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A STUDY OF NONLINEAR IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract. In this paper, we construct sufficient conditions for the existence and uniqueness of solutions to impulsive nonlinear neutral fractional integro-differential equations with infinite delay complemented with nonlocal boundary conditions with the aid of the Krasnoselskii fixed point theorem and the contraction mapping principle. Examples are also provided to illustrate the main results presented in this paper.
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1. Introduction

The the study of dynamic systems of impulsive differential equations (IDEs) of fractional order gained much attention recently due to the fact that many evolutions and dynamic processes in the real world are affected by abrupt changes, for instance, the population when sudden changes occur at specific instants like harvesting, earthquakes, diseases, and so on. Recently, numerous results were obtained to improve the impulsive controller systems in various fields, such as Lotka-Volterra models [1], biomedical models [2], electrical and mechanical models [3], neural networks models [4], economic models [5, 6], and so on. For theoretical aspects and applications of IDEs, we refer to [7, 8].

Recently, The authors in [9] derived the existence criteria for a class of IDEs with Hadmard fractional derivatives of multi-fractional order. Zhang et al. [10] utilized the lower and upper solutions method for the existence criteria of extremal solutions of muti-fractional order IDEs. Ahmad et al. [11] studied a class of IDEs that involves Caputo-generalized fractional derivatives

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and employed some fixed point theorems to deduce the existence results. In [12], the authors investigated a class of fractional IDEs supplemented with the Robin boundary conditions by utilizing the topological degree theory. Shah et al. [13] discussed the sufficient criteria for the existence of solutions of a multi-point implicit differential equation with impulsive effects. The existence results were obtained for the non-instantaneous impulsive fractional differential equations in [14]. The existence results for Atangana-Baleanu fractional IDEs were established by Kulandhivel et al. [15]. Furthermore, the solvability and stability of CH–fractional IDEs with the aid of coincidence theory were studied Zhao et al. [16].

On the other hand, delay effects on IDEs appear in several applications of mathematical modeling, for example, biomedical models [2], Neural Networks [17], dynamical systems [18], secure communication [19], etc. There is increasing interest in the study of fractional IDEs with unbounded delay. Hilal et al. [20] studied the existence results for a class of fractional neural IDEs with infinite delay. Zhou et al. [21] discussed the controllability of delayed fractional IDEs in Hilbert spaces. The existence and uniqueness results were obtained in [22] for a class of mutli-order Caputo-generalized fractional IDEs with infinite delay. Guo et al. [23] utilized Mönch's theorem for the existence and Hyers-Ulam for the stability of the solutions. The existence results were established of almost periodic solutions to a class of fractional neutral stochastic IDEs with infinite delay in [24]. Bao and Cao [25] derived sufficient criteria for the existence of solutions to a class of fractional stochastic IDEs with infinite delay. However, the results on fractional IDEs with infinite delay are still few, so further research is needed on this topic.

It should be mentioned that dealing with IDEs of fractional order derivatives differs from dealing with those of integer order due to the non-local nature of fractional derivatives, so we should be careful for the fractional derivatives with IDEs. For example, when we apply the Caputo fractional derivative to a function, it should be absolutely continuous or at least continuous and differentiable on the given interval. For this reason, the correct form of IDEs is to apply the derivative to each continuous subinterval, letting the lower bound of the fractional derivative be the lower bound of each subinterval; see [10, 22].

Inspired by the previous work, we, in this paper, investigate a new class of nonlinear IDEs with infinite delay and antiperiodic boundary conditions. Specifically, we discuss the following impulsive problem

$$\begin{cases} {}^{C}D^{\varsigma}_{t_l^+}[u(t)+\int_0^t p(r,u_r)dr] = \psi(t,u_t),\ t\in U',\ \ l=0,1,\dots,q, \\ \Delta u(t_l) = \Psi_l(u(t_l)),\ \ \Delta u'(t_l) = \Psi^*_l(u(t_l)),\ \ l=1,\dots,q, \\ u(t)=\omega(t),\ t\in(-\infty,0], \\ u(0)+u(b) = \sum_{l=0}^{q}\delta_l\, {}^{C}D^{\gamma}_{t_l^+}u(\mu_l)+\mathscr{A},\ t_l<\mu_l<t_{l+1}, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\varsigma}_{t_l^+}$ and ${}^{C}D^{\varkappa}_{t_l^+}$ denote the Caputo derivative operators of order $1<\varsigma\le 2$ and $0<\gamma<1$, respectively, $\psi,p: U\times\mathfrak{N}\to\mathbb{R}$, and $\omega\in\mathfrak{N}$, where $\mathfrak{N}$ is a phase space that will be defined in detail in Section 2, $\Psi_l,\Psi^*_l\in C(\mathbb{R},\mathbb{R}),\delta_l$ is positive constant for $l=0,1,\dots,q$, $U=[0,b], b>0, \mathscr{A}\in\mathbb{R}, 0=t_0<t_1<\dots<t_l<\dots<t_q<t_{q+1}=b, U'=U\backslash\{t_1,t_2,\dots,t_q\}$, and $\Delta u(t_l)=u(t_l^+)-$

$u(t_l^-)$, where $u(t_l^-)$ and $u(t_l^+)$ are the left and right limits of $u(t)$ at $t = t_l (l = 1,2,...,q)$, respectively, and $\Delta u'(t_l)$ have the same definition for $u'(t_l)$. We assume that $u_t : (-\infty, 0] \to \mathbb{R}$ belongs to the phase space $\mathfrak{N}$, with $u_t(r) = u(t+r)$, $r \leq 0$.

We organize the rest of the paper as follows. We recall, in Section 2, the preliminary materials related to our study and deduce the integral equation that is equivalent to the linear variant of (1.1). Next, in Section 3, we establish our main existence results. Finally, examples are given to demonstrate the obtained results.

2. Preliminaries

In this work, we introduce the space $(\mathfrak{N}, \|.\|_{\mathfrak{N}})$ as a seminormed space which contains all functions from $(-\infty, 0]$ into $\mathbb{R}$, such that the following axioms are satisfied (see [26, 27])

($\mathscr{C}_1$) For each $t \in [0,b]$, If $u : (-\infty, b] \to \mathbb{R}$, and $u_0 \in \mathfrak{N}$, then the following conditions hold:
(1) u_t is in $\mathfrak{N}$,
(2) $\|u_t\|_{\mathfrak{N}} \leq \nu(t) \sup\{|u(r)| : 0 \leq r \leq t\} + \rho(t)\|u_0\|_{\mathfrak{N}}$,
(3) $|u(t)| \leq \alpha \|u_t\|_{\mathfrak{N}}$,
where $\nu : [0,b] \to [0,\infty)$ is continuous, $\rho : [0,\infty) \to [0,\infty)$ is locally bounded, $\alpha \geq 0$ is a constant, and α, ν, and ρ are independent of $u(.)$ such that

$$\nu_b = \sup_{t\in[0,b]} |\nu(t)|, \ \rho_b = \sup_{t\in[0,b]} |\rho(t)| \tag{2.1}$$

($\mathscr{C}_2$) For a function $u(.)$ satisfying ($\mathscr{C}_1$), u_t is a $\mathfrak{N}-$valued continuous function on $[0,b]$,
($\mathscr{C}_3$) $\mathfrak{N}$ is a complete space.

Let us set $U_0 = [0,t_1], U_l = (t_l, t_{l+1}], l = 1,2,...,q$, with $t_{q+1} = b$, and define the following Banach spaces:

$PC(U,\mathbb{R}) = \{u : U \to \mathbb{R} : u \in C(U_l,\mathbb{R}), l = 0,1,...,q$, and $u(t_l^+)$ and $u(t_l^-)$ exist with $u(t_l^-) = u(t_l), l = 1,2,...,q\}$, with the norm $\|u\| = \sup_{t\in U} |u(t)|$, where $C(U,\mathbb{R})$ is the space of all continuous functions on U; and $PC^1(U,\mathbb{R}) = \{u' \in PC(U,\mathbb{R}) : u'(t_l^+), u'(t_l^-)$ exist and u' is left continuous at t_l, for $l = 1,2,...,q\}$, with the norm $\|u\| = \sup_{t\in U}\{|u(t)|_{PC}, |u'(t)|_{PC}\}$.

