\Gamma\to\Gamma$ is given by

$$\Upsilon(\mathscr{Y}(\check{f}))=\int_0^{\mathfrak{w}}\mathscr{G}(\check{f},\check{f}^*)\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}(\check{f}))\mathrm{d}\check{f} \tag{4.3}$$

Now, a fixed point of Υ(4.3) represents the solution to the dynamic market equilibrium problem, which is represented as (4.1). Indeed, (4.1) controls the current price $\mathscr{Y}(\check{f})$. Let $\mathscr{C}[0,\mathfrak{w}]$ symbolize the family of real continuous functions on $[0,\mathfrak{w}]$, and write $\Gamma=\mathscr{C}[0,\mathfrak{w}]$. We define $\varpi:\Gamma\times\Gamma\to\mathbb{R}^+$ by $\varpi(\nu,\nu')=\sup_{\check{f}\in[0,1]}|\nu(\check{f})-\nu'(\check{f})|$ for all $\nu,\nu'\in\Gamma$ and $\Omega(\check{g})=\beta(\check{g})=\vartheta(\check{g})=\check{g}$ for all $\check{g}\in[0,\infty)$. Then (Γ,ϖ) is a complete TCRMS.

Theorem 4.1. *Consider the operator $\Upsilon:\Gamma\to\Gamma$ (4.3) in a complete TCRMS (Γ,ϖ), satisfying*

(i) there exist $\check{f}\in[0,1],\rho\in(0,1)$ and $\nu,\nu'\in\Gamma$ s.t.

$$|\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_1(\check{f}))-\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_2(\check{f}))|\le\rho|\mathscr{Y}_1(\check{f}))-\mathscr{Y}_2(\check{f}))|;$$

(ii) there exists a continuous function $\mathscr{G}:\mathscr{U}^2\to\mathbb{R}$ that satisfies

$$\sup_{\mathfrak{s}\in[0,\mathfrak{w}]}\int_0^{\mathfrak{w}}\mathscr{G}(\check{f},\check{f}^*)d\check{f}\le 1;$$

Then, there exists only one solution to dynamic market equilibrium problem (4.1).

Proof. Observe that

$$\begin{aligned}
&|\Upsilon(\mathscr{Y}_1(\check{f}))-\Upsilon(\mathscr{Y}_2(\check{f}))|\\
&=\left|\int_0^{\mathfrak{w}}\mathscr{G}(\check{f},\check{f}^*)\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_1(\check{f}))\mathrm{d}\check{f}-\int_0^{\mathfrak{w}}\mathscr{G}(\check{f},\check{f}^*)\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_2(\check{f}))\mathrm{d}\check{f}\right|\\
&\le\int_0^{\mathfrak{w}}\mathscr{G}(\check{f},\check{f}^*)\mathrm{d}\check{f}\int_0^1\left|\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_1(\check{f}))-\mathscr{K}(\check{f}^*,\check{f},\mathscr{Y}_2(\check{f}))\right|d\check{f}\\
&\le\rho|\mathscr{Y}_1(\check{f}))-\mathscr{Y}_2(\check{f}))|.
\end{aligned}$$

Taking the supremum on both sides, we arrive at

$$\varpi(\Upsilon(\mathscr{Y}_1(\check{f})),\Upsilon(\mathscr{Y}_2(\check{f})))\le\rho\varpi(\mathscr{Y}_1(\check{f}),\mathscr{Y}_2(\check{f})).$$

Hence, all the hypothesis of Theorem 3.1 are fulfilled and therefore, initial value problem (4.2) has a unique solution. □

5. Application to Fractional Differential Equations

The Reiman-Liouville fractional derivative of order $\delta>0$ for a function $\nu\in\mathscr{C}[0,1]$ is provided by

$$\frac{1}{\Gamma(\xi-\delta)}\frac{d^\xi}{d\check{f}^\xi}\int_0^{\check{f}}\frac{\nu(\check{h})d\check{h}}{(\check{f}-\check{h})^{\delta-\xi+1}}=\mathscr{D}^\delta\nu(\check{f}),$$

where the right-hand side is pointwise defined on $[0,1]$, Γ is the Euler gamma function, and $[\delta]$ is the integer part of the number δ. Examine the following FDE

$$\begin{aligned}
&{}^{\check{h}}\mathscr{D}^{\check{s}}\nu(\check{f})+\mathfrak{f}(\check{f},\nu(\check{f}))=0,\quad 1\le\check{f}\le 0,\quad 2\le\check{s}>1;\\
&\nu(0)=\nu(1)=0,
\end{aligned}\tag{5.1}$$

