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NUMERICAL SIMULATION AND ANALYSIS FOR VARIABLE-ORDER FRACTAL-FRACTIONAL DERIVATIVES OF BIO-ETHANOL PRODUCTION MODELS WITH RECYCLING AND DEATH RATE EFFECTS

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Abstract. Bio-ethanol is a good choice for alternative energy instead of the use of fossil fuels and is still a hot topic for researchers at present. In this paper, a nonlinear mathematical model for bio-ethanol production under the fractal-fractional operators in the sense of Liouville-Caputo, Caputo-Fabrizio, and Atangana-Baleanu with variable order is considered. We use the fixed point theorem of Banach's type to prove the existence and uniqueness of the solution. The stability analysis of the proposed bioethanol model is examined by using the various types of Ulam's stability. The Adams-Bashforth method is applied to construct the numerical schemes for each type of kernel, as well as the graphically numerical simulations for some sets of fractional orders and fractal dimensions, which are conducted to support our analytic results.

Keywords. Ethanol model; Fractal-fractional operators; Fixed point theorems; Ulam's stability; Numerical scheme.
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1. INTRODUCTION

At present, fossil fuels, such as petroleum oil, natural gas, and coal, are the main contributors to environmental pollution and global climate change, which have an impact on living things. They are also expensive and nearing their end. Therefore, the development of renewable energy is one way to alleviate various problems as a result of the use of fossil fuels. Bio-ethanol

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is one of the most important renewable fuels, and it contributes to reducing the negative environmental consequences of worldwide fossil fuel use. Ethanol can be utilized in a variety of applications, including the medical field and as a substitute fuel inside gasoline engines in the form of gasohol, which is produced by mixing gasoline with bioethanol [1]. Bio-ethanol synthesis is obtained by fermenting yeast microorganisms with the substrate of agricultural crops such as sugar, starch, and fiber by-products, which can be found in sorghum, corn, sugarcane, carrot, and wheat [2, 3, 4, 5]. Understanding how bioethanol synthesis works in a bioreactor is the main focus of many scientists and researchers due to the processes that convert biomass into usable biomaterials having the potential to increase the economic value of currently discarded raw materials and also lower the amount of wastewater discharged by various industries. Mathematical modeling has the potential to save lives, help in policy and decision-making, and optimize economic growth. The construction of mathematical models for ethanol-produced processes can be a highly useful tool for forecasting the complicated phenomena that occur in continuous industrial ethanol procedures. To examine the dynamic behavior, a range of mathematical models based on differential equations were developed by many researchers. For example, in 2005, Mahecha-Botero *et al.* [6] investigated the dynamic behavior of the fermentor's bifurcation as well as the impact of ethanol membrane removal on the behavior, yield, and productivity of the process. In 2013, Isla *et al.* [7] studied the model describing the rates of biomass growth, substrate consumption, and ethanol production from wastewater in the soft drinks industry. In 2016, Comelli *et al.* [8] focused on the performance of commercial Saccharomyces yeast strains. The fermentation process of Saccharomyces bayanus was modeled by using a strict kinetic modeling methodology and the model described biomass growth, sugar consumption, and bioethanol production was developed from the An drews equation. Later, in 2018, Bhowmik and Alqahtani [9] studied the bio-ethanol production model as follows:

$$\begin{cases} \mathscr{V}\dfrac{d\mathscr{S}}{dt} = \mathscr{F}(\mathscr{S}_0-\mathscr{S}) - \dfrac{\mu_{\max}}{\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}}}\mathscr{M}_2(\mathscr{S},\mathscr{E})\mathscr{B}\mathscr{V}, \\ \mathscr{V}\dfrac{d\mathscr{B}}{dt} = -\mathscr{F}\mathscr{B} + \mu_{\max}\mathscr{M}_2(\mathscr{S},\mathscr{E})\mathscr{B}\mathscr{V} + \mathscr{R}\mathscr{F}(\mathscr{C}-1)\mathscr{B} - b_{\mathscr{H}}\mathscr{B}\mathscr{V}, \\ \mathscr{V}\dfrac{d\mathscr{E}}{dt} = -\mathscr{F}\mathscr{E} + \Upsilon_{\frac{\mathscr{E}}{\mathscr{B}}}\mu_{\max}\mathscr{M}_2(\mathscr{S},\mathscr{E})\mathscr{B}\mathscr{V} + \mathscr{R}\mathscr{V}(\mathscr{C}-1)\mathscr{E} + \mathscr{F}\gamma\mathscr{B}, \end{cases} \tag{1.1}$$

where $\mathscr{M}_2$ denotes the modified Andrew expression with inhibition by ethanol. $\mathscr{M}_2$ and residence time τ are given by

$$\mathscr{M}_2(\mathscr{S},\mathscr{E}) = \frac{\mathscr{S}}{\mathscr{K}_{\mathscr{S}} + \mathscr{S} + \mathscr{K}_{\mathscr{E}}\mathscr{E}^2}, \qquad \tau = \frac{\mathscr{V}}{\mathscr{F}},$$

and the initial conditions $(\mathscr{S}(0),\mathscr{B}(0),\mathscr{E}(0)) = (\mathscr{S}_0,\mathscr{B}_0,\mathscr{E}_0)$ with $\mathscr{S}_0 \geq 0$, $\mathscr{B}_0 \geq 0$, and $\mathscr{E}_0 \geq 0$. The three elements of model (1.1) are all non-negative. The descriptions of unknown symbols are described in Table 1. Next, (1.1) can be re-formulated by replacing the dimensionless variables as follows:

$$\mathscr{S}^* = \frac{\mathscr{S}}{\mathscr{K}_{\mathscr{S}}}, \quad \mathscr{B}^* = \frac{\mathscr{B}}{\Upsilon_{\frac{\mathscr{E}}{\mathscr{B}}}\mathscr{K}_{\mathscr{S}}}, \quad \mathscr{E}^* = \frac{\mathscr{E}}{\mathscr{K}_{\mathscr{S}}}, \quad \xi = \varepsilon_{\max} t.$$

TABLE 1. The details of unknown variables and parameters of the model (1.1)

Symbols	Descriptions
$\mathscr{S}(t), \mathscr{B}(t), \mathscr{E}(t)$	The substrate, biomass, and ethanol concentrations at time t, respectively.
τ	The residence time.
$\mathscr{V}$	The volume of the reactor.
$\mathscr{M}_2(\mathscr{S}, \mathscr{E})$	The rate of specific growth rate.
$\mathscr{F}$	The flow rate by means of reactor.
$\mu_{\max}$	The maximum specific growth rate.
$b_{\mathscr{H}}$	The coefficient of death.
γ	The ethanol production's kinetic constant.
$\mathscr{R}$	The recycling ratio, which is computed using volume flow rates.
$\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}}, \Upsilon_{\frac{\mathscr{E}}{\mathscr{S}}}$	The biomass and ethanol/biomass yield coefficients, respectively.
$\mathscr{K}_{\mathscr{S}}, \mathscr{K}_{\mathscr{E}}$	The saturation constant and the inhibition constant by ethanol, respectively.

Then, (1.1) can be re-written in dimensionless form by

$$\begin{cases} \dfrac{d\mathscr{S}^*}{d\xi} = \dfrac{\mathscr{S}_0^* - \mathscr{S}^*}{\tau^*} - \dfrac{\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2}, \\ \dfrac{d\mathscr{B}^*}{d\xi} = -\dfrac{\mathscr{B}^*}{\tau^*} + \dfrac{\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2} - b^*_{\mathscr{H}}\mathscr{B}^* + \dfrac{\mathscr{R}^*\mathscr{B}^*}{\tau^*}, \\ \dfrac{d\mathscr{E}^*}{d\xi} = -\dfrac{\mathscr{E}^*}{\tau^*} + \dfrac{\gamma_2\mathscr{B}^*}{\tau^*} + \dfrac{\gamma_3\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2} + \dfrac{\mathscr{R}^*\mathscr{E}^*}{\tau^*}, \end{cases} \tag{1.2}$$

where

$$b^*_{\mathscr{H}} = \frac{b_{\mathscr{H}}}{\mu_{\max}}, \quad \tau^* = \frac{\mathscr{V}\mu_{\max}}{\mathscr{F}}, \quad \gamma_1 = \mathscr{K}_{\mathscr{E}}\mathscr{K}_{\mathscr{S}}, \quad \gamma_2 = \gamma\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}}, \quad \gamma_3 = \Upsilon_{\frac{\mathscr{E}}{\mathscr{B}}}\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}}.$$

Note that all the parameters of model (1.2) are non-negative with $\mathscr{S}^*(0) = \mathscr{S}_0^* \geq 0$, $\mathscr{B}^*(0) = \mathscr{B}_0^* \geq 0$, and $\mathscr{E}^*(0) = \mathscr{E}_0^* \geq 0$. The details of the symbols utilized in model (1.1) are given in Table 2. In 2020, Ciesielski and Grzywacz [10] investigated the bifurcation analysis of the

TABLE 2. The details of unknown variables and parameters of the model (1.1)

Symbols	Descriptions
$\mathscr{S}^*(\xi), \mathscr{B}^*(\xi), \mathscr{E}^*(\xi)$	The dimensionless substrate, biomass, and ethanol concentrations at time ξ, respectively.
$\tau^*, b^*_{\mathscr{H}}$	The dimensionless residence time and death rate, respectively.
$\mathscr{R}^*$	The effective recycle.

oscillatory phenomena occurring in the process of bioethanol production by aerobic continuous fermentation with Saccharomyces cerevisiae. In 2021, Alqahtani *et al.* [11] studied the model of ethanol production in a continuously stirred bioreactor with recycling and analyzed the stability of the reactor-separator system. It has been noticed that this topic has attracted many researchers in the area of applied science and engineering.

Fractional calculus was invented and developed to describe a variety of complex problems. It is a huge knowledge resource that has drawn the interest of numerous academics and researchers

since the fractional order models provide more precise and reliable information on dynamical behavior than mathematical models with integer order [12, 13, 14]. Fractional operators based on the kernels utilized were widely applied in mathematical models to address a variety of real-world problems. Examples include the Liouville-Caputo operator, which has a power law kernel; the Caputo and Fabrizio operator, which has an exponential decay law kernel; and the Atangana-Baleanu operator, which has a Mittag-Leffler kernel [15, 16, 17]. Furthermore, Atangana [18] recently introduced the fractal-fractional operators, which integrated fractional and fractal derivatives. These operators with constant order were defined and applied in conjunction with differential systems in numerous literature. Despite the fact that fractional operators are widely used in many biochemical models [19, 20, 21], we could only identify a few pieces of literature applied them to models of the bioethanol phenomenon. For example, in 2021, Alqahtani *et al.* [22] investigated the stability of the fractional-order in the bioethanol production model under a generalized nonlocal operator with the sense of Caputo's type. Later, they [23] continued their work to study the fractal-fractional operators with three different kernels in the numerical analysis of the bioethanol production model with the effect of recycling and mortality rates. In addition, it can be seen that constant-order fractal-fractional calculus has significantly advanced. Here, we refer to [24, 25, 26, 27, 28]. Variable-order fractional derivatives [29] are very helpful in examining memory characteristics that vary with time and space, and they have received a lot of interest in their varying-order functions. Some studies demonstrated that variable-order fractional derivatives can be used to successfully represent a wide variety of difficult physical situations; see, e.g., [30, 31, 32, 33, 34] and the references cited therein. The qualitative results are essential highlights of the fundamental principles of mathematical model research in the discussion of the qualitative theory of each of the mentioned models, such as existence, uniqueness, and stability results. The qualitative theory of the solutions for these mentioned models helps us extend this to discover other qualities of solutions such as stability, equilibrium, numerical solutions, and their simulations. In this field, fixed point theory and Ulam's stability are usually effective, and their importance could be seen in boundary and initial value problems; see, e.g., [35, 36, 37, 38, 39] and the references therein. The fractal-fractional operators are the most recent, and there is no much research on their applications yet in the literature. We aim to study the bioethanol model (1.2) proposed by [9], who applied the model with the generalized nonlocal operator in the Caputo sense. To make it different and new, the spotlight in this study is to utilize the fractal-fractional derivatives with variable order for three different kernels, including a power law, an exponential decay law, and Mittag-Leffler. First, we demonstrate the existence and uniqueness of the solutions by using the fixed point result of Banach's. Second, we examine the stability of the solutions by using various Ulam's stability. Third, we find the approximated solutions by using the Adams-Bashforth method. Finally, the dynamics of the components of ethanol synthesis for all different kernels are demonstated by a graphical depiction of the results.

The rest of this work is structured as follows. Section 2 gives formulations of fixed-point theorems and the basic concepts of fractal-fractional operators with variable order in the sense of Liouville-Caputo, Caputo-Fabrizio, and Atangana-Baleanu. Section 3 presents the structure of models for bioethanol production under fractal-fractional operators via three kernels with variable order. The unique result for the considered model is established by using Banach's

fixed-point theory as well as various categories of Ulam's stability, which are extensively verified in Section 4. In Section 5, the numerical schemes utilized by the Adams-Bashforth method for the three kernels of the considered model are derived. Section 6 presents the discussion and numerical simulations in graphical illustrations, while Section 7, the last section, provides a summary of the entire study.

2. Preliminaries

Assume that $\alpha(t)$ and $\beta(t)$ are the fractional variable-order and fractal variable dimension, respectively, and suppose that $\mathscr{G}(t)$ is a continuous and fractal differentiable on (a,b) with order $\beta(t)$. Next, we provide some basic concepts of the fractal-fractional calculus with variable order that are required for our work.

Definition 2.1. ([29]). The fractal-fractional derivative of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ in the Liouville-Caputo sense with the power-law kernel is defined by

$$ {}^{\mathbb{FFP}}_{t}\mathfrak{D}_0^{(\alpha(t),\beta(t))}\mathscr{G}(t) = \frac{1}{\Gamma(m-\alpha(t))}\cdot\frac{d}{dt^{\beta(t)}}\int_0^t (t-s)^{m-\alpha(t)-1}\mathscr{G}(s)ds, \tag{2.1} $$

where $m-1<\alpha(t),\beta(t)\leq m$, $m\in\mathbb{N}$, $\Gamma(z)$ is the classical (Euler's) Gamma function givened, for $Re(z)>0$, by $\Gamma(z)=\int_0^\infty s^{z-1}e^{-s}ds$, and the fractal derivative of variable-order $\beta(t)$ is given by

$$ \frac{d\mathscr{G}(s)}{ds^{\beta(t)}} = \lim_{t\to s}\left\{\frac{\mathscr{G}(t)-\mathscr{G}(s)}{t^{\beta(t)}-s^{\beta(t)}}\right\}. $$

Definition 2.2. ([29]). The fractal-fractional derivative of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ in the Caputo-Fabrizio sense with the exponential decay kernel is defined by

$$ {}^{\mathbb{FFE}}_{t}\mathfrak{D}_0^{(\alpha(t),\beta(t))}\mathscr{G}(t) = \frac{\mathbb{M}(\alpha(t))}{1-\alpha(t)}\cdot\frac{d}{dt^{\beta(t)}}\int_0^t \exp\left(-\frac{\alpha(t)}{1-\alpha(t)}(t-s)\right)\mathscr{G}(s)ds, \tag{2.2} $$

where $\alpha(t)>0$, $\beta(t)\leq m$, $m\in\mathbb{N}$, and $\mathbb{M}(\alpha(t))=2/(2-\alpha(t))$, $\alpha(t)\in(0,1]$ with $\mathbb{M}(0)=\mathbb{M}(1)=1$.