Next, we define the space $\mathfrak{N}_b = \{u : (-\infty,b] \to \mathbb{R} : u|_{(-\infty,0]} \in \mathfrak{N}, u|_{[0,b]} \in PC(U,\mathbb{R})\}$, with a seminorm defined by $\|u\|_{\mathfrak{N}_b} = \|\omega\|_{\mathfrak{N}} + \sup_{r\in U} |u(r)|$, $u \in \mathfrak{N}_b$.

Definition 2.1. [28] The fractional integral for a function $\psi : [a,b] \to \mathbb{R}$, of order $\sigma > 0$ is defined by $I_{a^+}^{\sigma}\psi(t) = \int_a^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}\psi(s)ds$, $t > 0$. The Caputo derivative of order $\sigma \in \mathbb{R}^+$ on $[a,b]$ is defined by ${}^C D_{a^+}^{\sigma}\psi(x) = \frac{1}{\Gamma(n-\sigma)}\int_a^x (x-\tau)^{n-\sigma-1}\psi^{(n)}(\tau)d\tau$.

Lemma 2.2. [28] *Let $\sigma > 0$, $n \in \mathbb{N}$, $\sigma \in (n-1,n)$. If $y(x) \in AC^n[a,b]$, then*

$$I_a^{\sigma}({}^C D_{a^+}^{\sigma} y(x)) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k$$

Lemma 2.3. *For*

$$\Lambda_1 = b - \sum_{l=0}^{q} \frac{\delta_l(\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \neq 0, \tag{2.2}$$

let $G \in C(0,b)$, $\mathrm{B} \in AC(0,b)$, $u \in PC^1(U,\mathbb{R}) \cap AC^2(U_l)$, *and* $\Psi_l, \Psi_l^*(l=1,2,...,q)$ *be constants. Then the solution of the following impulsive boundary value problem:*

$$\begin{cases} {}^C D_{t_l^+}^{\varsigma}[u(t) + \int_0^t \mathrm{B}(r)dr] = G(t),\ t \in u', \\ \Delta u(t_l) = \Psi_l,\ \Delta u'(t_l) = \Psi_l^*, \\ u(t) = \omega(t),\ t \in (-\infty, 0], \\ u(0) + u(b) = \sum_{l=0}^{q} \delta_l {}^C D_{t_l}^{\gamma} u(\mu_l) + \mathscr{A},\ \mu_l \in (0,b), \end{cases} \tag{2.3}$$

is given by

$$u(t) = \begin{cases} \omega(t),\ t \in (-\infty, 0], \\ \int_0^t \mathrm{B}(r)dr + \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + \omega(0) + t\vartheta,\ t \in U_0, \\ \int_0^t \mathrm{B}(r)dr + \int_{t_l}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr \\ + \sum_{i=1}^{l} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + \Psi_k \Big] \\ + \sum_{i=1}^{l-1} (t_l - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + \Psi_i^* \Big] \\ + \sum_{i=1}^{l} (t - t_l) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + \Psi_i^* \Big] + \omega(0) + (t-t_l)\vartheta, \\ t \in U_l,\ l = 1,2,...,q, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} \vartheta = \frac{1}{\Lambda_1} \Big(& \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{-\gamma}}{\Gamma(1-\gamma)} \mathrm{B}(r)dr + \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)} \mathrm{G}(r)dr \\ & - \sum_{i=1}^{q} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)ds + \Psi_i \Big] - \sum_{i=1}^{q-1} (t_q - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + \Psi_i^* \Big] \\ & - \sum_{i=1}^{q} (b - t_q) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr \\ & + \Psi_i^* \Big] + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + \Psi_i^* \Big] - \int_0^b \mathrm{B}(r)dr \\ & - \int_{t_q}^{b} \frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr - 2\omega(0) + \mathscr{A} \Big). \end{aligned} \tag{2.5}$$

Proof. Applying $I_{t_l^+}^{\varsigma}$ to both sides of the equation in (2.3) and using Lemma 2.2, we obtain

$$u(t) \quad = \quad \int_0^t \mathrm{B}(r)dr + \int_{t_l}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + a_{1,l} + a_{2,l}(t - t_l),\ t \in U_l, \tag{2.6}$$

where $a_{1,l}, a_{2,l} \in \mathbb{R},\ l = 0, 1, ..., q$. Differentiating (2.6), we find

$$u'(t) = \mathrm{B}(t) + \int_{t_l}^{t} \frac{(t-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + a_{2,l},\ t \in U_l.$$

For $t \in U_0$, we have

$$u(t) = \int_0^t \mathrm{B}(r)dr + \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + a_{1,0} + a_{2,0}t, \tag{2.7}$$

and

$$u'(t) = \mathrm{B}(t) + \int_0^t \frac{(t-s)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + a_{2,0}. \tag{2.8}$$

Now, by substituting the condition $u(0) = \omega(0)$ in (2.7), we obtain $a_{1,0} = \omega(0)$. Thus (2.7) and (2.8) can be written as

$$u(t) = \int_0^t \mathrm{B}(r)dr + \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + \omega(0) + a_{2,0}t \tag{2.9}$$

and

$$u'(t) = S'(t) + \int_0^t \frac{(t-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} p(r)dr + a_{2,0}. \tag{2.10}$$

For $t \in U_1$, we find

$$u(t) = \int_0^t \mathrm{B}(r)dr + \int_{t_1}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + a_{1,1} + a_{2,1}(t-t_1), \tag{2.11}$$

and

$$u'(t) = \mathrm{B}(t) + \int_{t_1}^t \frac{(t-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + a_{2,1}. \tag{2.12}$$

Consequently, we have

$$u(t_1^-) = \int_0^{t_1} \mathrm{B}(r)dr + \int_0^{t_1} \frac{(t_1-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + \omega(0) + a_{2,0}t_1, u(t_1^+) = \int_0^{t_1} \mathrm{B}(r)dr + a_{1,1},$$
$$u'(t_1^-) = \mathrm{B}(t_1) + \int_0^{t_1} \frac{(t_1-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + a_{2,0}, u'(t_1^+) = \mathrm{B}(t_1) + a_{2,1},$$

where $\mathrm{B}(t_1^+) = \mathrm{B}(t_1^-) = \mathrm{B}(t_1)$. In view of the impulse conditions $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = \Psi_1$ and $\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \Psi_1^*$, we find that

$$a_{1,1} = \int_0^{t_1} \frac{(t_1-r)^{\varsigma-1}}{\Gamma(\varsigma)} \mathrm{G}(r)dr + \omega(0) + a_{2,0}t_1 + \Psi_1,$$

and

$$a_{2,1} = \int_0^{t_1} \frac{(t_1-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \mathrm{G}(r)dr + a_{2,0} + \Psi_1^*.$$

Substituting the values of $a_{1,1}$ and $a_{2,1}$ in (2.11), we see that

$$\begin{aligned} u(t) &= \int_0^t \mathrm{B}(r)dr + \int_{t_1}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr + \int_0^{t_1} \frac{(t_1-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr \\ &\quad +(t-t_1)\int_0^{t_1} \frac{(t_1-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr + \Psi_1 + (t-t_1)\Psi_1^* + \omega(0) + a_{2,0}t. \end{aligned}$$

By a similar process, for $t \in U_l,\ l = 1,2,...,q$, we obtain

$$\begin{aligned} u(t) &= \int_0^t \mathrm{B}(r)dr + \int_{t_l}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr + \sum_{i=1}^{l}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr + \Psi_i\Big] \\ &\quad + \sum_{i=1}^{l-1}(t_l-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr + \Psi_i^*\Big] \\ &\quad + \sum_{i=1}^{l}(t-t_l)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(s)ds + \Psi_i^*\Big] + \omega(0) + a_{2,0}t. \end{aligned} \tag{2.13}$$