The Caputo fractional derivative of order $\check{s}$ is represented as ${}^{\check{h}}\mathscr{D}^{\check{s}}$, while $\mathfrak{f}$ is a continuous function from $[0,1]\times\mathbb{R}$ to $\mathbb{R}$ as follows:

$$ {}^{\check{h}}\mathscr{D}^{\check{s}} = \frac{1}{\Gamma(\xi-\check{s})}\int_0^{\check{g}} \frac{\nu^{\xi}(\check{h})d\check{h}}{(\check{f}-\check{h})^{\check{s}-\xi+1}} $$

Let $\Gamma=(\mathscr{C}[0,1],\mathbb{R})$ be the set of all continuous functions defined on $[0,1]$. Let $\varpi:\Gamma\times\Gamma\to\mathbb{R}^{+}$ be defined by $\varpi(\nu,\nu^{'})=\sup_{\check{f}\in[0,1]}|\nu(\check{f})-\nu^{'}(\check{f})|$ for all $\nu,\nu^{'}\in\Gamma$ and $\Omega(\check{g})=\beta(\check{g})=\vartheta(\check{g})=\check{g}$ for all $\check{g}\in[0,\infty)$. Then (Γ,ϖ) is a complete TCRMS.

Theorem 5.1. *Consider fractional differential equation (5.1). Suppose that the following assumptions hold*

(i) there exist $\check{f}\in[0,1],\rho\in(0,1)$ and $\nu,\nu^{'}\in\Gamma$ such that $|\mathfrak{f}(\check{f},\nu)-\mathfrak{f}(\check{f},\nu^{'})|\leq\rho|\nu(\check{f})-\nu^{'}(\check{f})|$;

(ii) $\sup_{\check{f}\in[0,1]}\int_0^1|\mathscr{G}(\check{f},\check{h})|d\mathfrak{q}\leq 1$.

Then fractional differential equation (5.1) has a unique solution in Γ.

Proof. Observe that fractional differential equation (5.1) is equivalent to the integral equation $\nu(\check{f})=\int_0^1\mathscr{G}(\check{f},\check{h})\mathfrak{f}(\mathfrak{q},\nu(\check{h}))d\check{h}$, where

$$ \mathscr{G}(\check{f},\check{h})=\begin{cases}\frac{[\check{f}(1-\check{h})]^{\check{s}-1}-(\check{f}-\check{h})^{\check{s}-1}}{\Gamma(\check{s})}, & 0\leq\check{h}\leq\check{f}\leq 1,\\ \frac{[\check{f}(1-\check{h})]^{\check{s}-1}}{\Gamma(\check{s})}, & 0\leq\check{f}\leq\check{h}\leq 1.\end{cases} $$

Define the covariant mapping $\Upsilon:\Gamma\to\Gamma$ defined by $\Upsilon\nu(\check{f})=\int_0^1\mathscr{G}(\check{f},\check{h})\mathfrak{f}(\mathfrak{q},\nu(\check{h}))d\check{h}$. It is simple to observe that ν^{*} is a solution to e problem (5.1) if $\nu^{*}\in\Upsilon$ is a fixed point of Υ. Now, one has

$$ \begin{aligned}|\Upsilon\nu(\check{f})-\Upsilon\nu^{'}(\check{f})| &= \left|\int_0^1\mathscr{G}(\check{f},\check{h})\mathfrak{f}(\mathfrak{q},\nu(\check{h}))d\check{h}-\int_o^1\mathscr{G}(\check{f},\check{h})\mathfrak{f}(\mathfrak{q},\nu^{'}(\check{h}))d\check{h}\right|\\ &\leq\int_0^1|\mathscr{G}(\check{f},\check{h})|d\check{h}\int_0^1\left|\mathfrak{f}(\mathfrak{q},\nu(\check{h}))-\mathfrak{f}(\mathfrak{q},\nu^{'}(\check{h}))\right|d\check{h}\\ &\leq\rho\left|\nu(\check{f})-\nu^{'}(\check{f})\right|.\end{aligned} $$

Taking hte supremum on both sides, we have $\varpi(\Upsilon\nu,\Upsilon\nu^{'})\leq\rho\varpi(\nu,\nu^{'})$. Therefore, fractional differential equation (5.1) has a unique solution since all of the hypotheses of Theorem 3.1 are satisfied. □