Definition 2.3. ([29]). The fractal-fractional derivative of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ in the Atangana-Baleanu sense with the generalized Mittag-Leffler kernel is defined by

$$ {}^{\mathbb{FFM}}_{t}\mathfrak{D}_0^{(\alpha(t),\beta(t))}\mathscr{G}(t) = \frac{\mathbb{AB}(\alpha(t))}{1-\alpha(t)}\cdot\frac{d}{dt^{\beta(t)}}\int_0^t \mathbb{E}_{\alpha(t)}\left[-\frac{\alpha(t)}{1-\alpha(t)}(t-s)^{\alpha(t)}\right]\mathscr{G}(s)ds, \tag{2.3} $$

where $0<\alpha(t),\beta(t)\leq 1$, $\mathbb{AB}(\alpha(t))=1-\alpha(t)+\alpha(t)/\Gamma(\alpha(t))$ with $\mathbb{AB}(0)=\mathbb{AB}(1)=1$, and the Mittag-Leffler function is defined by

$$ \mathbb{E}_{\alpha(t)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha(t)k+1)}, \quad z,\alpha\in\mathbb{C}, \quad Re(\alpha(t))>0, \tag{2.4} $$

with $\mathbb{C}$ is the set of the complex number.

Definition 2.4. ([29]). The fractal-fractional integral of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ with the power-law kernel is defined by

$$ {}^{\mathbb{FFP}}_{t}\mathscr{I}_0^{\alpha(t),\beta(t)}\mathscr{G}(t) = \frac{\beta(t)}{\Gamma(\alpha(t))}\int_0^t s^{\alpha(t)-1}(t-s)^{\alpha(t)-1}\mathscr{G}(s)ds. \tag{2.5} $$

Definition 2.5. ([29]). The fractal-fractional integral of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ with the exponential decay kernel is defined by

$$ {}^{\mathbb{FFE}}_{t}\mathscr{I}_{0}^{\alpha(t),\beta(t)}\mathscr{G}(t)=\frac{(1-\alpha(t))\beta(t)t^{\beta(t)-1}\mathscr{G}(t)}{\mathbb{M}(\alpha(t))}+\frac{\alpha(t)\beta(t)}{\mathbb{M}(\alpha(t))}\int_{0}^{t}s^{\alpha(t)-1}\mathscr{G}(s)ds. \tag{2.6}$$

Definition 2.6. ([29]). The fractal-fractional integral of $\mathscr{G}(t)$ of variable-order $(\alpha(t),\beta(t))$ with the generalized Mittag-Leffler kernel is defined by

$$ {}^{\mathbb{FFM}}_{t}\mathscr{I}_{0}^{\alpha(t),\beta(t)}\mathscr{G}(t)=\frac{(1-\alpha(t))\beta(t)t^{\beta(t)-1}\mathscr{G}(t)}{\mathbb{AB}(\alpha(t))}+\frac{\alpha(t)\beta(t)}{\mathbb{AB}(\alpha(t))}\int_{0}^{t}s^{\alpha(t)-1}\mathscr{G}(s)ds. \tag{2.7}$$

Theorem 2.7. (Banach's fixed point theorem [40]). *Let $\mathbb{X}$ be a Banach space, and $\mathbb{W}\subset\mathbb{X}$, $\mathbb{W}\neq\emptyset$ be a closed subset. If the operator $\mathscr{Q}:\mathbb{W}\to\mathbb{W}$ holds the contraction condition, then $\mathscr{Q}$ has a unique fixed point in $\mathbb{W}$.*

3. Model Construction

This section proves both the existence and uniqueness results of model (1.2) in the sense of the power law (2.1), the exponential decay (2.2), and the generalized Mittag-Leffler kernels (2.3), and the Ulam's stability of the considered model. For ease of computation, we set the symbols

$$\begin{aligned}
\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) &= \frac{\mathscr{S}_0^*-\mathscr{S}^*}{\tau^*}-\frac{\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2}, &(3.1)\\
\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) &= -\frac{\mathscr{B}^*}{\tau^*}+\frac{\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2}-b^*_{\mathscr{H}}\mathscr{B}^*+\frac{\mathscr{R}^*\mathscr{B}^*}{\tau^*}, &(3.2)\\
\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) &= -\frac{\mathscr{E}^*}{\tau^*}+\frac{\gamma_2\mathscr{B}^*}{\tau^*}+\frac{\gamma_3\mathscr{S}^*\mathscr{B}^*}{1+\mathscr{S}^*+\gamma_1(\mathscr{E}^*)^2}+\frac{\mathscr{R}^*\mathscr{E}^*}{\tau^*}. &(3.3)
\end{aligned}$$

3.1. The $\mathbb{VOFF}$-$\mathbb{BEP}$ via the power law kernel.

The model (1.2) is modified by replacing the ordinary derivative with the fractal-fractional derivative under the power-law kernel as follows:

$$\begin{cases}
{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_{0}^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi)=\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*),\\
{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_{0}^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi)=\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*),\\
{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_{0}^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi)=\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*),
\end{cases} \tag{3.4}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), respectively. Model (3.4) can be re-constructed as:

$$\begin{cases}
{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_{0}^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi)=\mathscr{G}(\xi,\mathbb{U}(\xi)),\\
\mathbb{U}(0)=\mathbb{U}_0\geq 0,
\end{cases} \tag{3.5}$$

where

$$\mathbb{U}=\begin{pmatrix}\mathscr{S}^*(\xi)\\ \mathscr{B}^*(\xi)\\ \mathscr{E}^*(\xi)\end{pmatrix},\mathbb{U}(0)=\begin{pmatrix}\mathscr{S}^*(0)\\ \mathscr{B}^*(0)\\ \mathscr{E}^*(0)\end{pmatrix},\mathscr{G}(\xi,\mathbb{U}^*(\xi))=\begin{pmatrix}\mathscr{G}_1(\xi)\\ \mathscr{G}_2(\xi)\\ \mathscr{G}_3(\xi)\end{pmatrix}=\begin{pmatrix}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\\ \mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\\ \mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\end{pmatrix}. \tag{3.6}$$

Model (3.4) can be re-written to the Volterra type since the fractional integral is differentiable. Then the fractal-fractional derivative in the Riemann-Liouville sense can be transformed to the following model

$$\begin{cases} {}^{\mathbb{FFP}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFP}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFP}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \end{cases} \tag{3.7}$$

Model (3.7) can be re-formulated as:

$$\begin{cases} {}^{\mathbb{FFP}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}(\xi,\mathbb{U}(\xi)), \\ \mathbb{U}(0) = \mathbb{U}_0 \geq 0, \end{cases} \tag{3.8}$$

To apply the integral initial conditions, operator ${}^{\mathbb{FFP}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$ should be replaced by ${}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$. Using the fractal-fractional integral (2.5) to both sides of (3.8), we obtain the following equation:

$$\mathbb{U}(\xi) = \mathbb{U}_0 + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{U}(s))ds. \tag{3.9}$$

To analyze the qualitative results of the considered model, we provide a Banach space $\mathbb{X}_1 = \mathscr{C}([0,\mathbb{T}^*]\times\mathbb{R}^3,\mathbb{R})$ equipped with the norm $\|\mathbb{U}\| = \|(\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\| = \sup_{\xi\in[0,\mathbb{T}^*]}|\mathscr{S}^*(\xi) + \mathscr{B}^*(\xi) + \mathscr{E}^*(\xi)|$, If equation (3.9) has a fixed point, then it has a solution. Next, we define an operator $\mathscr{Q}_1 : \mathbb{X}_1 \to \mathbb{X}_1$. In view of (3.9), one can define $\mathscr{Q}_1$ as

$$(\mathscr{Q}_1\mathbb{U})(\xi) = \mathbb{U}_0 + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{U}(s))ds. \tag{3.10}$$

We then transform the considered model to the fixed point problem ($\mathbb{U} = \mathscr{Q}_1\mathbb{U}$).

3.2. **The $\mathbb{VOFF}$-$\mathbb{BEP}$ via the exponential decay kernel.** Model (1.2) is modified by replacing the ordinary derivative with the fractal-fractional derivative under the exponential decay kernel as follows:

$$\begin{cases} {}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi) = \mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi) = \mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi) = \mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \end{cases} \tag{3.11}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), $i = 1,2,3$, respectively. Model (3.14) can be re-constructed as:

$$\begin{cases} {}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi) = \mathscr{G}(\xi,\mathbb{V}(\xi)), \\ \mathbb{V}(0) = \mathbb{V}_0 \geq 0, \end{cases} \tag{3.12}$$

where

$$\mathbb{V} = \begin{pmatrix} \mathscr{S}^*(\xi) \\ \mathscr{B}^*(\xi) \\ \mathscr{E}^*(\xi) \end{pmatrix}, \mathbb{V}(0) = \begin{pmatrix} \mathscr{S}^*(0) \\ \mathscr{B}^*(0) \\ \mathscr{E}^*(0) \end{pmatrix}, \mathscr{G}(\xi,\mathbb{V}^*(\xi)) = \begin{pmatrix} \mathscr{G}_1(\xi) \\ \mathscr{G}_2(\xi) \\ \mathscr{G}_3(\xi) \end{pmatrix} = \begin{pmatrix} \mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\ \mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\ \mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \end{pmatrix}. \tag{3.13}$$

Model (3.11) can be re-written to the Volterra type since the fractional integral is differentiable, then the fractal-fractional derivative in the Riemann-Liouville sense can be transformed to the following model

$$\begin{cases} {}^{\mathbb{FFE}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFE}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFE}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \end{cases} \tag{3.14}$$

Model (3.14) can be re-formulated as:

$$\begin{cases} {}^{\mathbb{FFE}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}(\xi,\mathbb{V}(\xi)), \\ \mathbb{V}(0) = \mathbb{V}_0 \geq 0, \end{cases} \tag{3.15}$$

To apply the integral initial conditions, operator ${}^{\mathbb{FFE}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$ should be replaced by the operator ${}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$. Using the fractal-fractional integral (2.6) to both sides of (3.15), we obtain the following equation:

$$\mathbb{V}(\xi) = \mathbb{V}_0 + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{V}(\xi)) + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{V}(s))ds. \tag{3.16}$$

To analyze the qualitative results of the considered model, we define a Banach space $\mathbb{X}_2 = \mathscr{C}([0,\mathbb{T}^*]\times\mathbb{R}^3,\mathbb{R})$ equipped with the norm $\|\mathbb{V}\| = \|(\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\| = \sup_{\xi\in[0,\mathbb{T}^*]}|\mathscr{S}^*(\xi) + \mathscr{B}^*(\xi) + \mathscr{E}^*(\xi)|$, $\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^* \in \mathbb{X}_2$. If equation (3.16) has a fixed point, then it has a solution. Next, we define an operator $\mathscr{Q}_2 : \mathbb{X}_2 \to \mathbb{X}_2$. In view of (3.16), one can define $\mathscr{Q}_2$ as

$$(\mathscr{Q}_2\mathbb{V})(\xi) = \mathbb{V}_0 + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{V}(\xi)) + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{V}(s))ds. \tag{3.17}$$

We transform the considered model to the fixed point problem ($\mathbb{V} = \mathscr{Q}_2\mathbb{V}$).

3.3. **The $\mathbb{VOFF}$-$\mathbb{BEP}$ via the generalized Mittag-Leffler kernel.** Model (1.2) is modified by replacing the ordinary derivative with the fractal-fractional derivative under the generalized Mittag-Leffler kernel as follows:

$$\begin{cases} {}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi) = \mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi) = \mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi) = \mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \end{cases} \tag{3.18}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), $i = 1,2,3$, respectively. Model (3.18) can be re-constructed as:

$$\begin{cases} {}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi) = \mathscr{G}(\xi,\mathbb{W}(\xi)), \\ \mathbb{W}(0) = \mathbb{W}_0 \geq 0, \end{cases} \tag{3.19}$$

where

$$\mathbb{W} = \begin{pmatrix} \mathscr{S}^*(\xi) \\ \mathscr{B}^*(\xi) \\ \mathscr{E}^*(\xi) \end{pmatrix},\ \mathbb{W}(0) = \begin{pmatrix} \mathscr{S}^*(0) \\ \mathscr{B}^*(0) \\ \mathscr{E}^*(0) \end{pmatrix},\ \mathscr{G}(\xi,\mathbb{W}^*(\xi)) = \begin{pmatrix} \mathscr{G}_1(\xi) \\ \mathscr{G}_2(\xi) \\ \mathscr{G}_3(\xi) \end{pmatrix} = \begin{pmatrix} \mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\ \mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\ \mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \end{pmatrix}. \tag{3.20}$$

Model (3.18) can be re-written to the Volterra type since the fractional integral is differentiable, then the fractal-fractional derivative in the Riemann-Liouville sense can be transformed to the following model

$$\begin{cases} {}^{\mathbb{FFM}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{S}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFM}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{B}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \\ {}^{\mathbb{FFM}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathscr{E}^*(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*), \end{cases} \tag{3.21}$$

Model (3.21) can be re-formulated as:

$$\begin{cases} {}^{\mathbb{FFM}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi) = \beta(\xi)\xi^{\beta(\xi)-1}\mathscr{G}(\xi,\mathbb{W}(\xi)), \\ \mathbb{W}(0) = \mathbb{W}_0 \geq 0, \end{cases} \tag{3.22}$$

To apply the integral initial conditions, operator ${}^{\mathbb{FFM}_{\mathbb{RL}}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$ should be replaced by the operator ${}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}$. Using the fractal-fractional integral (2.7) to both sides of (3.22), we obtain the following equation:

$$\begin{aligned} \mathbb{W}(\xi) &= \mathbb{W}_0 + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{W}(\xi)) \\ &\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{W}(s))ds. \end{aligned} \tag{3.23}$$

To analyze the qualitative results of the considered model, we define a Banach space $\mathbb{X}_3 = \mathscr{C}([0,\mathbb{T}^*]\times\mathbb{R}^3,\mathbb{R})$ equipped with the norm $\|\mathbb{W}\| = \|(\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)\| = \sup_{\xi\in[0,\mathbb{T}^*]}|\mathscr{S}^*(\xi)+\mathscr{B}^*(\xi)+\mathscr{E}^*(\xi)|$, $\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^* \in \mathbb{X}_3$. If equation (3.23) has a fixed point, then it has a solution. Next, we define an operator $\mathscr{Q}_3 : \mathbb{X}_3 \to \mathbb{X}_3$. In view of (3.23), one can define $\mathscr{Q}_3$ as

$$\begin{aligned} (\mathscr{Q}_3\mathbb{W})(\xi) &= \mathbb{W}_0 + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{W}(\xi)) \\ &\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{W}(s))ds. \end{aligned} \tag{3.24}$$

Then, we transform the considered model to the fixed point problem ($\mathbb{W} = \mathscr{Q}_3\mathbb{W}$).