For $t \in U_l,\ l = 0,1,2,...,q$, we have

$$\begin{aligned} {}^{C}D_{t_l}^{\gamma}u(t) &= \int_{t_l}^t \frac{(t-r)^{-\gamma}}{\Gamma(1-\gamma)}\mathrm{B}(r)dr + \int_{t_l}^t \frac{(t-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}\mathrm{G}(r)dr \\ &\quad + \sum_{i=1}^{l}\frac{(t-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr + \Psi_i^*\Big] + a_{2,0}\frac{(t-t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \end{aligned}$$

and

$$\begin{aligned} u(b) &= \int_0^b \mathrm{B}(r)dr + \int_{t_q}^b \frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr + \sum_{i=1}^{q}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr + \Psi_i\Big] \\ &\quad + \sum_{i=1}^{q-1}(t_q-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr + \Psi_i^*\Big] \\ &\quad + \sum_{i=1}^{q}(b-t_q)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr + \Psi_i^*\Big] + \omega(0) + a_{2,0}b. \end{aligned}$$

Applying the boundary condition

$$u(0) + u(b) = \sum_{l=0}^{q} \delta_l {}^{C}D_{t_l}^{\gamma}u(\mu_l) + \mathscr{A},$$

we find

$$
\begin{aligned}
a_{2,0} = & \frac{1}{\left(b-\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\right)}\Bigg(\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{-\gamma}}{\Gamma(1-\gamma)}\mathrm{B}(r)dr \\
& +\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}\mathrm{G}(r)dr-\sum_{i=1}^{q}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr+\Psi_i\Big] \\
& -\sum_{i=1}^{q-1}(t_q-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr+\Psi_i^*\Big] \\
& -\sum_{i=1}^{q}(b-t_q)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr+\Psi_i^*\Big] \\
& +\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\mathrm{G}(r)dr+\Psi_i^*\Big] \\
& -\int_0^b\mathrm{B}(r)dr-\int_{t_q}^{b}\frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}\mathrm{G}(r)dr-2\omega(0)+\mathscr{A}\Bigg).
\end{aligned}
$$

Thus, by substituting the value of $a_{2,0}$ in (2.9) and (2.13), we obtain the solution (2.4). The converse can be proved by direct calculation. □

The following hypotheses are needed in the forthcoming analysis:

($\mathscr{H}_1$) There exist two constants L_1 and L_2 such that

$$|p(t,u)-p(t,s)| \le L_1\|u-s\|_{\mathfrak{N}}, \quad \text{for } t\in U \text{ and every } u,s\in\mathfrak{N},$$

and

$$|\psi(t,u)-\psi(t,s)| \le L_2\|u-s\|_{\mathfrak{N}}, \quad \text{for } t\in U \text{ and every } u,s\in\mathfrak{N}.$$

($\mathscr{H}_2$) for each $l=1,...,q$, there exist $d_1,d_2>0$ such that

$$\|\Psi_l(u)-\Psi_l(s)\| \le d_1\|u-s\|,\ \|\Psi_l^*(u)-\Psi_l^*(s)\| \le d_2\|u-s\|,\ \forall u,s\in\mathbb{R}.$$

($\mathscr{H}_3$) The functions $\psi,p : U\times\mathfrak{N}_b\to\mathbb{R}$ are continuous and there exist continuous nonnegative functions $\kappa_1,\kappa_2 : U\to(0,\infty)$ such that $|\psi(t,u)|\le\kappa_1(t)$, $|p(t,u)|\le\kappa_2(t)$ and $\kappa_i^* = sup_{t\in[0,b]}\kappa_i(t), i=1,2.$

($\mathscr{H}_4$) $\Psi_l:\mathbb{R}\to\mathbb{R}$, $\Psi_l^*:\mathbb{R}\to\mathbb{R}$, $l=1,...,q$ are continuous functions and there are two constants η_1,η_2 such that $||\Psi_l(u)||\le\eta_1$ and $||\Psi_l^*(u)||\le\eta_2$.

3. Main Results

To establish the existence and uniqueness results for problem (1.1), we transform, with the aid of Lemma 2.3, problem (1.1) into a fixed point problem. Define the operator $\mathscr{G} : \mathfrak{N}_b \to \mathfrak{N}_b$

by

$$(\mathscr{G}u)(t)=\begin{cases}\omega(t),\ t\in(-\infty,0],\\ \displaystyle\int_0^t p(r,u_r)dr+\int_0^t\frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,u_r)dr+\omega(0)+t\vartheta,\ t\in U_0,\\ \displaystyle\int_0^t p(r,u_r)dr+\int_{t_l}^t\frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,u_r)dr\\ \displaystyle+\sum_{i=1}^{l}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,u_r)dr+\Psi_i(u(t_i))\Big]\\ \displaystyle+\sum_{i=1}^{l-1}(t_l-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,u_r)ds+\Psi_i^*(u(t_i))\Big]\\ \displaystyle+\sum_{i=1}^{l}(t-t_l)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,u_r)dr+\Psi_i^*(u(t_i))\Big]+\omega(0)+(t-t_l)\vartheta,\\ t\in U_l,\ l=1,2,\dots,q,\end{cases}$$

where ϑ is defined by (2.5) with $p(t,u_t)$ instead of $\mathrm{B}(t)$ and $\psi(t,u_t)$ instead of $\mathrm{G}(t)$. Let $s(.):(-\infty,b]\to\mathbb{R}$ be a function defined by

$$s(t)=\begin{cases}\omega(t),\ t\in(-\infty,0],\\ 0,\ \ t\in(0,b],\end{cases}$$

Then $s_0=\omega$. For every $\theta\in C([0,b],\mathbb{R})$ with $\theta(0)=0$, we define

$$\bar{\theta}(t)=\begin{cases}0,\ t\in(-\infty,0],\\ \theta(t),\ t\in(0,b].\end{cases}$$

If $u(.)$ satisfies (1.1), then we can split $u(.)$ as $u(t)=s(t)+\bar{\theta}(t)$, which yields $u_t=s_t+\bar{\theta}_t$ for $t\in U$, and a function $\theta(.)$ satisfying

$$\theta(t)=\begin{cases}\int_0^t p(r,s_r+\bar{\theta}_r)dr+\int_0^t\dfrac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr+\omega(0)+t\vartheta,\ t\in U_0,\\ \int_0^t p(r,s_r+\bar{\theta}_r)dr+\int_{t_l}^t\dfrac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr\\ +\sum_{i=1}^{l}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr+\Psi_i(s(t_i)+\bar{\theta}(t_i))\Big]\\ +\sum_{i=1}^{l-1}(t_l-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr+\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\Big]\\ \displaystyle+\sum_{i=1}^{l}(t-t_l)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr+\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\Big]\\ +\omega(0)+(t-t_l)\vartheta,\ t\in U_l,\ l=1,2,\dots,q.\end{cases}$$

Set $\mathfrak{N}'_b=\{\theta\in\mathfrak{N}_b$ such that $\theta_0=0\}$, and consider $\|.\|_{\mathfrak{N}'_b}$ to be a seminorm on $\mathfrak{N}'_b$ which is defined by

$$\|\theta\|_{\mathfrak{N}'_b}=\sup_{t\in[0,b]}|\theta(t)|+\|\theta_0\|_{\mathfrak{N}}=\sup_{t\in[0,b]}|\theta(t)|,\ \theta\in\mathfrak{N}'_b.$$