Example 5.2. According to (5.1), we consider

$$ \begin{cases}{}^{\mathrm{C}}\mathscr{D}^{j}\mathfrak{q}(\check{f})+\frac{\sqrt{7}\ln(\check{f}+1)\tan(\mathfrak{q}(\check{f}))}{\check{f}^{3}+1}=0, & 0\leq\check{f}\leq 1,\\ \mathfrak{q}(0)=\mathfrak{q}(1)=0,\end{cases} $$

with three cases $j = \{\frac{13}{30}, \frac{7}{8}, \frac{17}{30}\}$, where $\mathfrak{f}(\check{f}, \mathfrak{q}(\check{f})) = \frac{\sqrt{7}\ln(\check{f}+1)\tan(\mathfrak{q}(\check{f}))}{\check{f}^3+1}$. Then, for $\mathfrak{q}(\check{f})$, $\mathfrak{q}'(\check{f}) \in C[0,1]$, we have

$$\begin{aligned}\left|\mathfrak{f}(\check{f}, \mathfrak{q}(\check{f})) - \mathfrak{f}(\check{f}, \mathfrak{q}'(\check{f}))\right| &= \left|\frac{\sqrt{7}\ln(\check{f}+1)\tan(\mathfrak{q}(\check{f}))}{\check{f}^3+1} - \frac{\sqrt{7}\ln(\check{f}+1)\tan(\mathfrak{q}'(\check{f}))}{\check{f}^3+1}\right| \\ &= \left|\frac{\sqrt{7}\ln(\check{f}+1)}{\check{f}^3+1}\right| \left|\tan(\mathfrak{q}(\check{f})) - \tan(\mathfrak{q}'(\check{f}))\right| \\ &\leq \sqrt{7}\ln 3 \left|\mathfrak{q}(\check{f}) - \mathfrak{q}'(\check{f})\right| = \sqrt{\rho}\left|\mathfrak{q}(\check{f}) - \mathfrak{q}'(\check{f})\right|,\end{aligned}$$

where $\rho = \left(\sqrt{7}\ln 3\right)^2$. Hence, all the conditions of Theorem 5.1 are satisfied.

6. Conclusion

In this paper, we introduced the triple composed rectangular metric space and proved some fixed point theorems in this space. In addition, we presented Some examples and an application to support our main results, which expand and generalize some recent known results obtained in the literature.

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References

[1] I. Ayoob, N.Z. Chuan, N. Mlaiki, Double-composed metric spaces, Mathematics, 11 (2023) 1866.
[2] R.P. Agarwal, E. Karapınar, Farshid Khojasteh, Ćirić and Meir-Keeler fixed point results in super metric spaces, Appl. Set-Valued Anal. Optim. 4 (2022), 271-275.
[3] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. (Debr.) 57 (2000) 31-37.
[4] P. Das, A fixed point theorem on a class of generalized metric spaces, Korean J. Math. Sci. 9 (2022) 29-33.
[5] M. Gabeleh, P.R. Patle, M. De La Sen, Noncyclic φ-contractions in hyperbolic uniformly convex metric spaces, J. Nonlinear Var. Anal. 7 (2023), 251-265.
[6] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl. 2014 (2014) 38.
[7] Z. Kadeburg, S. Radenovic, On generalized metric spaces: A survey, TWMS J. Pure Appl. Math. 5 (2014) 3-13.
[8] X. Kang, N. Fang, Some common coupled fixed point results for the mappings with a new contractive condition in a Menger PbM-metric space, J. Nonlinear Funct. Anal. 2023 (2023) 9.
[9] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71-76.
[10] W.A. Kirk, N. Shahzad, Generalized metrics and Caristi's theorem, Fixed Point Theory Appl. 2013 (2013) 129.
[11] H. Lakzian, B. Samet, Fixed points for (ψ, ϕ)-weakly contractive mappings in generalized metric spaces, Appl. Math. Lett. 25 (2012) 902-906.
[12] N. Mlaiki, Double-controlled metric-like spaces, J. Inequal. Appl. 2020 (2020) 189.
[13] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric-type spaces and the related contraction principle, Mathematics 6 (2018) 194.

[14] O. Popescu, Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2014 (2014) 190.
[15] S. Reich, A. J. Zaslavski, Convergence of inexact iterates of strict contractions in metric spaces with graphs, J. Appl. Numer. Optim. 4 (2022) 215-220.
[16] T. Suzuki, Generalized metric space do not have the compatible topology, Abstr. Appl. Anal. 2014 (2014), 458098.
[17] A. J. Zaslavski, Convergence results for set-valued strict contractions, Commun. Optim. Theory 2024 (2024) 37.