4. Qualitative Analysis

4.1. **Existence-uniqueness Results.**

Theorem 4.1. *Assume that $\mathscr{G} \in \mathbb{X}_1$ verifies the following assumption:*

($\mathscr{A}_1$) *There exists a positive constant $\mathscr{L}_{\mathscr{G}_{\mathbb{U}}} < 1$ so that*

$$|\mathscr{G}(\xi,\mathbb{U}_1(\xi)) - \mathscr{G}(\xi,\mathbb{U}_2(\xi))| \leq \mathscr{L}_{\mathscr{G}_{\mathbb{U}}}|\mathbb{U}_1(\xi)-\mathbb{U}_2(\xi)|,$$

for all $\mathbb{U}_1$, $\mathbb{U}_2 \in \mathbb{X}_1$ and $\xi \in [0,\mathbb{T}^*]$.

If $\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)\mathscr{L}_{\mathscr{G}_{\mathbb{U}}} < \Gamma(\alpha^*+\beta^*)$, then problem (3.8) has a unique solution, which implies that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.4) has a unique solution.

Proof. Let $\mathscr{G}^*$ be a non-negative constant such that $\sup_{\xi\in[0,\mathbb{T}^*]}|\mathscr{G}(\xi,0)| = \mathscr{G}_1^* < +\infty$, $\alpha^* = \sup_{\xi\in[0,\mathbb{T}^*]}\{\alpha(\xi)\}$ and $\beta^* = \sup_{\xi\in[0,\mathbb{T}^*]}\{\beta(\xi)\}$. Define a bounded, closed, and convex subset of $D_{r_1} = \left\{\mathbb{U}\in\mathbb{X}_1 : \|\mathbb{U}\|_{\mathbb{X}_1}\le r_1\right\}$. Fix a constant

$$r_1 \ge \frac{\|\mathbb{U}_0\|_{\mathbb{X}_1} + \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{G}_1^*}{1-\frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}}.$$

The proof is split into two steps.

Step I: Prove $\mathscr{Q}_1 D_{r_1}\subset D_{r_1}$.

For any $\mathbb{U}\in D_{r_1}$, we have

$$\begin{aligned}
&|(\mathscr{Q}_1\mathbb{U})(\xi)|\\
&\le |\mathbb{U}_0| + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\mathscr{G}(s,\mathbb{U}(s))|\,ds\\
&\le |\mathbb{U}_0| + \frac{\beta(\xi)}{\Gamma(\alpha^*)}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\Big[|\mathscr{G}(s,\mathbb{U}(s))-\mathscr{G}(s,0)| + |\mathscr{G}(s,\mathbb{U}(s))|\Big]ds\\
&\le \|\mathbb{U}_0\|_{\mathbb{X}_1} + \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\left[\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}r_1 + \mathscr{G}_1^*\right],
\end{aligned}$$

which implies that $\mathscr{Q}_1 D_{r_1}\subset D_{r_1}$.

Step II: Prove that $\mathscr{Q}_1$ is a contraction.

For any $\mathbb{U}_1, \mathbb{U}_2\in D_{r_1}$ and $\xi\in[0,\mathbb{T}^*]$, we have

$$\begin{aligned}
&|(\mathscr{Q}_1\mathbb{U}_1)(\xi)-(\mathscr{Q}_1\mathbb{U}_2)(\xi)|\\
&\le \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\mathscr{G}(s,\mathbb{U}_1(s))-\mathscr{G}(s,\mathbb{U}_2(s))|\,ds\\
&\le \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}\|\mathbb{U}_1-\mathbb{U}_2\|_{\mathbb{X}_1},
\end{aligned}$$

which implies that

$$\|\mathscr{Q}_1\mathbb{U}_1-\mathscr{Q}_1\mathbb{U}_2\|_{\mathbb{X}_1}\le\frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}\|\mathbb{U}_1-\mathbb{U}_2\|_{\mathbb{X}_1}.$$

Since $[(\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*))/\Gamma(\alpha^*+\beta^*)]\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}<1$, by the conditions of Banach contraction principle (Lemma 2.7), then $\mathscr{Q}_1$ is said to be a contraction. Hence $\mathscr{Q}_1$ has the unique fixed point which implies that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.4) has a unique solution. □

Theorem 4.2. *Assume that $\mathscr{G}\in\mathbb{X}_2$ verifies the following assumption:*

($\mathscr{A}_2$) *There exists a positive constant $\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}<1$ so that*

$$|\mathscr{G}(\xi,\mathbb{V}_1(\xi))-\mathscr{G}(\xi,\mathbb{V}_2(\xi))|\le\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}|\mathbb{V}_1(\xi)-\mathbb{V}_2(\xi)|,$$

for all $\mathbb{V}_1$, $\mathbb{V}_2\in\mathbb{X}_2$ and $\xi\in[0,\mathbb{T}^*]$.

If $\left(\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)+\alpha^*(\mathbb{T}^*)^{\beta^*}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}<\mathbb{M}(\alpha^*)$, then problem (3.15) has a unique solution which implies that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) has a unique solution.

Proof. Suppose that $\mathscr{G}_2^*$ is a non-negative constant such that $\sup_{\xi\in[0,\mathbb{T}^*]}|\mathscr{G}(\xi,0)|=\mathscr{G}_2^*<+\infty$, $\alpha^*=\sup_{\xi\in[0,\mathbb{T}^*]}\{\alpha(\xi)\}$, $\alpha_*=\inf_{\xi\in[0,\mathbb{T}^*]}\{\alpha(\xi)\}$, and $\beta^*=\sup_{\xi\in[0,\mathbb{T}^*]}\{\beta(\xi)\}$. Set a bounded, closed, and convex subset of $D_{r_2}=\left\{\mathbb{V}\in\mathbb{X}_2:\|\mathbb{V}\|_{\mathbb{X}_2}\leq r_2\right\}$. Fix a constant

$$r_2\geq\frac{\|\mathbb{V}_0\|_{\mathbb{X}_2}+\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{G}_2^*}{1-\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}}.$$

The steps of proof are divided into two steps.

Step I: Prove $\mathscr{Q}_2D_{r_2}\subset D_{r_2}$.

For any $\mathbb{V}\in D_{r_2}$, we have

$$\begin{aligned}
&|(\mathscr{Q}_2\mathbb{V})(\xi)|\\
\leq\;&|\mathbb{V}_0|+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\mathscr{G}(\xi,\mathbb{V}(\xi))|+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}|\mathscr{G}(s,\mathbb{V}(s))|\,ds\\
\leq\;&|\mathbb{V}_0|+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\Big[|\mathscr{G}(\xi,\mathbb{V}(\xi))-\mathscr{G}(\xi,0)|+|\mathscr{G}(\xi,\mathbb{V}(\xi))|\Big]\\
&+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}\Big[|\mathscr{G}(s,\mathbb{V}(s))-\mathscr{G}(s,0)|+|\mathscr{G}(s,\mathbb{V}(s))|\Big]ds\\
\leq\;&\|\mathbb{V}_0\|_{\mathbb{X}_2}+\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}\Big[\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}r_2+\mathscr{G}_2^*\Big]+\frac{\alpha^*\beta^*}{\mathbb{M}(\alpha^*)}\int_0^\xi s^{\beta(\xi)-1}ds\Big[\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}r_2+\mathscr{G}_2^*\Big],
\end{aligned}$$

which yields that $\mathscr{Q}_2D_{r_2}\subset D_{r_2}$.

Step II: Prove that $\mathscr{Q}_2$ is a contraction.

For any $\mathbb{V}_1,\mathbb{V}_2\in D_{r_2}$ and $\xi\in[0,\mathbb{T}^*]$, we have

$$\begin{aligned}
|(\mathscr{Q}_2\mathbb{V}_1)(\xi)-(\mathscr{Q}_2\mathbb{V}_2)(\xi)|\;\leq\;&\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\mathscr{G}(\xi,\mathbb{V}_1(\xi))-\mathscr{G}(\xi,\mathbb{V}_2(\xi))|\\
&+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}|\mathscr{G}(s,\mathbb{V}_1(s))-\mathscr{G}(s,\mathbb{V}_2(s))|\,ds\\
\leq\;&\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}\|\mathbb{V}_1-\mathbb{V}_2\|_{\mathbb{X}_2},
\end{aligned}$$

which yields that

$$\|\mathscr{Q}_2\mathbb{V}_1-\mathscr{Q}_2\mathbb{V}_2\|_{\mathbb{X}_2}\leq\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}\|\mathbb{V}_1-\mathbb{V}_2\|_{\mathbb{X}_2}.$$

Since $[(\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*))/\mathbb{M}(\alpha^*)+(\alpha^*(\mathbb{T}^*)^{\beta^*})/\mathbb{M}(\alpha^*)]\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}<1$, by the conditions of Banach contraction principle (Lemma 2.7), then $\mathscr{Q}_2$ is said to be a contraction. Hence $\mathscr{Q}_2$ has the unique fixed point which implies that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) has a unique solution. □

Theorem 4.3. *Suppose that $\mathscr{G}\in\mathbb{X}_3$ satisfies the following assumption:*

($\mathscr{A}_3$) *There exists a positive constant $\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}<1$ so that*

$$|\mathscr{G}(\xi,\mathbb{W}_1(\xi))-\mathscr{G}(\xi,\mathbb{W}_2(\xi))|\leq\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}|\mathbb{W}_1(\xi)-\mathbb{W}_2(\xi)|,$$

for all $\mathbb{W}_1, \mathbb{W}_2 \in \mathbb{X}_3$ *and* $\xi \in [0, \mathbb{T}^*]$.

If

$$\left(\frac{\beta^*(\mathbb{T}^*)^{\beta^*-1}(1-\alpha^*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \mathscr{L}_{\mathscr{G}_{\mathbb{W}}} < 1$$

then problem (3.22) *has a unique solution which implies that the* $\mathbb{VOFF}$-$\mathbb{BEP}$ *model* (3.18) *has a unique solution.*

Proof. Assume that $\mathscr{G}_3^*$ is a non-negative constant such that $\sup_{\xi\in[0,\mathbb{T}^*]} |\mathscr{G}(\xi,0)| = \mathscr{G}_3^* < +\infty$, $\alpha^* = \sup_{\xi\in[0,\mathbb{T}^*]}\{\alpha(\xi)\}$, $\alpha_* = \inf_{\xi\in[0,\mathbb{T}^*]}\{\alpha(\xi)\}$, and $\beta^* = \sup_{\xi\in[0,\mathbb{T}^*]}\{\beta(\xi)\}$. Set a bounded, closed, and convex subset of $D_{r_3} = \left\{\mathbb{W} \in \mathbb{X}_3 : \|\mathbb{V}\|_{\mathbb{X}_3} \leq r_3\right\}$. Fix a constant

$$r_3 \geq \frac{\|\mathbb{W}_0\|_{\mathbb{X}_3} + \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right)}{1 - \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \mathscr{L}_{\mathscr{G}_{\mathbb{W}}}}.$$

The steps of proof are divided into two steps.

Step I: Prove $\mathscr{Q}_3 D_{r_3} \subset D_{r_3}$.

For any $\mathbb{W} \in D_{r_3}$, we have

$$\begin{aligned}
&|(\mathscr{Q}_3\mathbb{W})(\xi)| \\
\leq\ & |\mathbb{W}_0| + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))} |\mathscr{G}(\xi,\mathbb{W}(\xi))| \\
&+ \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))} \int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1} |\mathscr{G}(s,\mathbb{W}(s))|\, ds \\
\leq\ & |\mathbb{W}_0| + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))} \Big[|\mathscr{G}(\xi,\mathbb{W}(\xi)) - \mathscr{G}(\xi,0)| + |\mathscr{G}(\xi,\mathbb{W}(\xi))| \Big] \\
&+ \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))} \int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1} \Big[|\mathscr{G}(s,\mathbb{W}(s)) - \mathscr{G}(s,0)| + |\mathscr{G}(s,\mathbb{W}(s))| \Big] ds \\
\leq\ & \|\mathbb{W}_0\|_{\mathbb{X}_3} + \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \Big[\mathscr{L}_{\mathscr{G}_{\mathbb{W}}} r_3 + \mathscr{G}_3^* \Big],
\end{aligned}$$

which yields that $\mathscr{Q}_3 D_{r_3} \subset D_{r_3}$.

Step II: Prove that $\mathscr{Q}_2$ is a contraction.