Consequently, $(\mathfrak{N}'_b, \|.\|_{\mathfrak{N}'_b})$ is a Banach space. Now, we define the operator $\mathfrak{T}: \mathfrak{N}'_b \to \mathfrak{N}'_b$ as

$$\mathfrak{T}\theta(t) = \begin{cases} \int_0^t p(r, s_r + \bar{\theta}_r)dr + \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \psi(r, s_r + \bar{\theta}_r)dr + \omega(0) + t\vartheta,\ t \in U_0, \\ \int_0^t p(r, s_r + \bar{\theta}_r)dr + \int_{t_l}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} \psi(r, s_r + \bar{\theta}_r)dr \\ + \sum_{i=1}^{l} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-1}}{\Gamma(\varsigma)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i(s(t_i) + \bar{\theta}(t_i)) \Big] \\ + \sum_{i=1}^{l-1} (t_l - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i^*(s(t_i) + \bar{\theta}(t_i)) \Big] \\ + \sum_{i=1}^{l} (t - t_l) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i^*(s(t_i) + \bar{\theta}(t_i)) \Big] \\ + \omega(0) + (t - t_l)\vartheta,\ t \in U_l,\ l = 1, 2, ..., q, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} \vartheta \ = \ & \frac{1}{\Lambda_1} \Bigg\{ \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{-\gamma}}{\Gamma(1-\gamma)} p(r, s_r + \bar{\theta}_r)dr \\ & + \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)} \psi(r, s_r + \bar{\theta}_r)dr \\ & - \sum_{i=1}^{q} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-1}}{\Gamma(\varsigma)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i(s(t_i) + \bar{\theta}(t_i)) \Big] \\ & - \sum_{i=1}^{q-1} (t_q - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i^*(s(t_i) + \bar{\theta}(t_i)) \Big] \\ & - \sum_{i=1}^{q} (b - t_q) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i^*(s(t_i) + \bar{\theta}(t_i)) \Big] \\ & + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} \psi(r, s_r + \bar{\theta}_r)dr + \Psi_i^*(s(t_i) + \bar{\theta}(t_i)) \Big] \\ & - \int_0^b p(r, s_r + \bar{\theta}_r)dr - \int_{t_q}^b \frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)} \psi(r, s_r + \bar{\theta}_r)dr - 2\omega(0) + \mathscr{A} \Bigg\}. \end{aligned} \tag{3.2}$$

Clearly, $\mathscr{G}$ possesses a fixed point if and only if $\mathfrak{T}$ does.

For computational convenience, we set

$$\begin{aligned} \Lambda_2 \ = \ & \frac{\max_{0 \le l \le q} (t_{l+1} - t_l)}{|\Lambda_1|} \Big(\sum_{l=0}^{q} \frac{\delta_l (\mu_l - t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)} + \sum_{i=1}^{q} \frac{(t_i - t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} \\ & + \sum_{i=1}^{q-1} \frac{(t_q - t_i)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \sum_{i=1}^{q} \frac{(b - t_q)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\ & + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma} (t_i - t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)} + \frac{(b - t_q)^{\varsigma}}{\Gamma(\varsigma+1)} \Big), \end{aligned} \tag{3.3}$$

$$\begin{aligned}\bar{\Lambda}_2 &= \frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)^{\varsigma}}{\Gamma(\varsigma+1)}+\sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)}+\sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}\\ &\quad+\sum_{i=1}^{q}\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)},\end{aligned} \tag{3.4}$$

$$\Lambda_3=b+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}+b\Big), \tag{3.5}$$

and

$$\begin{aligned}\Lambda_4 &= (q-1)t_q-\sum_{i=1}^{q-1}t_i+q\max_{0\le l\le q}(t_{l+1}-t_l)\\ &\quad+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(q\,b-\sum_{i=1}^{q}t_i+\sum_{l=1}^{q}\frac{l\,\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big).\end{aligned} \tag{3.6}$$

In the first existence result, we employ Krasnoselskii's fixed point theorem [29].

Theorem 3.1. *Let $(\mathscr{H}_2)$, $(\mathscr{H}_3)$, and $(\mathscr{H}_4)$ be satisfied. Then, there exists at least one solution to problem (1.1) on $(-\infty,b]$, provided that*

$$\left\{d_1\Big(q+q\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)+d_2\Lambda_4\right\}<1, \tag{3.7}$$

where Λ_4 is defined by (3.6).

Proof. Define $\mathscr{S}_r=\{\theta\in\mathfrak{N}'_b:\|\theta\|_{\mathfrak{N}'_b}\le r\}$, with

$$\begin{aligned} r &> \kappa_1^*(\bar{\Lambda}_2+\Lambda_2)+\kappa_2^*\Lambda_3+\eta_1\Big(q+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)+\eta_2\Lambda_4+\Big(1+\frac{2\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)|\omega(0)|\\ &\quad+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}|,\end{aligned}$$

where $\kappa_1^*,\kappa_2^*,\eta_1$ and η_2 are given in $(\mathscr{H}_3)$ and $(\mathscr{H}_4)$, respectively, and $\Lambda_2,\bar{\Lambda}_2,\Lambda_3$ are defined by (3.3), (3.4), (3.5), respectively. Consider the operators $\mathscr{N}:\mathfrak{N}'_b\to\mathfrak{N}'_b$ and $\mathscr{J}:\mathfrak{N}'_b\to\mathfrak{N}'_b$

which are defined on $\mathscr{S}_r$ by

$$\begin{aligned}
&(\mathscr{N}\theta)(t)\\
&=\int_0^t p(r,s_r+\bar{\theta}_r)dr+\int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr+\omega(0)\\
&+\frac{t}{\Lambda_1}\left\{\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{-\gamma}}{\Gamma(1-\gamma)}p(r,s_r+\bar{\theta}_r)dr+\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}\psi(r,s_r+\bar{\theta}_r)dr\right.\\
&-\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr-\sum_{i=1}^{q-1}(t_q-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr-\sum_{i=1}^{q}(b-t_q)\\
&\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr\\
&\left.-\int_0^b p(r,s_r+\bar{\theta}_r)dr-\int_{t_q}^{b}\frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr-2\omega(0)+\mathscr{A}\right\},
\end{aligned}$$

for $t\in U_0$ and if $t\in U_l,\ l=1,2,...,q$. Then

$$\begin{aligned}
&(\mathscr{N}\theta)(t)\\
&=\int_0^t p(r,s_r+\bar{\theta}_r)dr+\int_{t_l}^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr\\
&+\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr+\sum_{i=1}^{l-1}(t_l-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr\\
&+\sum_{i=1}^{l}(t-t_l)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr++\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}\psi(r,s_r+\bar{\theta}_r)dr\\
&+\omega(0)+\frac{(t-t_l)}{\Lambda_1}\left\{\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{-\gamma}}{\Gamma(1-\gamma)}p(r,s_r+\bar{\theta}_r)dr-\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr\right.\\
&-\sum_{i=1}^{q-1}(t_q-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr-\sum_{i=1}^{q}(b-t_q)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr\\
&+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}\psi(r,s_r+\bar{\theta}_r)dr\\
&\left.-\int_0^b p(r,s_r+\bar{\theta}_r)dr-\int_{t_q}^{b}\frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r,s_r+\bar{\theta}_r)dr-2\omega(0)+\mathscr{A}\right\},
\end{aligned}$$

Also, for $t\in U_0$, we define

$$\begin{aligned}
(\mathscr{J}\theta)(t)=-\frac{t}{\Lambda_1}&\left\{\sum_{i=1}^{q}\Psi_i(s(t_i)+\bar{\theta}(t_i))+\sum_{i=1}^{q-1}(t_l-t_i)\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\right.\\
&\left.+\sum_{i=1}^{q}(b-t_q)\Psi_i^*(s(t_i)+\bar{\theta}(t_i))-\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\right\},
\end{aligned}$$

and, for $t \in U_l,\ l = 1,2,...,q,$

$$
\begin{aligned}
&(\mathscr{J}\theta)(t)\\
&= \sum_{i=1}^{l}\Psi_i(s(t_i)+\bar{\theta}(t_i)) + \sum_{i=1}^{l-1}(t_l-t_i)\Psi_i^*(s(t_i)+\bar{\theta}(t_i)) + \sum_{i=1}^{l}(t-t_l)\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\\
&\quad -\frac{(t-t_l)}{\Lambda_1}\Bigg\{\sum_{i=1}^{q}\Psi_i(s(t_i)+\bar{\theta}(t_i)) + \sum_{i=1}^{q-1}(t_q-t_i)\Psi_i^*(s(t_i)+\bar{\theta}(t_i)) + \sum_{i=1}^{q}(b-t_q)\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\\
&\quad -\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Psi_i^*(s(t_i)+\bar{\theta}(t_i))\Bigg\}.
\end{aligned}
$$