For any $\mathbb{W}_1, \mathbb{W}_2 \in D_{r_3}$ and $\xi \in [0, \mathbb{T}^*]$, we have

$$\begin{aligned}
&|(\mathscr{Q}_3\mathbb{W}_1)(\xi) - (\mathscr{Q}_3\mathbb{W}_2)(\xi)| \\
\leq\ & \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))} |\mathscr{G}(\xi,\mathbb{W}_1(\xi)) - \mathscr{G}(\xi,\mathbb{W}_2(\xi))| \\
&+ \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))} \int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1} |\mathscr{G}(s,\mathbb{W}_1(s)) - \mathscr{G}(s,\mathbb{W}_2(s))|\, ds \\
\leq\ & \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \mathscr{L}_{\mathscr{G}_{\mathbb{W}}} \|\mathbb{W}_1 - \mathbb{W}_2\|_{\mathbb{X}_3},
\end{aligned}$$

which yields that

$$\|\mathscr{Q}_3\mathbb{W}_1-\mathscr{Q}_3\mathbb{W}_2\|_{\mathbb{X}_3}\leq\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)}+\frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}\|\mathbb{W}_1-\mathbb{W}_2\|_{\mathbb{X}_3}.$$

Since $[(\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*))/\mathbb{AB}(\alpha^*)+(\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*))/(\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*))]\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}<1$, by the conditions of Banach contraction principle (Lemma 2.7), then $\mathscr{Q}_3$ is said to be a contraction. Hence $\mathscr{Q}_3$ has the unique fixed point which implies that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) has a unique solution. □

4.2. **Ulam's stability.** In this section, we show that various type of Ulam's stability for $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4), (3.11), and (3.18). Let $\kappa_{\mathscr{G}_{\mathbb{U}}}$, $\kappa_{\mathscr{G}_{\mathbb{V}}}$, $\kappa_{\mathscr{G}_{\mathbb{W}}}>0$ be constants and functions $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}$, $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}$, $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}\in\mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$. Next, we consider the following inequalities

$$\left|{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi)-\mathscr{G}(\xi,\mathbb{U}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{U}}}, \tag{4.1}$$

$$\left|{}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi)-\mathscr{G}(\xi,\mathbb{V}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{V}}}, \tag{4.2}$$

$$\left|{}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi)-\mathscr{G}(\xi,\mathbb{W}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{W}}}, \tag{4.3}$$

$$\left|{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi)-\mathscr{G}(\xi,\mathbb{U}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi), \tag{4.4}$$

$$\left|{}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi)-\mathscr{G}(\xi,\mathbb{V}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi), \tag{4.5}$$

$$\left|{}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi)-\mathscr{G}(\xi,\mathbb{W}(\xi))\right| \leq \kappa_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi), \tag{4.6}$$

$$\left|{}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi)-\mathscr{G}(\xi,\mathbb{U}(\xi))\right| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi), \tag{4.7}$$

$$\left|{}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi)-\mathscr{G}(\xi,\mathbb{V}(\xi))\right| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi), \tag{4.8}$$

$$\left|{}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi)-\mathscr{G}(\xi,\mathbb{W}(\xi))\right| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi), \tag{4.9}$$

where $\xi\in[0,\mathbb{T}^*]$, $\kappa_{\mathscr{G}_{\mathbb{U}}}=\max(\kappa_{\mathscr{G}_{\mathbb{U}_j}})^T$, $\kappa_{\mathscr{G}_{\mathbb{V}}}=\max(\kappa_{\mathscr{G}_{\mathbb{V}_j}})^T$, and $\kappa_{\mathscr{G}_{\mathbb{W}}}=\max(\kappa_{\mathscr{G}_{\mathbb{W}_j}})^T$, $j=1,2,3$.

Definition 4.4. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is called Ulam-Hyers stable if there exists a constant $\Phi_{\mathscr{G}_{\mathbb{U}}}>0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{U}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{U}}\in\mathbb{X}_1$ of (4.1), there exists a solution $\mathbb{U}\in\mathbb{X}_1$ of (3.5) with $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)|\leq\Phi_{\mathscr{G}_{\mathbb{U}}}\kappa_{\mathscr{G}_{\mathbb{U}}}$, where $\xi\in[0,\mathbb{T}^*]$ and $\Phi_{\mathscr{G}_{\mathbb{U}}}=\max(\Phi_{\mathscr{G}_{\mathbb{U}_i}})^T$, $i=1,2,3$.

Definition 4.5. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is called Ulam-Hyers stable if there exists a constant $\Phi_{\mathscr{G}_{\mathbb{V}}}>0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{V}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{V}}\in\mathbb{X}_2$ of (4.2), there exists a solution $\mathbb{V}\in\mathbb{X}_2$ of (3.12) with $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)|\leq\Phi_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}$, where $\xi\in[0,\mathbb{T}^*]$ and $\Phi_{\mathscr{G}_{\mathbb{V}}}=\max(\Phi_{\mathscr{G}_{\mathbb{V}_i}})^T$, $i=1,2,3$.

Definition 4.6. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.18) is called Ulam-Hyers stable if there exists a constant $\Phi_{\mathscr{G}_{\mathbb{W}}}>0$ such that for, every $\kappa_{\mathscr{G}_{\mathbb{W}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{W}}\in\mathbb{X}_3$ of (4.3), there exists a solution $\mathbb{W}\in\mathbb{X}_3$ of (3.19) with $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)|\leq\Phi_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}$, where $\xi\in[0,\mathbb{T}^*]$ and $\Phi_{\mathscr{G}_{\mathbb{W}}}=\max(\Phi_{\mathscr{G}_{\mathbb{W}_i}})^T$, $i=1,2,3$.

Definition 4.7. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is called generalized Ulam-Hyers stable if there exists a function $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ via $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(0)=0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{U}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$ of (4.4), there exists a solution $\mathbb{U} \in \mathbb{X}_1$ of (3.5) with $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\kappa_{\mathscr{G}_{\mathbb{U}}})$, where $\xi \in [0,\mathbb{T}^*]$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{U}_i}})^T$, $i=1,2,3$.

Definition 4.8. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is called generalized Ulam-Hyers stable if there exists a function $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ via $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(0)=0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{V}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$ of (4.5), there exists a solution $\mathbb{V} \in \mathbb{X}_2$ of (3.12) with $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\kappa_{\mathscr{G}_{\mathbb{V}}})$, where $\xi \in [0,\mathbb{T}^*]$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{V}_i}})^T$, $i=1,2,3$.

Definition 4.9. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.18) is called generalized Ulam-Hyers stable if there exists a function $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ via $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(0)=0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{W}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{W}} \in \mathbb{X}_3$ of (4.6), there exists a solution $\mathbb{W} \in \mathbb{X}_3$ of (3.19) with $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)| \leq \mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\kappa_{\mathscr{G}_{\mathbb{W}}})$, where $\xi \in [0,\mathbb{T}^*]$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{W}_i}})^T$, $i=1,2,3$.

Definition 4.10. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is called Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{U}}}>0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{U}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$ of (4.4), there exists a solution $\mathbb{U} \in \mathbb{X}_1$ of (3.5) with $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{U}}}\kappa_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{U}}} = \max(\Omega_{\mathscr{G}_{\mathbb{U}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{U}_i}})^T$, $i=1,2,3$.

Definition 4.11. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is called Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{V}}}>0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{V}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$ of (4.5), there exists a solution $\mathbb{V} \in \mathbb{X}_2$ of (3.12) with $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{V}}} = \max(\Omega_{\mathscr{G}_{\mathbb{V}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{V}_i}})^T$, $i=1,2,3$.

Definition 4.12. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.18) is called Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{W}}}>0$ such that, for every $\kappa_{\mathscr{G}_{\mathbb{W}}}>0$ and for every solution $\mathbb{Z}_{\mathbb{W}} \in \mathbb{X}_3$ of (4.6), there exists a solution $\mathbb{W} \in \mathbb{X}_3$ of (3.19) with $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{W}}} = \max(\Omega_{\mathscr{G}_{\mathbb{W}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{W}_i}})^T$, $i=1,2,3$.

Definition 4.13. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is called generalized Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{U}}}>0$ such that, for every solution $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$ of (4.7), there exists a solution $\mathbb{U} \in \mathbb{X}_1$ of (3.5) with $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{U}}} = \max(\Omega_{\mathscr{G}_{\mathbb{U}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{U}_i}})^T$, $i=1,2,3$.

Definition 4.14. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is called generalized Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{V}}}>0$ such that, for every solution $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$ of (4.8), there exists a solution $\mathbb{V} \in \mathbb{X}_2$ of (3.12) with $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{V}}} = \max(\Omega_{\mathscr{G}_{\mathbb{V}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{V}_i}})^T$, $i=1,2,3$.

Definition 4.15. $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.18) is called generalized Ulam-Hyers-Rassias stable with respect to $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} \in \mathscr{C}([0,\mathbb{T}^*],\mathbb{R}^+)$ if there exists a constant $\Omega_{\mathscr{G}_{\mathbb{W}}}>0$ such that, for every solution $\mathbb{Z}_{\mathbb{W}} \in \mathbb{X}_3$ of (4.9), there exists a solution $\mathbb{W} \in \mathbb{X}_3$ of (3.19) with $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$, where $\xi \in [0,\mathbb{T}^*]$, $\Omega_{\mathscr{G}_{\mathbb{W}}} = \max(\Omega_{\mathscr{G}_{\mathbb{W}_i}})^T$, and $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{W}_i}})^T$, $i=1,2,3$.

Remark 4.16. $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$, $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$, $\mathbb{Z}_{\mathbb{W}} \in \mathbb{X}_3$ be solutions of (4.1), (4.2), and (4.3), respectively, if and only if there exist $\chi_{\mathbb{U}} \in \mathbb{X}_1$, $\chi_{\mathbb{V}} \in \mathbb{X}_2$, $\chi_{\mathbb{W}} \in \mathbb{X}_3$ such that $\chi_{\mathbb{U}}(0)=0$, $\chi_{\mathbb{V}}(0)=0$, $\chi_{\mathbb{W}}(0)=0$, with the following properties

($\mathscr{B}_1$) $|\chi_{\mathbb{U}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{U}}}$, with $\chi_{\mathbb{U}} = \max(\chi_{\mathbb{U}_i})^T$, $(i=1,2,3)$, for $\kappa_{\mathscr{G}_{\mathbb{U}}} > 0$, and ${}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi) = \mathscr{G}(\xi,\mathbb{U}(\xi)) + \chi_{\mathbb{U}}(\xi)$.

($\mathscr{B}_2$) $|\chi_{\mathbb{V}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{V}}}$, with $\chi_{\mathbb{V}} = \max(\chi_{\mathbb{V}_i})^T$, $(i=1,2,3)$, for $\kappa_{\mathscr{G}_{\mathbb{V}}} > 0$, and ${}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi) = \mathscr{G}(\xi,\mathbb{V}(\xi)) + \chi_{\mathbb{V}}(\xi)$.

($\mathscr{B}_3$) $|\chi_{\mathbb{W}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{W}}}$, with $\chi_{\mathbb{W}} = \max(\chi_{\mathbb{W}_i})^T$, $(i=1,2,3)$, for $\kappa_{\mathscr{G}_{\mathbb{W}}} > 0$, and ${}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi) = \mathscr{G}(\xi,\mathbb{W}(\xi)) + \chi_{\mathbb{W}}(\xi)$.

Remark 4.17. Let $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$, $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$, $\mathbb{Z}_{\mathbb{W}} \in \mathbb{X}_3$ be solutions of (4.4), (4.5), and (4.6), respectively, if and only if there exist $\psi_{\mathbb{U}} \in \mathbb{X}_1$, $\psi_{\mathbb{V}} \in \mathbb{X}_2$, and $\psi_{\mathbb{W}} \in \mathbb{X}_3$ such that the following properties

($\mathscr{C}_1$) $|\psi_{\mathbb{U}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$, $\psi_{\mathbb{U}} = \max(\psi_{\mathbb{U}_i})^T$, $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{U}_i}})^T$, for $i = 1,2,3$, and ${}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{U}(\xi) = \mathscr{G}(\xi,\mathbb{U}(\xi)) + \psi_{\mathbb{U}}(\xi)$.

($\mathscr{C}_2$) $|\psi_{\mathbb{V}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$, $\psi_{\mathbb{V}} = \max(\psi_{\mathbb{V}_i})^T$, $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{V}_i}})^T$, for $i = 1,2,3$, and ${}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{V}(\xi) = \mathscr{G}(\xi,\mathbb{V}(\xi)) + \psi_{\mathbb{V}}(\xi)$.

($\mathscr{C}_3$) $|\psi_{\mathbb{W}}(\xi)| \leq \kappa_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$, $\psi_{\mathbb{W}} = \max(\psi_{\mathbb{W}_i})^T$, $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} = \max(\mathscr{P}_{\mathscr{G}_{\mathbb{W}_i}})^T$, for $i = 1,2,3$, and ${}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{W}(\xi) = \mathscr{G}(\xi,\mathbb{W}(\xi)) + \psi_{\mathbb{W}}(\xi)$.

4.2.1. *Ulam-Hyers stability.* First, we show some essential results that are required for considering Ulam's stability for $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4), (3.11), and (3.18).

Lemma 4.18. *Let $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$ be a solution to (4.1). Then*

$$\left|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi)\right| \leq \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\kappa_{\mathscr{G}_{\mathbb{U}}},$$

where

$$\mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi) = \mathbb{Z}_{\mathbb{U}_0} + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{U}}(s))ds. \tag{4.10}$$

Proof. Suppose that $\mathbb{Z}_{\mathbb{U}}$ is the solution to (4.1). By applying ($\mathscr{B}_1$) as in Remark 4.16, we have

$$\begin{cases} {}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{Z}_{\mathbb{U}}(\xi) = \mathscr{G}(\xi,\mathbb{Z}_{\mathbb{U}}(\xi)) + \chi_{\mathbb{U}}(\xi), \\ \mathbb{Z}_{\mathbb{U}}(0) = \mathbb{Z}_{\mathbb{U}_0} \geq 0. \end{cases} \tag{4.11}$$

Then, the solution of problem (4.11) can be written as

$$\begin{aligned} \mathbb{Z}_{\mathbb{U}}(\xi) &= \mathbb{Z}_{\mathbb{U}_0} + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{U}}(s))ds \\ &\quad + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\chi_{\mathbb{U}}(s)ds. \end{aligned}$$

This yields that

$$\begin{aligned} \left|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi)\right| &\leq \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\left|\chi_{\mathbb{U}}(s)\right|ds \\ &\leq \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\kappa_{\mathscr{G}_{\mathbb{U}}}. \end{aligned}$$

Thus inequality (4.10) is required. □

Theorem 4.19. *If ($\mathscr{A}_1$) and Lemma 4.18 hold, then $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is Ulam-Hyers stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{U}}} \in \mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{U}}$ be a solution to (4.1). Assume that $\mathbb{U} \in \mathbb{X}_1$ is a unique solution to the considered problem. Then,

$$\begin{aligned}
&|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{U}(\xi)| \\
&\leq \left| \mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{U}_0 - \frac{\beta(\xi)}{\Gamma(\alpha(\xi))} \int_0^{\xi} s^{\beta(s)-1} (\xi - s)^{\alpha(s)-1} \mathscr{G}(s, \mathbb{U}(s)) ds \right| \\
&\leq \left| \mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{Z}_{\mathbb{U}_0} - \frac{\beta(\xi)}{\Gamma(\alpha(\xi))} \int_0^{\xi} s^{\beta(s)-1} (\xi - s)^{\alpha(s)-1} \mathscr{G}(s, \mathbb{Z}_{\mathbb{U}}(s)) ds \right| \\
&+ \frac{\beta(\xi)}{\Gamma(\alpha(\xi))} \int_0^{\xi} s^{\beta(s)-1} (\xi - s)^{\alpha(s)-1} |\mathscr{G}(s, \mathbb{Z}_{\mathbb{U}}(s)) - \mathscr{G}(s, \mathbb{U}(s))| ds \\
&\leq \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)} \Big(\kappa_{\mathscr{G}_{\mathbb{U}}} + \mathscr{L}_{\mathscr{G}_{\mathbb{U}}} |\mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{U}(\xi)| \Big).
\end{aligned}$$

By choosing a constant

$$\Phi_{\mathscr{G}_{\mathbb{U}}} := \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)} \left(1 - \frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)} \mathscr{L}_{\mathscr{G}_{\mathbb{U}}} \right)^{-1},$$

one sees that $|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{U}(\xi)| \leq \Phi_{\mathscr{G}_{\mathbb{U}}} \kappa_{\mathscr{G}_{\mathbb{U}}}$. Hence, $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is Ulam-Hyers stable. □

By choosing a function $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi) = \Phi_{\mathscr{G}_{\mathbb{U}}} \kappa_{\mathscr{G}_{\mathbb{U}}}$ with $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(0) = 0$ in Theorem 4.19, we obtain that $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.4) is generalized Ulam-Hyers stable.

Lemma 4.20. *Let $\mathbb{Z}_{\mathbb{V}} \in \mathbb{X}_2$ be a solution of (4.2) then*

$$\begin{aligned}
&\left| \mathbb{Z}_{\mathbb{V}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))} \int_0^{\xi} s^{\beta(\xi)-1} \mathscr{G}(s, \mathbb{Z}_{\mathbb{V}}(s)) ds \right| \\
\leq &\left(\frac{\beta^* \xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)} + \frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)} \right) \kappa_{\mathscr{G}_{\mathbb{V}}},
\end{aligned}$$

where

$$\mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi) = \mathbb{Z}_{\mathbb{V}_0} + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))} \mathscr{G}(\xi, \mathbb{Z}_{\mathbb{V}}(\xi)). \tag{4.12}$$

Proof. Suppose that $\mathbb{Z}_{\mathbb{V}}$ is the solution of (4.2). By applying ($\mathscr{B}_2$) as in Remark 4.16, we have

$$\begin{cases} {}^{\mathbb{FFE}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)} \mathbb{Z}_{\mathbb{V}}(\xi) = \mathscr{G}(\xi, \mathbb{Z}_{\mathbb{V}}(\xi)) + \chi_{\mathbb{V}}(\xi), \\ \mathbb{Z}_{\mathbb{V}}(0) = \mathbb{Z}_{\mathbb{V}_0} \geq 0. \end{cases} \tag{4.13}$$

Then, the solution of problem (4.13) can be written as

$$\begin{aligned}
&\mathbb{Z}_{\mathbb{V}}(\xi)\\
&= \mathbb{Z}_{\mathbb{V}_0} + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{V}}(\xi)) + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\\
&+ \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\chi_{\mathbb{V}}(\xi) + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\chi_{\mathbb{V}}(s)ds,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\left|\mathbb{Z}_{\mathbb{V}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\right|\\
\leq\ & \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\chi_{\mathbb{V}}(\xi)| + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}|\chi_{\mathbb{V}}(s)|ds\\
\leq\ & \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)} + \frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\kappa_{\mathscr{G}_{\mathbb{V}}}.
\end{aligned}$$

Thus (4.12) is required. □

Theorem 4.21. *If ($\mathscr{A}_2$) and Lemma 4.20 hold, then $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is Ulam-Hyers stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{V}}} \in \mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{V}}$ be a solution of (4.2). Assume that $\mathbb{V} \in \mathbb{X}_2$ is a unique solution of the considered problem. Then,

$$\begin{aligned}
&|\mathbb{Z}_{\mathbb{V}}(\xi) - \mathbb{V}(\xi)|\\
\leq\ & \left|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathbb{V}_0 - \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{V}(\xi)) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{V}(s))ds\right|\\
\leq\ & \left|\mathbb{Z}_{\mathbb{V}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\right|\\
&+ \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{V}}(\xi)) - \mathscr{G}(\xi,\mathbb{V}(\xi))|\\
&+ \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}|\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s)) - \mathscr{G}(s,\mathbb{V}(s))|ds\\
\leq\ & \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)} + \frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\left(\kappa_{\mathscr{G}_{\mathbb{V}}} + \mathscr{L}_{\mathscr{G}_{\mathbb{V}}}|\mathbb{Z}_{\mathbb{V}}(\xi) - \mathbb{V}(\xi)|\right).
\end{aligned}$$

By taking

$$\Phi_{\mathscr{G}_{\mathbb{V}}} := \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)} + \frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\left(1 - \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)} + \frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}\right)^{-1},$$

one sees hat

$$|\mathbb{Z}_{\mathbb{V}}(\xi) - \mathbb{V}(\xi)| \leq \Phi_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}.$$

Hence, $\mathbb{VOFF}$-$\mathbb{BEP}$ models (3.11) is Ulam-Hyers stable. □

By choosing a function $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)=\Phi_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}$ with $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(0)=0$ in Theorem 4.21, we obtain that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) is generalized Ulam-Hyers stable.

Lemma 4.22. *Let $\mathbb{Z}_{\mathbb{W}}\in\mathbb{X}_3$ be a solution to (4.3). Then*

$$\begin{aligned}&\left|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi)-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\right|\\ \leq\ &\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)}+\frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}\right)\kappa_{\mathscr{G}_{\mathbb{W}}},\end{aligned}$$

where

$$\mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi)=\mathbb{Z}_{\mathbb{W}_0}+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi)).\tag{4.14}$$

Proof. Suppose that $\mathbb{Z}_{\mathbb{W}}$ is the solution to (4.3). By applying $(\mathscr{B}_3)$ as in Remark 4.16, we have

$$\begin{cases}{}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{Z}_{\mathbb{W}}(\xi)=\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi))+\chi_{\mathbb{W}}(\xi),\\ \mathbb{Z}_{\mathbb{W}}(0)=\mathbb{Z}_{\mathbb{W}_0}\geq 0.\end{cases}\tag{4.15}$$

Then, the solution of (4.15) can be written as

$$\begin{aligned}&\mathbb{Z}_{\mathbb{W}}(\xi)\\ &=\mathbb{Z}_{\mathbb{W}_0}+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi))\\ &\quad+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\\ &\quad+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\chi_{\mathbb{W}}(\xi)+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\chi_{\mathbb{W}}(s)ds,\end{aligned}$$

which yields that

$$\begin{aligned}&\left|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi)-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\right|\\ \leq\ &\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}|\chi_{\mathbb{W}}(\xi)|+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\chi_{\mathbb{W}}(s)|ds\\ \leq\ &\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)}+\frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}\right)\kappa_{\mathscr{G}_{\mathbb{W}}}.\end{aligned}$$

Thus (4.14) is obtained. □

Theorem 4.23. *If $(\mathscr{A}_3)$ and Lemma 4.22 hold, then $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is Ulam-Hyers stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{W}}} \in \mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{W}}$ be a solution to (4.3). Assume that $\mathbb{W} \in \mathbb{X}_3$ is a unique solution to the considered problem. Then,

$$\begin{aligned}
&|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}(\xi)| \\
\leq\ & \left| \mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}_0 - \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi, \mathbb{W}(\xi)) \right. \\
& \left. - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s, \mathbb{W}(s))ds \right| \\
\leq\ & \left| \mathbb{Z}_{\mathbb{W}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s, \mathbb{Z}_{\mathbb{W}}(s))ds \right| \\
& + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}|\mathscr{G}(\xi, \mathbb{Z}_{\mathbb{W}}(\xi)) - \mathscr{G}(\xi, \mathbb{W}(\xi))| \\
& + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\mathscr{G}(s, \mathbb{Z}_{\mathbb{W}}(s)) - \mathscr{G}(s, \mathbb{W}(s))|ds \\
\leq\ & \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \left(\kappa_{\mathscr{G}_{\mathbb{W}}} + \mathscr{L}_{\mathscr{G}_{\mathbb{W}}}|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}(\xi)| \right).
\end{aligned}$$

By choosing a constant

$$\Phi_{\mathscr{G}_{\mathbb{W}}} := \frac{\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}}{1 - \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)} \right) \mathscr{L}_{\mathscr{G}_{\mathbb{W}}}},$$

one sees that $|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}(\xi)| \leq \Phi_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}$. Hence, $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is Ulam-Hyers stable. □

By choosing a function $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi) = \Phi_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}$ with $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(0) = 0$ in Theorem 4.23, we obtain that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is generalized Ulam-Hyers stable.

4.2.2. *Ulam-Hyers-Rassias stability results.* Next, we give another assumption that are needed in the sequel.

- ($\mathscr{H}_1$) For any $\xi \in [0, \mathbb{T}^*]$, there exists an increasing function $\mathscr{P}_{\mathscr{G}_{\mathbb{U}}} \in \mathbb{X}_1$ and $\lambda_{\mathscr{G}_{\mathbb{U}}} > 0$ such that ${}^{\mathbb{FFP}}_{t}\mathscr{J}_0^{\alpha(t),\beta(t)}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi) \leq \lambda_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$.
- ($\mathscr{H}_2$) For any $\xi \in [0, \mathbb{T}^*]$, there exits an increasing function $\mathscr{P}_{\mathscr{G}_{\mathbb{V}}} \in \mathbb{X}_2$ and $\lambda_{\mathscr{G}_{\mathbb{V}}} > 0$ such that ${}^{\mathbb{FFE}}_{t}\mathscr{J}_0^{\alpha(t),\beta(t)}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi) \leq \lambda_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$.
- ($\mathscr{H}_3$) For any $\xi \in [0, \mathbb{T}^*]$, there exist an increasing function $\mathscr{P}_{\mathscr{G}_{\mathbb{W}}} \in \mathbb{X}_2$ and $\lambda_{\mathscr{G}_{\mathbb{W}}} > 0$ such that ${}^{\mathbb{FFM}}_{t}\mathscr{J}_0^{\alpha(t),\beta(t)}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi) \leq \lambda_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$.

Lemma 4.24. *Let $\mathbb{Z}_{\mathbb{U}} \in \mathbb{X}_1$ be a solution of (4.7) then $\left|\mathbb{Z}_{\mathbb{U}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi)\right| \leq \kappa_{\mathscr{G}_{\mathbb{U}}}\lambda_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$, where $\mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi)$ is given by (4.10).*

Proof. Assume that $\mathbb{Z}_{\mathbb{U}}$ is the solution to (4.7). By using ($\mathscr{C}_1$) as in Remark 4.17, we have

$$\begin{cases} {}^{\mathbb{FFP}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{Z}_{\mathbb{U}}(\xi) = \mathscr{G}(\xi, \mathbb{Z}_{\mathbb{U}}(\xi)) + \psi_{\mathbb{U}}(\xi), \\ \mathbb{Z}_{\mathbb{U}}(0) = \mathbb{Z}_{\mathbb{U}_0} \geq 0. \end{cases} \tag{4.16}$$

Then, the solution of problem (4.16) can be solved as

$$\begin{aligned}\mathbb{Z}_{\mathbb{U}}(\xi) &= \mathbb{Z}_{\mathbb{U}_0}+\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{U}}(s))ds\\ &\quad+\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\psi_{\mathbb{U}}(s)ds,\end{aligned}$$

which implies that

$$\left|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{U}}}(\xi)\right|\leq\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\left|\psi_{\mathbb{U}}(s)\right|ds\leq\kappa_{\mathscr{G}_{\mathbb{U}}}\lambda_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi).$$

This completes the proof. □

Theorem 4.25. *If $(\mathscr{A}_1)$, $(\mathscr{D}_1)$ and Lemma 4.24 hold, then $\mathbb{V}\mathbb{O}\mathbb{F}\mathbb{F}$-$\mathbb{B}\mathbb{E}\mathbb{P}$ models (3.4) is Ulam-Hyers-Rassias stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{U}}}\in\mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{U}}$ be a solution to (4.7). Assume that $\mathbb{U}\in\mathbb{X}_1$ is a unique solution of the considered problem. By applying Lemma 4.24 with $(\mathscr{A}_1)$, and $(\mathscr{D}_1)$, we have

$$\begin{aligned}|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)| &\leq \left|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}_0-\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{U}(s))ds\right|\\ &\leq \left|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{Z}_{\mathbb{U}_0}-\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{U}}(s))ds\right|\\ &\quad+\frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(s)-1}(\xi-s)^{\alpha(s)-1}\left|\mathscr{G}(s,\mathbb{Z}_{\mathbb{U}}(s))-\mathscr{G}(s,\mathbb{U}(s))\right|ds\\ &\leq \kappa_{\mathscr{G}_{\mathbb{U}}}\lambda_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)+\frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)|.\end{aligned}$$

By setting a constant $\Omega_{\mathscr{G}_{\mathbb{U}}}:=\lambda_{\mathscr{G}_{\mathbb{U}}}\left(1-\frac{\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\Gamma(\alpha^*+\beta^*)}\mathscr{L}_{\mathscr{G}_{\mathbb{U}}}\right)^{-1}$, one has $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)|\leq\Omega_{\mathscr{G}_{\mathbb{U}}}\kappa_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$. Hence, $\mathbb{V}\mathbb{O}\mathbb{F}\mathbb{F}$-$\mathbb{B}\mathbb{E}\mathbb{P}$ models (3.4) is Ulam-Hyers-Rassias stable. □

By taking $\kappa_{\mathscr{G}_{\mathbb{U}}}=1$ into $|\mathbb{Z}_{\mathbb{U}}(\xi)-\mathbb{U}(\xi)|\leq\Omega_{\mathscr{G}_{\mathbb{U}}}\kappa_{\mathscr{G}_{\mathbb{U}}}\mathscr{P}_{\mathscr{G}_{\mathbb{U}}}(\xi)$ as in Theorem 4.25, we obtain that $\mathbb{V}\mathbb{O}\mathbb{F}\mathbb{F}$-$\mathbb{B}\mathbb{E}\mathbb{P}$ models (3.4) is generalized Ulam-Hyers-Rassias stable.