Notice that $\mathscr{N}+\mathscr{J}=\mathfrak{T}$, where $\mathfrak{T}:\mathfrak{N}'_b \to \mathfrak{N}'_b$ is defined by (3.1). For $\theta,\ \theta^* \in \mathscr{S}_r$, when $t \in U_0$, we have

$$
\begin{aligned}
&|\mathscr{N}\theta(t)+\mathscr{J}\theta^*(t)|\\
&\le \kappa_1^*\Bigg\{\int_0^t\frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}dr + \frac{t}{|\Lambda_1|}\Bigg\{\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}dr + \sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}dr\\
&+\sum_{i=1}^{q-1}(t_q-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}dr + \sum_{i=1}^{q}(b-t_q)\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}dr\\
&+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\delta-2}}{\Gamma(\varsigma-1)}dr + \int_{t_q}^{b}\frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}dr\Bigg\}\Bigg\} + \kappa_2^*\Bigg\{t+\frac{t}{|\Lambda_1|}\\
&\Bigg\{\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{-\gamma}}{\Gamma(1-\gamma)}dr + b\Bigg\}\Bigg\} + \frac{t}{|\Lambda_1|}\Bigg\{\eta_1 q + \eta_2\Big(\sum_{i=1}^{q-1}(t_q-t_i) + q(b-t_m)\\
&+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\Bigg\} + |\omega(0)| + \frac{2t}{|\Lambda_1|}|\omega(0)| + \frac{t}{|\Lambda_1|}|\mathscr{A}|\\
&\le \kappa_1^*\Bigg\{\frac{t_1^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{t_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)} + \sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}\\
&+\sum_{i=1}^{q}\frac{(b-t_q)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)} + \frac{(b-t_q)^{\varsigma}}{\Gamma(\varsigma+1)}\Big)\Bigg\}\\
&+\kappa_2^*\Bigg\{t_1 + \frac{t_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)} + b\Big)\Bigg\} + \frac{t_1 q}{|\Lambda_1|}\eta_1 + \frac{t_1}{|\Lambda_1|}\Big(q\,b - \sum_{i=1}^{q}t_i\\
&+\sum_{l=1}^{q}\frac{l\ \delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\eta_2 + (1+\frac{2t_1}{|\Lambda_1|})|\omega(0)| + \frac{t_1}{|\Lambda_1|}|\mathscr{A}|\\
&\le \kappa_1^*(\bar{\Lambda}_2+\Lambda_2) + \kappa_2^*\Lambda_3 + \frac{t_1 q}{|\Lambda_1|}\eta_1 + t_1\eta_2\Lambda_4 + (1+\frac{2t_1}{|\Lambda_1|})|\omega(0)| + \frac{t_1}{|\Lambda_1|}|\mathscr{A}| < r.
\end{aligned}
$$

Also, for θ, $\theta^* \in \mathscr{S}_r$, and $t \in U_l$, $l = 1,2,...,q$, we have

$$
\begin{aligned}
&|\mathscr{N}\theta(t) + \mathscr{J}\theta^*(t)| \\
&\leq \kappa_1^* \Bigg\{ \int_{t_l}^{t} \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)} dr + \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)} dr + \sum_{i=1}^{l-1} (t_l - t_i) \int_{t_i}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr \\
&+ \sum_{i=1}^{l} (t - t_l) \int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \frac{t-t_l}{|\Lambda_1|} \Bigg\{ \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)} dr + \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\varsigma-1}}{\Gamma(\varsigma)} dr \\
&+ \sum_{i=1}^{q-1} (t_q - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \sum_{i=1}^{q} (b - t_q) \int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \\
&\int_{t_{i-1}}^{t_i} \frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \int_{t_q}^{b} \frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)} dr \Bigg\} \Bigg\} + \kappa_2^* \Bigg\{ t + \frac{t-t_l}{|\Lambda_1|} \Big(\sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{-\gamma}}{\Gamma(1-\gamma)} dr + b \Big) \Bigg\} \\
&+ (q + \frac{(t-t_l)q}{|\Lambda_1|}) \eta_1 + \Big(\sum_{i=1}^{l-1} (t_l - t_i) + \sum_{i=1}^{l} (t - t_l) + \frac{t-t_l}{|\Lambda_1|} (\sum_{i=1}^{q-1} (t_q - t_i) \\
&+ \sum_{i=1}^{q} (b - t_l) + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)}) \Big) \eta_2 + (1 - \frac{2(t-t_l)}{|\Lambda_1|}) |\omega(0)| + \frac{t-t_l}{|\Lambda_1|} |\mathscr{A}| \\
&\leq \kappa_1^* \Bigg\{ \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q} \frac{(t_i - t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q-1} \frac{(t_q - t_i)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\
&+ \sum_{i=1}^{q} \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\
&+ \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} \Bigg\{ \sum_{l=0}^{q} \frac{\delta_l (\mu_l - t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)} + \sum_{i=1}^{q} \frac{(t_i - t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q-1} \frac{(t_q - t_i)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\
&+ \sum_{i=1}^{q} \frac{(b - t_q)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma} (t_i - t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)} + \frac{(b - t_q)^{\varsigma}}{\Gamma(\varsigma+1)} \Bigg\} \Bigg\} \\
&+ \kappa_2^* \Big(b + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} \Big(\sum_{l=0}^{q} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} + b \Big) \Big) + \Big(q + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l) q}{|\Lambda_1|} \Big) \eta_1 \\
&+ \Bigg\{ (q-1) t_m - \sum_{i=1}^{q-1} t_i + q \max_{0\leq l\leq q} (t_{l+1} - t_l) + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} \Big(q\, b - \sum_{i=1}^{q} t_i \\
&+ \sum_{l=1}^{q} \frac{l\ \delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \Big) \Bigg\} \eta_2 + (1 + \frac{2 \max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|}) |\omega(0)| + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} |\mathscr{A}| \\
&= \kappa_1^* (\bar{\Lambda}_2 + \Lambda_2) + \kappa_2^* \Lambda_3 + \eta_1 \Big(q + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l) q}{|\Lambda_1|} \Big) \\
&+ \eta_2 \Lambda_4 + \Big(1 + \frac{2 \max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} \Big) |\omega(0)| + \frac{\max_{0\leq l\leq q} (t_{l+1} - t_l)}{|\Lambda_1|} |\mathscr{A}| < r.
\end{aligned}
$$