Lemma 4.26. *Let $\mathbb{Z}_{\mathbb{V}}\in\mathbb{X}_2$ be a solution to (4.8). Then*

$$\left|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi)-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\right|\leq\kappa_{\mathscr{G}_{\mathbb{V}}}\lambda_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi),\tag{4.17}$$

where $\mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi)$ is given by (4.12).

Proof. Assume that $\mathbb{Z}_{\mathbb{V}}$ is the solution of (4.8). By using $(\mathscr{C}_2)$ as in Remark 4.17, we have

$$\begin{cases}{}^{\mathbb{F}\mathbb{F}\mathbb{E}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{Z}_{\mathbb{V}}(\xi)=\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{V}}(\xi))+\psi_{\mathbb{V}}(\xi),\\ \mathbb{Z}_{\mathbb{V}}(0)=\mathbb{Z}_{\mathbb{V}_0}\geq 0.\end{cases}$$

Then, its can be solved as

$$\mathbb{Z}_{\mathbb{V}}(\xi)=\mathbb{Z}_{\mathbb{V}_0}+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{V}}(\xi))+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds$$
$$+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\psi_{\mathbb{V}}(\xi)+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}\psi_{\mathbb{V}}(s)ds,$$

which implies that

$$\begin{aligned}&\left|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi)-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\right|\\ \leq\ &\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\psi_{\mathbb{V}}(\xi)|+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}|\psi_{\mathbb{V}}(s)|ds\\ \leq\ &\kappa_{\mathscr{G}_{\mathbb{V}}}\lambda_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi).\end{aligned}$$

Thus (4.17) is obtained. □

Theorem 4.27. *If ($\mathscr{A}_2$), ($\mathscr{D}_2$) and Lemma 4.26 hold, then $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) is Ulam-Hyers-Rassias stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{V}}}\in\mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{V}}$ be a solution to (4.8). Assume that $\mathbb{V}\in\mathbb{X}_2$ is a unique solution of the considered problem. By applying Lemma 4.26 with ($\mathscr{A}_2$), and ($\mathscr{D}_2$), we have

$$\begin{aligned}&|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)|\\ \leq\ &\left|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}_0-\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{V}(\xi))-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{V}(s))ds\right|\\ \leq\ &\left|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{V}}}(\xi)-\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))ds\right|\\ &+\frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}|\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{V}}(\xi))-\mathscr{G}(\xi,\mathbb{V}(\xi))|\\ &+\frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^{\xi}s^{\beta(\xi)-1}|\mathscr{G}(s,\mathbb{Z}_{\mathbb{V}}(s))-\mathscr{G}(s,\mathbb{V}(s))|ds\\ \leq\ &\kappa_{\mathscr{G}_{\mathbb{V}}}\lambda_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)+\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)|.\end{aligned}$$

By setting a constant $\Omega_{\mathscr{G}_{\mathbb{V}}}:=\lambda_{\mathscr{G}_{\mathbb{V}}}\left(1-\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{M}(\alpha^*)}+\frac{\alpha^*(\mathbb{T}^*)^{\beta^*}}{\mathbb{M}(\alpha^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{V}}}\right)^{-1}$, one sees that that $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)|\leq\Omega_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$. Hence, $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) is Ulam-Hyers-Rassias stable. □

By taking $\kappa_{\mathscr{G}_{\mathbb{V}}}=1$ into $|\mathbb{Z}_{\mathbb{V}}(\xi)-\mathbb{V}(\xi)|\leq\Omega_{\mathscr{G}_{\mathbb{V}}}\kappa_{\mathscr{G}_{\mathbb{V}}}\mathscr{P}_{\mathscr{G}_{\mathbb{V}}}(\xi)$ as in Theorem 4.27, we obtain the $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.11) is generalized Ulam-Hyers-Rassias stable.

Lemma 4.28. *Let $\mathbb{Z}_{\mathbb{W}}\in\mathbb{X}_3$ be a solution of (4.9). Then*

$$\left|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi)\right|\leq\kappa_{\mathscr{G}_{\mathbb{W}}}\lambda_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi),\tag{4.18}$$

where $\mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi)$ is given by (4.14).

Proof. Assume that $\mathbb{Z}_{\mathbb{W}}$ is the solution of (4.9). By using $(\mathscr{C}_3)$ as in Remark 4.17, we have

$$\begin{cases} {}^{\mathbb{FFM}}_{\xi}\mathfrak{D}_0^{\alpha(\xi),\beta(\xi)}\mathbb{Z}_{\mathbb{W}}(\xi) = \mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi)) + \psi_{\mathbb{W}}(\xi), \\ \mathbb{Z}_{\mathbb{W}}(0) = \mathbb{Z}_{\mathbb{W}_0} \geq 0. \end{cases}$$

Then its solution can be solved as

$$\begin{aligned} & \mathbb{Z}_{\mathbb{W}}(\xi) \\ = \; & \mathbb{Z}_{\mathbb{W}_0} + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi)) \\ & + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds \\ & + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\psi_{\mathbb{W}}(\xi) + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\psi_{\mathbb{W}}(s)ds, \end{aligned}$$

which implies that

$$\begin{aligned} & \left|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\right| \\ \leq \; & \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}|\psi_{\mathbb{W}}(\xi)| + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\psi_{\mathbb{W}}(s)|ds \\ \leq \; & \kappa_{\mathscr{G}_{\mathbb{W}}}\lambda_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi). \end{aligned}$$

Thus (4.18) is obtained. □

Theorem 4.29. *If $(\mathscr{A}_3)$, $(\mathscr{D}_3)$ and Lemma 4.28 hold, then $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is Ulam-Hyers-Rassias stable.*

Proof. Let $\kappa_{\mathscr{G}_{\mathbb{W}}} \in \mathbb{R}^+$ and $\mathbb{Z}_{\mathbb{W}}$ be a solution of (4.9). Assume that $\mathbb{W} \in \mathbb{X}_3$ is a unique solution of the considered problem. By applying Lemma 4.28 with $(\mathscr{A}_3)$, and $(\mathscr{D}_3)$, we have

$$\begin{aligned} & |\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}(\xi)| \\ \leq \; & \left|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{Z}_{\mathbb{W}_0} - \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi))\right. \\ & \left. - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\right| \\ \leq \; & \left|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathscr{G}_{\mathbb{Z}_{\mathbb{W}}}(\xi) - \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s))ds\right| \\ & + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}|\mathscr{G}(\xi,\mathbb{Z}_{\mathbb{W}}(\xi)) - \mathscr{G}(\xi,\mathbb{W}(\xi))| \\ & + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}|\mathscr{G}(s,\mathbb{Z}_{\mathbb{W}}(s)) - \mathscr{G}(s,\mathbb{W}(s))|ds \\ \leq \; & \kappa_{\mathscr{G}_{\mathbb{W}}}\lambda_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi) + \left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)} + \frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}|\mathbb{Z}_{\mathbb{W}}(\xi) - \mathbb{W}(\xi)|. \end{aligned}$$

By setting a constant $\Omega_{\mathscr{G}_{\mathbb{W}}} := \lambda_{\mathscr{G}_{\mathbb{W}}}\left(1-\left(\frac{\beta^*\xi_{\min}^{\beta^*-1}(1-\alpha_*)}{\mathbb{AB}(\alpha^*)}+\frac{\alpha^*\beta^*(\mathbb{T}^*)^{\alpha^*+\beta^*-1}\Gamma(\beta^*)}{\mathbb{AB}(\alpha^*)\Gamma(\alpha^*+\beta^*)}\right)\mathscr{L}_{\mathscr{G}_{\mathbb{W}}}\right)^{-1}$, one sees that $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$. Hence, $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is Ulam-Hyers-Rassias stable. □

By taking $\kappa_{\mathscr{G}_{\mathbb{W}}} = 1$ into $|\mathbb{Z}_{\mathbb{W}}(\xi)-\mathbb{W}(\xi)| \leq \Omega_{\mathscr{G}_{\mathbb{W}}}\kappa_{\mathscr{G}_{\mathbb{W}}}\mathscr{P}_{\mathscr{G}_{\mathbb{W}}}(\xi)$ as in Theorem 4.29, we obtain that $\mathbb{VOFF}$-$\mathbb{BEP}$ model (3.18) is generalized Ulam-Hyers-Rassias stable.

5. Numerical Schemes for the $\mathbb{VOFF}$-$\mathbb{BEP}$ Model

In this section, we apply Lagrangian piece-wise interpolation [31] to generate numerical schemes for $\mathbb{VOFF}$-$\mathbb{BEP}$ model via power law (2.1), exponential decay (2.2), and generalized Mittag-Leffler kernels (2.3).

5.1. **Adams-Bashforth method for the power law kernel.**

Taking the fractal-fractional integral (2.5) into both sides of (3.7), we have the following equations:

$$\mathscr{S}^*(\xi) = \mathscr{S}_0^* + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_1(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \tag{5.1}$$

$$\mathscr{B}^*(\xi) = \mathscr{B}_0^* + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_2(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \tag{5.2}$$

$$\mathscr{E}^*(\xi) = \mathscr{E}_0^* + \frac{\beta(\xi)}{\Gamma(\alpha(\xi))}\int_0^{\xi} s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_3(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \tag{5.3}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), respectively. Next, we will derive the numerical results of the above model at $\xi = \xi_{n+1}$, the equations (5.1)-(5.3) can be written

$$\mathscr{S}_{n+1}^* = \mathscr{S}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_1(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds, \tag{5.4}$$

$$\mathscr{B}_{n+1}^* = \mathscr{B}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_2(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds, \tag{5.5}$$

$$\mathscr{E}_{n+1}^* = \mathscr{E}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_3(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds. \tag{5.6}$$

Hence, the approximation of system (5.4)-(5.6) is provided below

$$\mathscr{S}_{n+1}^* = \mathscr{S}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_1(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)ds, \tag{5.7}$$

$$\mathscr{B}_{n+1}^* = \mathscr{B}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_2(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)ds, \tag{5.8}$$

$$\mathscr{E}_{n+1}^* = \mathscr{E}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_3(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)ds. \tag{5.9}$$

We apply the Lagrangian piece-wise interpolation in a finite interval $[\xi_n, \xi_{n+1}]$ to approximate the kernal inside the integrals as follows:

$$\begin{aligned}
\mathscr{P}_j^{\mathscr{S}}(s) &= \frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_j^{\beta(\xi_j)-1}\mathscr{G}_1(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*) \\
&\quad -\frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_1(s,\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*), \qquad (5.10)\\
\mathscr{P}_j^{\mathscr{B}}(s) &= \frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_j^{\beta(\xi_j)-1}\mathscr{G}_2(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*) \\
&\quad -\frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_2(s,\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*), \qquad (5.11)\\
\mathscr{P}_j^{\mathscr{E}}(s) &= \frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_j^{\beta(\xi_j)-1}\mathscr{G}_3(s,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*) \\
&\quad -\frac{s-\xi_{j-1}}{\xi_j-\xi_{j-1}}\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_3(s,\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*). \qquad (5.12)
\end{aligned}$$

Substituting (5.10)-(5.12) into (5.7)-(5.9), we have

$$\begin{aligned}
\mathscr{S}_{n+1}^* &= \mathscr{S}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{P}_j^{\mathscr{S}}(s)ds, \qquad (5.13)\\
\mathscr{B}_{n+1}^* &= \mathscr{B}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{P}_j^{\mathscr{B}}(s), \qquad (5.14)\\
\mathscr{E}_{n+1}^* &= \mathscr{E}_0^* + \frac{\beta(\xi_n)}{\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{P}_j^{\mathscr{E}}(s). \qquad (5.15)
\end{aligned}$$

The following numerical scheme was achieved by calculating the right-hand sides of the integral equations (5.13)-(5.15). We have the following formulas

$$\begin{aligned}
\mathscr{S}_{n+1}^* &= \mathscr{S}_0^* + \frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_1(\xi_j,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)\Theta_1(n,j) \\
&\quad -\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_1(\xi_{j-1},\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*)\Theta_2(n,j)\Big], \\
\mathscr{B}_{n+1}^* &= \mathscr{B}_0^* + \frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_2(\xi_j,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)\Theta_1(n,j) \\
&\quad -\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_2(\xi_{j-1},\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*)\Theta_2(n,j)\Big], \\
\mathscr{E}_{n+1}^* &= \mathscr{E}_0^* + \frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_3(\xi_j,\mathscr{S}_j^*,\mathscr{B}_j^*,\mathscr{E}_j^*)\Theta_1(n,j) \\
&\quad -\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_3(\xi_{j-1},\mathscr{S}_{j-1}^*,\mathscr{B}_{j-1}^*,\mathscr{E}_{j-1}^*)\Theta_2(n,j)\Big],
\end{aligned}$$

where

$$\Theta_1(n,j) = (n+1-j)^{\alpha(\xi_j)}(n-j+2+\alpha(\xi_j)) - (n-j)^{\alpha(\xi_j)}(n-j+2+2\alpha(\xi_j)) \qquad (5.16)$$

and

$$\Theta_2(n,j) = (n+1-j)^{\alpha(\xi_{j-1})+1} - (n-j)^{\alpha(\xi_{j-1})}(n-j+1+\alpha(\xi_{j-1})), \tag{5.17}$$

and $\Delta\xi$ is the step size.