Thus, for θ, $\theta^* \in \mathscr{S}_r$ and $t \in [0,b]$, we have

$$\|\mathscr{N}\theta+\mathscr{J}x^*\|_{\mathfrak{N}'_b}\le\kappa_1^*(\bar{\Lambda}_2+\Lambda_2)+\kappa_2^*\Lambda_3+\eta_1\Big(q+\frac{\max_{0\le l\le q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)+\eta_2\Lambda_4$$
$$+\Big(1+\frac{2\max_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)|\omega(0)|+\frac{\max_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}|<\mathrm{r},$$

which yields that $\mathscr{N}\theta+\mathscr{J}\theta^*\in\mathscr{S}_r$. Using condition $(\mathscr{H}_2)$ and relation (3.7), we prove that $\mathscr{J}$ is a contraction. For $\theta,\,\theta^*\in\mathscr{S}_r$ and $t\in U_0$, we obtain

$$\sup_{t\in[0,b]}|\mathscr{J}\theta(t)-\mathscr{J}\theta^*(t)|$$
$$\le\sup_{t\in[0,b]}\Big\{\frac{t}{|\Lambda_1|}\Big\{\sum_{i=1}^{q}|\Psi_i(s(t_i)+\bar{\theta}(t_i))-\Psi_i(s(t_i)+\bar{\theta}^*(t_i))|+\sum_{i=1}^{q-1}(t_q-t_i)|\Psi_i^*(s(t_i)+\bar{\theta}(t_i))$$
$$-\Psi_i^*(s(t_i)+\bar{\theta}(t_i))|+\sum_{i=1}^{q}(b-t_q)|\Psi_i^*(s(t_i)+\bar{\theta}(t_i))-\Psi_i^*(s(t_i)+\bar{\theta}(t_i))|$$
$$+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}|\Psi_i^*(s(t_i)+\bar{\theta}(t_i))-\Psi_i^*(s(t_i)+\bar{\theta}^*(t_i))|\Big\}\Big\}$$
$$\le\frac{t_1}{|\Lambda_1|}\Big\{d_1q\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|+d_2\Big(q\,b-\sum_{i=1}^{q}t_i+\sum_{l=1}^{q}\frac{l\;\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|\Big\}$$
$$\le\Big\{\frac{t_1q}{|\Lambda_1|}d_1+d_2\Lambda_4\Big\}\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|.$$

Similarly, for $\theta,\,\theta^*\in\mathscr{S}_r$ and $t\in U_l$, we have

$$\sup_{t\in[0,b]}|\mathscr{J}\theta(t)-\mathscr{J}\theta^*(t)|$$
$$\le\Bigg\{\Big(q+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)d_1+\Big\{(q-1)t_q-\sum_{i=1}^{q-1}t_i+q\max_{0\le l\le q}(t_{l+1}-t_l)$$
$$+\frac{\max_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(q\,b-\sum_{i=1}^{q}t_i+\sum_{l=1}^{q}\frac{l\;\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\Big\}d_2\Bigg\}\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|$$
$$\le\Big\{d_1\Big(q+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)+d_2\Lambda_4\Big\}\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|.$$

Therefore, for $\theta,\theta^*\in\mathscr{S}_r$ and $t\in[0,b]$, we have

$$\|\mathscr{J}\theta-\mathscr{J}\theta^*\|_{\mathfrak{N}'_b}\le\Big\{d_1\Big(q+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)+d_2\Lambda_4\Big\}\|\theta-\theta^*\|_{\mathfrak{N}'_b}.$$

Furthermore, the continuity of ψ implies that $\mathscr{N}$ is continuous. Also, $\mathscr{N}$ is uniformly bounded on $\mathscr{S}_r$ as

$$\|\mathscr{N}\theta\|_{\mathfrak{N}'_b}\;\le\;\kappa_1^*(\bar{\Lambda}_2+\Lambda_2)+\kappa_2^*\Lambda_3+\Big(1+\frac{2\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)|\omega(0)|+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}|.$$

Next, to establish the compactness of $\mathscr{N}$, let $\theta \in \mathscr{S}_r$. Then, by the condition $(\mathscr{H}_3)$, for $\tau_1, \tau_2 \in U_0$ with $\tau_1 < \tau_2$, we find

$$\begin{aligned}
&|(\mathscr{N}\theta)(\tau_2) - (\mathscr{N}\theta)(\tau_1)| \\
&\leq \kappa_1^*(t)\Bigg\{ \int_0^{\tau_1} |(\tau_2 - r)^{\varsigma-1} - (\tau_1 - r)^{\varsigma-1}| dr + \int_{\tau_1}^{\tau_2} |(\tau_2 - r)^{\varsigma-1}| dr \\
&+ \frac{\tau_2 - \tau_1}{|\Lambda_1|}\Bigg\{ \sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)} dr + \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-1}}{\Gamma(\varsigma)} dr \\
&+ \sum_{i=1}^{q-1} (t_q - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \sum_{i=1}^{q} (b - t_q) \int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr \\
&+ \sum_{l=1}^{q} \sum_{i=1}^{l} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} \int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{\varsigma-2}}{\Gamma(\varsigma-1)} dr + \int_{t_q}^{b} \frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)} dr \Bigg\}\Bigg\} \\
&+ \kappa_2^*(t)\Bigg\{ (\tau_2 - \tau_1) + \frac{\tau_2 - \tau_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q} \delta_l \int_{t_l}^{\mu_l} \frac{(\mu_l - r)^{-\gamma}}{\Gamma(1-\gamma)} dr + b \Big) \Bigg\} \\
&\leq \kappa_1^*(t)\Bigg\{ \frac{2(\tau_2 - \tau_1)^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{\tau_2^{\varsigma} - \tau_1^{\varsigma}}{\Gamma(\varsigma+1)} + (\tau_2 - \tau_1)\Lambda_2 \Bigg\} \\
&+ \kappa_2^*(t)\Bigg\{ (\tau_2 - \tau_1) + \frac{\tau_2 - \tau_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} + b \Big) \Bigg\}.
\end{aligned}$$

For $\tau_1, \tau_2 \in U_l, l = 0, 1, ..., q$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&|(\mathscr{N}\theta)(\tau_2) - (\mathscr{N}\theta)(\tau_1)| \\
&\leq \kappa_1^*(t)\Big\{ \frac{2(\tau_2 - \tau_1)^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{\tau_2^{\varsigma} - \tau_1^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{l} \frac{(\tau_2 - \tau_1)(t_i - t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\
&+ (\tau_2 - \tau_1)\Lambda_2 \Big\} + \kappa_2^*(t)\Big\{ (\tau_2 - \tau_1) + \frac{\tau_2 - \tau_1}{|\Lambda_1|}\big(\sum_{l=0}^{q} \frac{\delta_l (\mu_l - t_l)^{1-\gamma}}{\Gamma(2-\gamma)} + b \big) \Big\}.
\end{aligned}$$

Based on the inequalities above, we deduce that

$$|(\mathscr{N}\theta)(\tau_2) - (\mathscr{N}\theta)(\tau_1)| \to 0$$

as $\tau_2 \to \tau_1$ for all $\tau_1, \tau_2 \in U_l, l = 0, 1, ..., q$ independent of $\theta \in \mathscr{S}_r$. This shows that $\mathscr{N}$ is equicontinuous. Consequently, $\mathscr{N}$ is relatively compact on $\mathscr{S}_r$. Hence, from the Arzelá-Ascoli theorem, we deduce that $\mathscr{N}$ is compact on $\mathscr{S}_r$. Thus all the hypotheses of Krasnoselskii's fixed point theorem [29] are fulfilled, and then problem (1.1) has at least one solution defined on $(-\infty, b]$. □

The next result is based on the Banach fixed point theorem for demonstrating the uniqueness of the solution to problem (1.1).

Theorem 3.2. *Let $\psi, p \in C(U\times\mathfrak{N},\mathbb{R})$ satisfying $(\mathscr{H}_1),(\mathscr{H}_2)$ and $(\mathscr{H}_4)$. Then, there exists a unique solution to problem (1.1) on $(-\infty,b]$, provided*

$$\left\{\nu_b\Big(L_1\Lambda_3+L_2(\bar{\Lambda}_2+\Lambda_2)\Big)+d_1\,q\left(1+\frac{\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\right)+d_2\Lambda_4\right\}<1,$$

where ν_b is defined by (2.1), and Λ_2, $\bar{\Lambda}_2$, Λ_3 and Λ_4 are given by (3.3), (3.4), (3.5), and (3.6), respectively.

Proof. Let $\sup_{t\in[0,b]}|\psi(t,0)|=M$, $\sup_{t\in[0,b]}|p(t,0)|=\bar{M}$, and define

$$\mathscr{S}_{\bar{r}}=\{\theta\in\mathfrak{N}'_b:\|\theta\|_{\mathfrak{N}'_b}\le\bar{r}\}$$

with

$$\begin{aligned}\bar{r}>&\Big(1-L_1\nu_b\Lambda_3-L_2\nu_b(\bar{\Lambda}_2+\Lambda_2)\Big)^{-1}\Big\{(L_1\rho_b||\omega||_{\mathfrak{N}}+\bar{M})\Lambda_3+(L_2\rho_b||\omega||_{\mathfrak{N}}+M)(\bar{\Lambda}_2+\Lambda_2)\\&+\eta_2\Lambda_4+\Big(1+\frac{2\max\limits_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)|\omega(0)|+\frac{\max_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}|\Big\}\\&+\eta_1\Big(q+q\frac{\max_{0\le l\le q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big),\end{aligned}$$

where ρ_b is introduced in (2.1). Firstly, we show that $\mathfrak{T}\mathscr{S}_{\bar{r}}\subset\mathscr{S}_{\bar{r}}$. For $\theta\in\mathscr{S}_{\bar{r}}$ and $t\in U_0$, we easily find

$$\begin{aligned}&|(\mathfrak{T}\theta)(t)|\\\le\;&\big[L_1(\rho_b||\omega||_{\mathfrak{N}}+\nu_b\bar{r})+\bar{M}\big]\left\{t_1+\frac{t_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}+b\Big)\right\}\\&+\big[L_2(\rho_b||\omega||_{\mathfrak{N}}+\nu_b\bar{r})+M\big]\left\{\frac{t_1^{\varsigma}}{\Gamma(\varsigma+1)}+\frac{t_1}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)}+\sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)}\right.\\&+\sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}+\sum_{i=1}^{q}\frac{(b-t_q)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}\\&\left.+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)}+\frac{(b-t_q)^{\varsigma}}{\Gamma(\varsigma+1)}\Big)\right\}\\&+\frac{t_1q}{|\Lambda_1|}\eta_1+\frac{t_1}{|\Lambda_1|}\Big(q\,b-\sum_{i=1}^{q}t_i+\sum_{l=1}^{q}\frac{l\;\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\eta_2+\Big(1+\frac{2t_1}{|\Lambda_1|}\Big)|\omega(0)|+\frac{t_1}{|\Lambda_1|}|\mathscr{A}|\\\le\;&\big[L_1(\rho_b||\omega||_{\mathfrak{N}}+\nu_b\bar{r})+\bar{M}\big]\Lambda_3+\big[L_2(\rho_b||\omega||_{\mathfrak{N}}+\nu_b\bar{r})+M\big]\Big(\bar{\Lambda}_2+\Lambda_2\Big)\\&+\frac{t_1q}{|\Lambda_1|}\eta_1+\Lambda_4\eta_2+\Big(1+\frac{2t_1}{|\Lambda_1|}\Big)|\omega(0)|+\frac{t_1}{|\Lambda_1|}|\mathscr{A}|<\bar{r}.\end{aligned}$$

Hence, by taking the norm for $t\in U_0$, we have $\|\mathfrak{T}\theta\|_{\mathfrak{N}'_b}<\bar{r}$. For $t\in[0,b]$,

$$\|s_t+\bar{\theta}_t\|_{\mathfrak{N}}\le\rho_b\|\omega\|_{\mathfrak{N}}+\nu_b\sup\{|\theta(r)|:r\in[0,t]\}\le\rho_b\|\omega\|_{\mathfrak{N}}+\nu_b\bar{r}.$$

Similarly, for $\theta \in \mathscr{S}_r$ and $t \in U_l$, we obtain

$$\begin{aligned}
&|(\mathfrak{T}\theta)(t)| \\
\leq\ & \big[L_1(\rho_b||\omega||_{\mathfrak{N}} + \nu_b\bar{r}) + \bar{M}\big]\Bigg\{ b + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)} + b\Big)\Bigg\} \\
& + \big[L_2(\rho_b||\omega||_{\mathfrak{N}} + \nu_b\bar{r}) + M\big]\Bigg\{ \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} + \sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} \\
& + \sum_{i=1}^{q}\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Bigg\{\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)} + \sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)} \\
& + \sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \sum_{i=1}^{q}\frac{(b-t_q)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)} + \sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)} \\
& + \frac{(b-t_q)^{\varsigma}}{\Gamma(\varsigma+1)}\Bigg\}\Bigg\} + \Big(q + \frac{q\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)\eta_1 + \Bigg\{(q-1)t_m + \sum_{i=1}^{q-1}t_i + q\max_{0\leq l\leq q}(t_{l+1}-t_l) \\
& + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(q\,b - \sum_{i=1}^{q}t_i + \sum_{l=1}^{q}\frac{l\ \delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\Bigg\}\eta_2 \\
& + (1 + \frac{2\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|})|\omega(0)| + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}| \\
\leq\ & \big[L_1(\rho_b||\omega||_{\mathfrak{N}} + \nu_b\bar{r}) + \bar{M}\big]\Lambda_3 + \big[L_2(\rho_b||\omega||_{\mathfrak{N}} + \nu_b\bar{r}) + M\big](\bar{\Lambda}_2 + \Lambda_2) \\
& + \eta_1\Big(q + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big) + \eta_2\Lambda_4 + \Big(1 + \frac{2\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big)|\omega(0)| + \frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}|\mathscr{A}| \\
& < \bar{r}.
\end{aligned}$$

Consequently, for $t \in U_l, l = 0, 1, ..., q$, we have $\|\mathfrak{T}\theta\|_{\mathfrak{N}'_b} < \bar{r}$. Hence, $\mathfrak{T}\mathscr{S}_{\bar{r}} \subset \mathscr{S}_{\bar{r}}$.

Now, we prove the contraction property of $\mathfrak{T} : \mathfrak{N}'_b \to \mathfrak{N}'_b$. Let $\theta, \theta^* \in \mathfrak{N}'_b$. Then, for each $t \in U_0$,

$$\begin{aligned}
&|\mathfrak{T}\theta(t) - \mathfrak{T}\theta^*(t)| \\
\leq\ & \int_0^t L_1\nu_b \sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr + \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr \\
& + \frac{t}{|\Lambda_1|}\Bigg\{\sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{-\gamma}}{\Gamma(1-\gamma)}L_1\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr \\
& + \sum_{l=0}^{q}\delta_l\int_{t_l}^{\mu_l}\frac{(\mu_l-r)^{\varsigma-\gamma-1}}{\Gamma(\varsigma-\gamma)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr \\
& + \sum_{i=1}^{q}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-1}}{\Gamma(\varsigma)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr + d_1\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{q-1}(t_q-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr+d_2\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|\Big]\\
&+\sum_{i=1}^{q}(b-t_q)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr+d_2\sup_{t\in[0,b]}|\theta(t)-\theta^*(t)|\Big]\\
&+\sum_{l=1}^{q}\sum_{i=1}^{1}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{\varsigma-2}}{\Gamma(\varsigma-1)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr+d_2\sup_{t\in[0,b]}|\theta(t)\\
&-\theta^*(t)|\Big]+\int_0^b L_1\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr+\int_{t_q}^b\frac{(b-r)^{\varsigma-1}}{\Gamma(\varsigma)}L_2\nu_b\sup_{r\in[0,b]}|\theta(r)-\theta^*(r)|dr\Bigg\}\\
\leq\;&\Bigg\{\nu_b\Bigg[L_2\Big(\bar{\Lambda}_2+\Lambda_2\Big)+L_1\Lambda_3\Bigg]+\frac{t_1q}{|\Lambda_1|}d_1+d_2\Lambda_4\Bigg\}||\theta-\theta^*||_{\mathfrak{N}'_b}.
\end{aligned}
$$