5.2. **Adams-Bashforth method for the exponential decay kernel.** Taking the fractal-fractional integral (2.6) into both sides of (3.14), we have the following equations:

$$\begin{aligned}
\mathscr{S}^*(\xi) &= \mathscr{S}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}\mathscr{G}_1(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \qquad (5.18)\\
\mathscr{B}^*(\xi) &= \mathscr{B}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}\mathscr{G}_2(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \qquad (5.19)\\
\mathscr{E}^*(\xi) &= \mathscr{E}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{M}(\alpha(\xi))}\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{M}(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}\mathscr{G}_3(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \qquad (5.20)
\end{aligned}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), respectively. Next, we set $\xi = \xi_{n+1}$, and equations (5.18)-(5.20) are led to the following results

$$\begin{aligned}
\mathscr{S}_{n+1}^* &= \mathscr{S}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_1(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds, \qquad (5.21)\\
\mathscr{B}_{n+1}^* &= \mathscr{B}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_2(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds, \qquad (5.22)\\
\mathscr{E}_{n+1}^* &= \mathscr{E}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_3(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds. \qquad (5.23)
\end{aligned}$$

Hence, the approximation of system (5.21)-(5.23) is as below

$$\begin{aligned}
\mathscr{S}^*_{n+1} &= \mathscr{S}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_1(s,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)ds,\\
\mathscr{B}^*_{n+1} &= \mathscr{B}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_2(s,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)ds,\\
\mathscr{E}^*_{n+1} &= \mathscr{E}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{M}(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{n+1})-1}\mathscr{G}_3(s,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)ds.
\end{aligned}$$

By applying the Lagrange polynomial piece-wise interpolation, we achieve

$$\begin{aligned}
\mathscr{S}^*_{n+1} &= \mathscr{S}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\sum_{j=1}^{n}\left[\Xi_1(j)\mathscr{G}_1(\xi_j,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)-\Xi_2(j)\mathscr{G}_1(\xi_{j-1},\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\right],\\
\mathscr{B}^*_{n+1} &= \mathscr{B}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\sum_{j=1}^{n}\left[\Xi_1(j)\mathscr{G}_2(\xi_j,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)-\Xi_2(j)\mathscr{G}_2(\xi_{j-1},\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\right],\\
\mathscr{E}^*_{n+1} &= \mathscr{E}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{M}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\sum_{j=1}^{n}\left[\Xi_1(j)\mathscr{G}_3(\xi_j,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)-\Xi_2(j)\mathscr{G}_3(\xi_{j-1},\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\right],
\end{aligned}$$

where

$$\begin{aligned}
\Xi_1(j) &= \frac{\alpha(\xi_n)\beta(\xi_n)}{\beta(\xi_{n+1})(\beta(\xi_{n+1}+1))(\Delta\xi)\mathbb{M}(\alpha(\xi_n))}\\
&\quad\times\left\{\xi_{j+1}^{\beta(\xi_{n+1})}\left[2(\Delta\xi)\beta(\xi_{n+1})-\xi_{j-1}\right]-\xi_j^{\beta(\xi_{n+1})}\left[(\Delta\xi)\beta(\xi_{n+1})-\xi_{j-1}\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Xi_2(j) &= \frac{\alpha(\xi_n)\beta(\xi_n)}{\beta(\xi_{n+1})(\beta(\xi_{n+1}+1))(\Delta\xi)\mathbb{M}(\alpha(\xi_n))}\\
&\quad\times\left\{\xi_{j+1}^{\beta(\xi_{n+1})}\left[(\Delta\xi)\beta(\xi_{n+1})-\xi_j\right]+\xi_j^{\beta(\xi_{n+1})+1}\right\}.
\end{aligned}$$

5.3. **Adams-Bashforth method for the generalized Mittag-Leffler kernel.** Taking the fractal-fractional integral (2.7) into both sides of (3.21), we have the following equations:

$$\begin{aligned}
\mathscr{S}^*(\xi) &= \mathscr{S}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}_1(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_1(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \quad (5.24)\\
\mathscr{B}^*(\xi) &= \mathscr{B}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}_2(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_2(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \quad (5.25)\\
\mathscr{E}^*(\xi) &= \mathscr{E}_0^* + \frac{\beta(\xi)\xi^{\beta(\xi)-1}(1-\alpha(\xi))}{\mathbb{AB}(\alpha(\xi))}\mathscr{G}_3(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*) \\
&\quad + \frac{\alpha(\xi)\beta(\xi)}{\mathbb{AB}(\alpha(\xi))\Gamma(\alpha(\xi))}\int_0^\xi s^{\beta(\xi)-1}(\xi-s)^{\alpha(\xi)-1}\mathscr{G}_3(s,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)ds, \quad (5.26)
\end{aligned}$$

where $\mathscr{G}_i(\xi,\mathscr{S}^*,\mathscr{B}^*,\mathscr{E}^*)$ are defined by (3.1)-(3.3), respectively. By setting $\xi=\xi_{n+1}$, the equations (5.24)-(5.26) becomes

$$\begin{aligned}
\mathscr{S}_{n+1}^* &= \mathscr{S}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_1(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds,
\end{aligned} \quad (5.27)$$

$$\begin{aligned}
\mathscr{B}_{n+1}^* &= \mathscr{B}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_2(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds,
\end{aligned} \quad (5.28)$$

and

$$\begin{aligned}
\mathscr{E}_{n+1}^* &= \mathscr{E}_0^* + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*) \\
&\quad + \frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\int_0^{\xi_{n+1}} s^{\beta(\xi_{n+1})-1}(\xi_{n+1}-s)^{\alpha(\xi_{n+1})-1}\mathscr{G}_3(s,\mathscr{S}_n^*,\mathscr{B}_n^*,\mathscr{E}_n^*)ds.
\end{aligned} \quad (5.29)$$

Then, the approximate of the above equations (5.27)-(5.29) is

$$\begin{aligned}
\mathscr{S}^*_{n+1} &= \mathscr{S}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_1(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)ds,\\
\mathscr{B}^*_{n+1} &= \mathscr{B}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_2(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)ds,\\
\mathscr{E}^*_{n+1} &= \mathscr{E}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\alpha(\xi_n)\beta(\xi_n)}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n))}\sum_{j=1}^{n}\int_{\xi_j}^{\xi_{j+1}} s^{\beta(\xi_{j+1})-1}(\xi_{j+1}-s)^{\alpha(\xi_{j+1})-1}\mathscr{G}_3(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)ds.
\end{aligned}$$

By applying the Lagrange polynomial piece-wise interpolation, we achieve

$$\begin{aligned}
\mathscr{S}^*_{n+1} &= \mathscr{S}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_1(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_1(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)\Theta_1(n,j)\\
&\quad-\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_1(s,\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\Theta_2(n,j)\Big],\\
\mathscr{B}^*_{n+1} &= \mathscr{B}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_2(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_2(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)\Theta_1(n,j)\\
&\quad-\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_2(s,\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\Theta_2(n,j)\Big],\\
\mathscr{E}^*_{n+1} &= \mathscr{E}^*_0 + \frac{\beta(\xi_n)\xi_n^{\beta(\xi_n)-1}(1-\alpha(\xi_n))}{\mathbb{AB}(\alpha(\xi_n))}\mathscr{G}_3(\xi_n,\mathscr{S}^*_n,\mathscr{B}^*_n,\mathscr{E}^*_n)\\
&\quad+\frac{\beta(\xi_n)(\Delta\xi)^{\alpha(\xi_n)}}{\mathbb{AB}(\alpha(\xi_n))\Gamma(\alpha(\xi_n)+2)}\sum_{j=1}^{n}\Big[\xi_j^{\beta(\xi_j)-1}\mathscr{G}_3(s,\mathscr{S}^*_j,\mathscr{B}^*_j,\mathscr{E}^*_j)\Theta_1(n,j)\\
&\quad-\xi_{j-1}^{\beta(\xi_{j-1})-1}\mathscr{G}_3(s,\mathscr{S}^*_{j-1},\mathscr{B}^*_{j-1},\mathscr{E}^*_{j-1})\Theta_2(n,j)\Big],
\end{aligned}$$

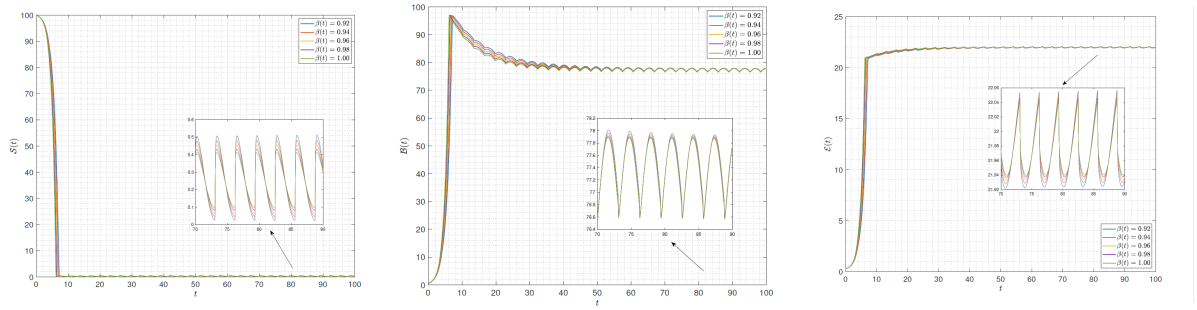
where $\Theta_i(n,j)$, $(i=1,2)$, are given by (5.16)-(5.17), respectively.

6. Graphical Illustrations for the $\mathbb{VOFF}$-$\mathbb{BEP}$ Model

We show the graphical illustrations to analyze the dynamic behavior of the suggested dimensionless nonlinear $\mathbb{VOFF}$-$\mathbb{BEP}$ model with all the different kernels mentioned above, which

are separated into three subsections. Here, we consider kinetics utilizing sugar concentrations frequently used in ethanol synthesis, ranging from 100 to 250 g/L. The parameter values and initial conditions are applied in every subsection as follows: $\gamma = 0.338 g_{ethanol}/g_{biomass}$, $\mu_{\max} = 0.333 h^{-1}$, $b_{\mathscr{H}} = 0.00916 h^{-1}$, $\Upsilon_{\frac{\mathscr{E}}{\mathscr{B}}} = 3.817 g_{ethanol}/g_{biomass}$, $\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}} = 0.054 g_{biomass}/g_{ethanol}$, $\mathscr{K}_{\mathscr{E}} = 0.048 L g_{substrate}/g^2_{ethanol}$, $\mathscr{K}_{\mathscr{S}} = 0.032 g_{substrate}/L$, $\mathscr{S}_0^* = 100$, $\mathscr{B}_0^* = 0.5$, and $\mathscr{E}_0^* = 0.3$. The other parameter values are as follows: $b_{\mathscr{H}^*} = b_{\mathscr{H}}/\mu_{\max} = 0.0275$, $\gamma_1 = \mathscr{K}_{\mathscr{E}}\mathscr{K}_{\mathscr{S}} = 1.5$, $\tau^* = \mathscr{V}\mu_{\max}/\mathscr{F} = 10.5$, $\gamma_2 = \gamma\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}} = 6.2593$, and $\gamma_3 = \Upsilon_{\frac{\mathscr{E}}{\mathscr{B}}}\Upsilon_{\frac{\mathscr{B}}{\mathscr{S}}} = 0.2061$. The investigation of how the evolution of $\mathscr{S}(t)$, $\mathscr{B}(t)$, and $\mathscr{E}(t)$ change in period time are discussed to determine the fairness and make a comparison in this section.

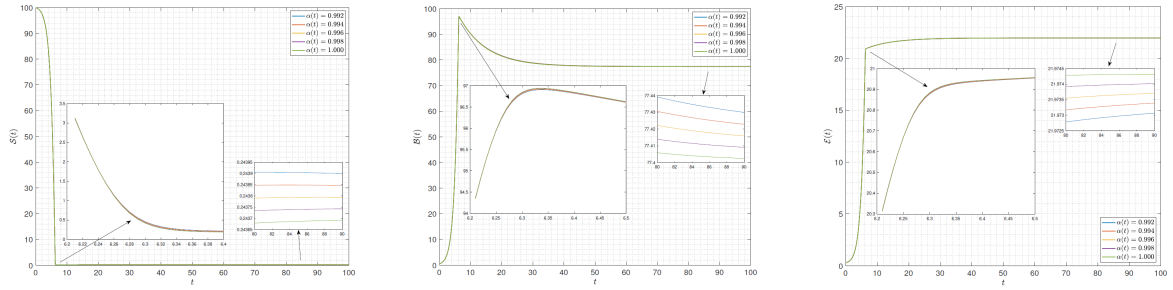
6.1. **The $\mathbb{VOFF}$-$\mathbb{BEP}$ model via the power law kernel.** In this subsection, we split the study into three cases to see the dynamical behavior of the three groups of the $\mathbb{VOFF}$-$\mathbb{BEP}$ model under the power law kernel for various values of $\alpha(t)$ and $\beta(t)$. As seen in Figure 1, we set $\alpha(t) = 0.985 + \frac{1}{100}\left|\cos\left(t + \frac{\pi}{4}\right)\right|$ and $\beta(t)$ varies as $0.92, 0.94, 0.96, 0.98, 1.00$. It can be observed in the beginning that the substrate concentration decreases very rapidly while the biomass concentration quickly increases and then decreases. The ethanol concentration increases very fast before rising a little further, and all three groups become oscillatory stable finally. In Figure 2, we set $\alpha(t)$ varies as $0.992, 0.994, 0.996, 0.998, 1.000$ and $\beta(t) = 0.98 + \sqrt{\frac{e^{-t}}{t+100}}$. We see the dynamic behavior for the substrate, biomass, and ethanol concentration behaving similarly to their behavior in Figure 1, but the graphics do not oscillate, which is the difference. However, all of them achieve stability smoothly as time passes. In Figure 3, we set more complicate variable order as $\alpha(t) = 0.990 + \sqrt{\frac{e^{-t}}{t^2+100}}$, $\alpha(t) = 0.985 + \frac{1}{1000}\sqrt{|\sin(t + \frac{\pi}{6})|}$, $\alpha(t) = 0.995 + \frac{t+1}{t^2+1000}$, $\alpha(t) = 0.995 + \frac{1}{20}\sqrt{\frac{1}{100}\log(t+1)}$, and $\alpha(t) = 0.988 + \frac{e^{-t}}{e^{-t}+100}$ varies and $\beta(t) = 0.985 + \frac{1}{100}|\cos(t + \frac{\pi}{4})|$. In this case, the dynamic behaviors of the three groups are still the same as in the other two instances discussed before. As time pass, they reach stability but are more complex and oscillatory.



(A) (B) (C)

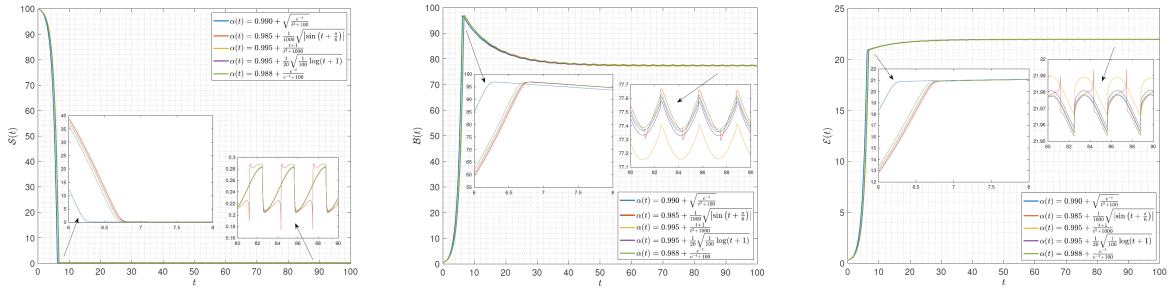
FIGURE 1. Dynamic behavior of $(\mathscr{S}(t), \mathscr{B}(t), \mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the power law kernel when $\alpha(t) = 0.985 + \frac{1}{100}|\cos(t + \frac{\pi}{4})|$ and $\beta(t)$ increases.

6.2. **The $\mathbb{VOFF}$-$\mathbb{BEP}$ via the exponential decay kernel.** In this subsection, we study the dynamical behavior of the three groups of the $\mathbb{VOFF}$-$\mathbb{BEP}$ model under the exponential decay kernel for various values of $\alpha(t)$ and $\beta(t)$. In the same way, we separate the study into three



(A) (B) (C)

FIGURE 2. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the power law kernel when $\alpha(t)$ increases and $\beta(t)=0.98+\sqrt{\frac{e^{-t}}{t+100}}$.

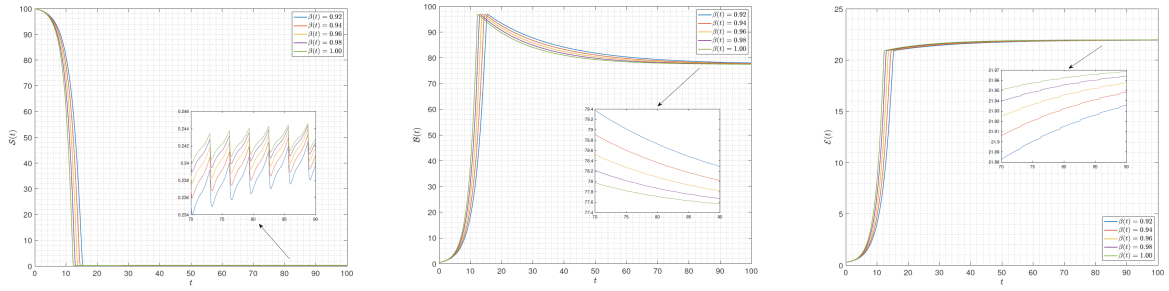


(A) (B) (C)

FIGURE 3. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the power law kernel when $\alpha(t)$ varies and $\beta(t)=0.985+\frac{1}{100}|\cos(t+\frac{\pi}{4})|$.

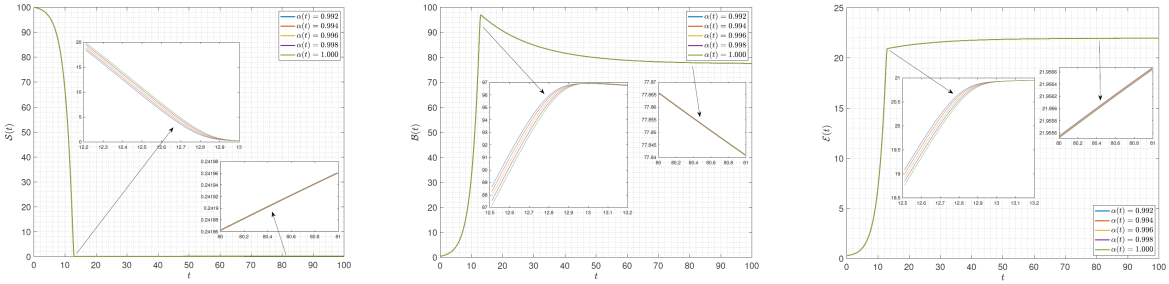
cases and set the value of $\alpha(t)$ and $\beta(t)$ the same as in the previous subsection. Figure 4 shows the dynamic behavior of the substrate, biomass, and ethanol concentration for $\alpha(t)=0.985+\frac{1}{100}|\cos(t+\frac{\pi}{4})|$ and $\beta(t)\in\{0.92,0.94,0.96,0.98,1.00\}$. We observe that the behavior of the substrate concentration rapidly decreases and then stays oscillatory stable. The biomass concentration quickly increases from the start day to the peak values and then decreases until stable and does not oscillate. The ethanol concentration rapidly increases and then rise again a little more until it stable smoothly. However, we notice that their behaviors in this subsection are similar to the previous subsection in the case-by-case, but they take longer to reach a stable state. Figure 5 shows the dynamical behavior of the system for $\alpha(t)\in\{0.992,0.994,0.996,0.998,1.000\}$ increases and $\beta(t)=0.98+\sqrt{\frac{e^{-t}}{t+100}}$. It can be seen that the behaviors of all groups from the start day until they reach a stable state are the same as in the previous case but do not oscillate, and when α is close to 1, all graphs are very close. Figure 6 shows the dynamical behavior of $\mathscr{S}(t)$, $\mathscr{B}(t)$ and $\mathscr{E}(t)$ concentration for $\alpha(t)=0.990+\sqrt{\frac{e^{-t}}{t^2+100}}$, $\alpha(t)=0.985+\frac{1}{1000}\sqrt{|\sin(t+\frac{\pi}{6})|}$, $\alpha(t)=0.995+\frac{t+1}{t^2+1000}$, $\alpha(t)=0.995+\frac{1}{20}\sqrt{\frac{1}{100}\log(t+1)}$, and $\alpha(t)=0.988+\frac{e^{-t}}{e^{-t}+100}$ varies and $\beta(t)=0.985+\frac{1}{100}|\cos(t+\frac{\pi}{4})|$. For this case, the behaviors of all groups from the beginning until they reach a stable state are the same as in the previous case, but all of them are

oscillating stable. Additionally, although they behave similarly to the other two cases, they take longer to stabilize.



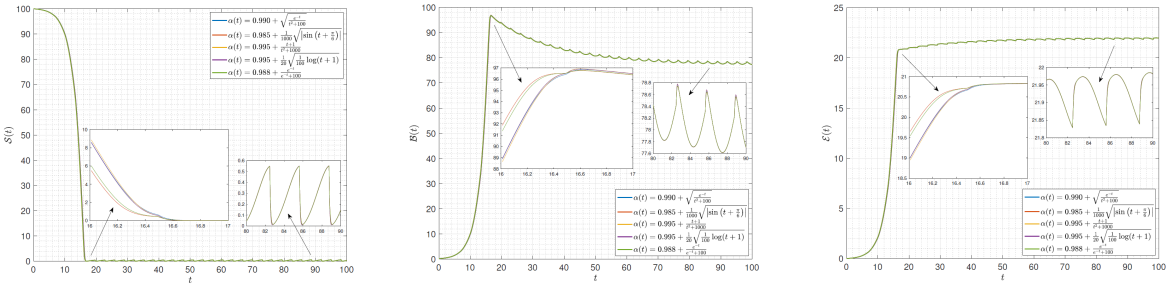
(A) (B) (C)

FIGURE 4. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the exponential decay kernel when $\alpha(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$ and $\beta(t)$ increases.



(A) (B) (C)

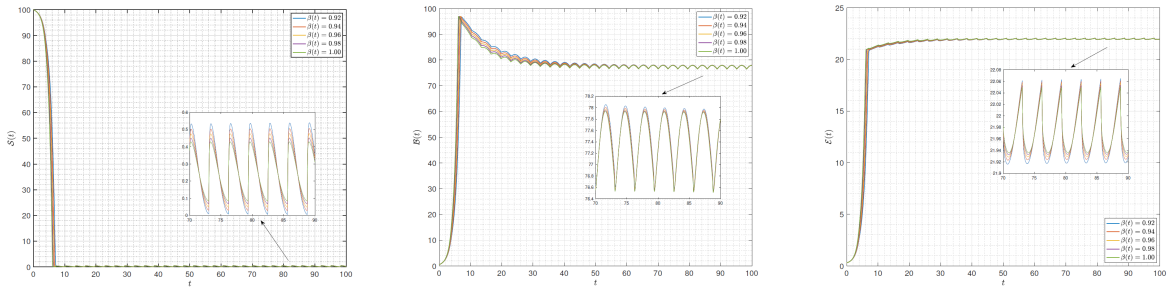
FIGURE 5. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the exponential decay kernel when $\alpha(t)$ increases and $\beta(t) = 0.98 + \sqrt{\frac{e^{-t}}{t+100}}$.



(A) (B) (C)

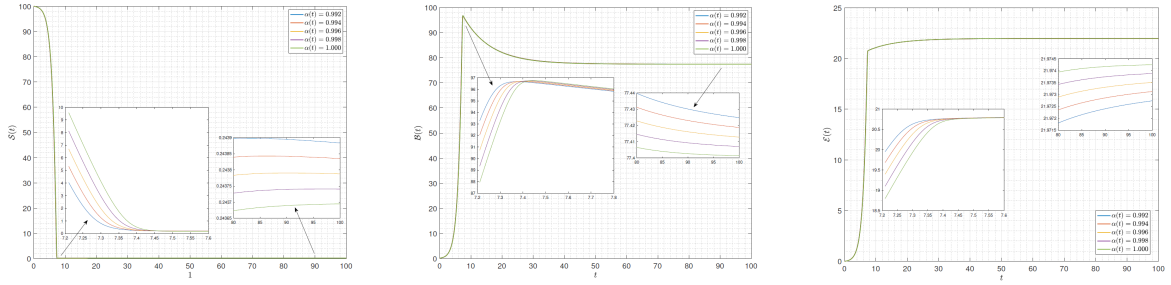
FIGURE 6. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the exponential decay kernel when $\alpha(t)$ varies and $\beta(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$.

6.3. **The $\mathbb{VOFF}$-$\mathbb{BEP}$ via the generalized Mittag-Leffler kernel.** Similar to the other subsections, in this one we investigate the dynamical behavior of the three groups of the $\mathbb{VOFF}$-$\mathbb{BEP}$ model for different values of $\alpha(t)$ and $\beta(t)$ under the generalized Mittag-Leffler kernel. We set $\alpha(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$ and $\beta(t) \in \{0.92, 0.94, 0.96, 0.98, 1.00\}$ in Figure 7, set $\alpha(t) \in \{0.992, 0.994, 0.996, 0.998, 1.000\}$ and $\beta(t) = 0.98 + \sqrt{\frac{e^{-t}}{t+100}}$ in Figure 8, and in Figure 9 we set $\alpha(t) = 0.990 + \sqrt{\frac{e^{-t}}{t^2+100}}$, $\alpha(t) = 0.985 + \frac{1}{1000}\sqrt{|\sin(t+\frac{\pi}{6})|}$, $\alpha(t) = 0.995 + \frac{t+1}{t^2+1000}$, $\alpha(t) = 0.995 + \frac{1}{20}\sqrt{\frac{1}{100}\log(t+1)}$, and $\alpha(t) = 0.988 + \frac{e^{-t}}{e^{-t}+100}$ varies and $\beta(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$. The dynamic behaviors of all groups in Figure 7 to Figure 9 are similar: the substrate concentration decreases quickly from the beginning until it reaches a stable state as oscillatory. The biomass concentration speedily increases and decreases before going to a steady state. The ethanol concentration rapidly increases and gains further until it achieves stable oscillation as time passes.



(A) (B) (C)

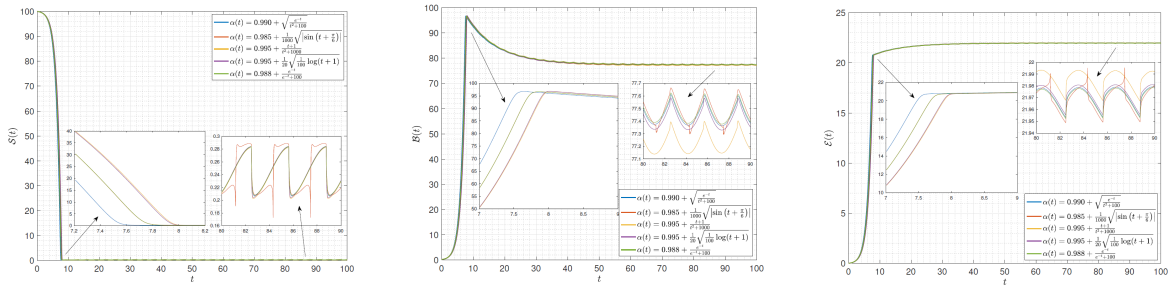
FIGURE 7. Dynamic behavior of $(\mathscr{S}(t), \mathscr{B}(t), \mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the generalized Mittag-Leffler kernel when $\alpha(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$ and $\beta(t)$ increases.



(A) (B) (C)

FIGURE 8. Dynamic behavior of $(\mathscr{S}(t), \mathscr{B}(t), \mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the generalized Mittag-Leffler kernel when $\alpha(t)$ increases and $\beta(t) = 0.98 + \sqrt{\frac{e^{-t}}{t+100}}$.

For the discussion, after studying the $\mathbb{VOFF}$-$\mathbb{BEP}$ model under the three kernels, we discovered that the system behaves somewhat differently when the fractional order has a tiny variation.



(A) (B) (C)

FIGURE 9. Dynamic behavior of $(\mathscr{S}(t),\mathscr{B}(t),\mathscr{E}(t))$ for the $\mathbb{VOFF}$-$\mathbb{BEP}$ via the generalized Mittag-Leffler kernel when $\alpha(t)$ varies and $\beta(t) = 0.985 + \frac{1}{100}|\cos(t+\frac{\pi}{4})|$.

Variable order has an important role in the smoothness or oscillation of the graphical simulation. One notice is when the values of α are integers, all graphs in this setting do not oscillate and reach a steady state when time passes, as seen in Figures 2, 5, and 8. Another notice is when α and β are in variable order, the graphs in each subsection oscillate stably over time, as seen in Figures 3, 6, and 9.

7. CONCLUSIONS

It is known that bio-ethanol is a vital renewable and alternative energy instead of fossil fuels. Moreover, fractional differential equations have been applied to alleviate better some defects associated with the system's dynamic behavior in mathematical models than differential equations with integer order. In this paper, a numerical scheme based on the Adams-Bashforth method interpolation was suggested to obtain numerical solutions for the bio-ethanol model with variable-order fractal-fractional equations involving a power law, an exponential decay law, and Mittag-Leffler kernels. We accomplished the main aim by constructing the designed models via different triple kernels. Then, for each model, we proved the existence and uniqueness of the solutions using Banach's fixed point theorem. We verified the stability analysis by utilizing the various types of Ulam's stability to guarantee the current results. Finally, some illustrations were provided with the help of the Adams-Bashforth method to analyze the dynamic behavior. These graphic simulations and numerical results reveal that the proposed models under these three kernels produce similar outcomes with minor differences depending on the given variable orders. For future-work suggestions, it will be possible to examine real-world situations by using various mathematical models that employ variable-order fractal fractional operators. One may also apply the proposed or other bio-ethanol models in the context of other fascinating fractional operators. It would be very successful if this work would assist researchers in various applied sciences and engineering fields to enrich the works of literature in this area of study.

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