In the same way, for $t\in U_l$, we obtain that

$$
\begin{aligned}
&|\mathfrak{T}\theta(t)-\mathfrak{T}\theta^*(t)|\\
\leq\;&\Bigg\{\nu_b\Bigg[L_2\Big(\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)^{\varsigma}}{\Gamma(\varsigma+1)}+\sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)}+\sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}\\
&+\sum_{i=1}^{q}\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}+\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{\varsigma-\gamma}}{\Gamma(\varsigma-\gamma+1)}\\
&+\sum_{i=1}^{q}\frac{(t_i-t_{i-1})^{\varsigma}}{\Gamma(\varsigma+1)}+\sum_{i=1}^{q-1}\frac{(t_q-t_i)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}\\
&+\sum_{i=1}^{q}\frac{(b-t_q)(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(\varsigma)}+\sum_{l=1}^{q}\sum_{i=1}^{l}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}(t_i-t_{i-1})^{\varsigma-1}}{\Gamma(2-\gamma)\Gamma(\varsigma)}\\
&+\frac{(b-t_q)^{\varsigma}}{\Gamma(\varsigma+1)}\Big)\Big)+L_1\Big(b+\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(\sum_{l=0}^{q}\frac{\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}+b\Big)\Big)\Bigg]\\
&+\Big(q+\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)q}{|\Lambda_1|}\Big)d_1+\Big((q-1)t_m-\sum_{i=1}^{q-1}t_i+q\max_{0\leq l\leq q}(t_{l+1}-t_l)\\
&+\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big(q\,b-\sum_{i=1}^{q}t_i+\sum_{l=1}^{q}\frac{l\;\delta_l(\mu_l-t_l)^{1-\gamma}}{\Gamma(2-\gamma)}\Big)\Big)d_2\Bigg\}||\theta-\theta^*||_{\mathfrak{N}'_b}.
\end{aligned}
$$

Therefore, for $t\in U_l$, $l=0,1,\dots,q$, we have

$$
\begin{aligned}
&||\mathfrak{T}\theta-\mathfrak{T}\theta^*||_{\mathfrak{N}'_b}=\sup_{t\in[0,b]}|\mathfrak{T}\theta(t)-\mathfrak{T}\theta^*(t)|\\
&\leq\Bigg\{\nu_b\Big(L_2(\bar{\Lambda}_2+\Lambda_2)+L_1\Lambda_3\Big)+d_1(q+\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)q}{|\Lambda_1|})+d_2\Lambda_4\Bigg\}||\theta-\theta^*||_{\mathfrak{N}'_b},
\end{aligned}
$$

which, with the aid of the condition (3.2), implies that $\mathfrak{T}$ is a contraction. As a consequence, $\mathfrak{T}$ has a unique fixed point by the contraction mapping principle, which correspond to the existence of a unique solution to problem (1.1) on $(-\infty, b]$.

Next, we provide an example. Consider the following problem

$$\begin{cases} {}^{C}D^{3/2}_{t_l^+}[u(t)+\int_0^t p(r,u_r)dr] = \psi(t,u_t),\ t \in U' := [0,1], t \neq 5/7,\ l=0,1, \\ \Delta u(5/7) = \frac{|u(5/7)|}{8+|u(5/7)|},\ \Delta u'(5/7) = \frac{1}{4}\sin u(5/7), \\ u(t) = \omega(t),\ t \in (-\infty, 0], \\ u(0)+u(1) = \frac{1}{3}{}^{C}D^{1/2}_{0^+}u(3/7) + \frac{1}{10}{}^{C}D^{1/2}_{\frac{5}{7}^+}u(4/5)+2, \end{cases} \tag{3.8}$$

where $\varsigma = 3/2$, $\gamma = 1/2$, $q = 1$, $\mu_0 = 3/7$, $\mu_1 = 4/5$, $\delta_0 = 1/3$, $\delta_1 = 1/10$, $\mathscr{A} = 2$, $t_1 = 5/7$, and $p(t,u_t)$, $\psi(t,u_t)$, $\omega(t)$ will be fixed later. By the above data, we find that $\Lambda_1 = 0.7207319194$, $\Lambda_2 = 1.01524922$, $\bar{\Lambda}_2 = 1.005776297$, $\Lambda_3 = 2.244671571$ and $\Lambda_4 = 1.251060122$, where Λ_1, Λ_2, $\bar{\Lambda_2}$, Λ_3, and Λ_4 are respectively given by (2.2), (3.3), (3.4), (3.5), and (3.6). Let us define $\mathfrak{N}_z = \{u \in C((-\infty,0],\mathbb{R}) : \lim_{\varpi\to-\infty} e^{z\varpi}u(\varpi) \text{ exists in } \mathbb{R}\}$, where z is a positive real constant. it clear that the space $\mathfrak{N}_z$ is a phase space with the norm $\|u\|_z = \sup_{-\infty<\varpi\leq 0} e^{z\varpi}|u(\varpi)|$, and $\nu = \rho = \alpha = 1$.

Let $\omega(t)$ be a continuous function and $\lim_{t\to-\infty} e^{zt}\omega(t) < \infty$, i.e, $\omega \in \mathfrak{N}_z$. For instance, we can choose $\omega(t) = e^{\sqrt[5]{t}}$ which yields $\omega(0) = 1$. Obviously, $\omega \in \mathfrak{N}_z$. To illustrate Theorem 3.1, we assume

$$p(t,u_t) = \frac{e^{-zt}}{5\sqrt{900+t}}\Big(\frac{|u_t|}{|u_t|+1} + \tan^{-1} t\Big),\ (t,u_t) \in [0,1]\times\mathfrak{N}_z, \tag{3.9}$$

and

$$\psi(t,u_t) = \frac{e^{-zt}}{30(1+t^2)}\sin u_t + \frac{\cos t}{55},\ (t,u_t) \in [0,1]\times\mathfrak{N}_z. \tag{3.10}$$

Note that the conditions $(\mathscr{H}_2), (\mathscr{H}_3)$ and $(\mathscr{H}_4)$ hold with $d_1 = 1/8, d_2 = 1/4$, $\eta_1 = 1/8, \eta_2 = 1/4$, $\kappa_1(t) = \frac{e^{-zt(1+\tan^{-1} t)}}{5\sqrt{900+t}}$ and $\kappa_2(t) = \frac{e^{-zt}}{30(1+t^2)} + \frac{\cos t}{55}$. Furthermore,

$$\left\{ d_1\Big(q + q\frac{\max\limits_{0\leq l\leq q}(t_{l+1}-t_l)}{|\Lambda_1|}\Big) + d_2\Lambda_4 \right\} \approx 0.5616470342 < 1.$$

Thus, all the conditions of Theorem 3.1 are verified, and as a result, problem (3.8) with $p(t,u_t)$ and $\psi(t,u_t)$ defined by (3.9) and (3.10) has at least one solution on $(-\infty, 2]$.
Furthermore, the applicability of Theorem 3.2 can be demonstrate by choosing

$$p(t,u_t) = \frac{e^{-zt}}{3\sqrt{400+t}}\Big(\tan^{-1} u_t + \frac{e^t}{5}\Big),\ (t,u_t) \in [0,1]\times\mathfrak{N}_z, \tag{3.11}$$

and

$$\psi(t,u_t) = \frac{e^{-zt}}{(t+7)^2}\Big(\frac{|u_t|}{1+2|u_t|} + u_t\Big),\ (t,u_t) \in [0,1]\times\mathfrak{N}_z. \tag{3.12}$$

Clearly, the conditions $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\mathscr{H}_4)$ hold true with $L_1 = 1/60, L_2 = 2/49$, $d_1 = 1/8, d_2 = 1/4$, $\eta_1 = 1/8$ and $\eta_2 = 1/4$. Furthermore,

$$\left\{ \nu_b \Big(L_1\Lambda_3 + L_2(\bar{\Lambda}_2 + \Lambda_2) \Big) + d_1\, q \left(1 + \frac{\max\limits_{0\le l\le q} (t_{l+1} - t_l)}{|\Lambda_1|} \right) + d_2\Lambda_4 \right\} \approx 0.7053714234 < 1.$$

Thus we conclude that the problem (3.8) with $p(t,u_t)$ and $\psi(t,u_t)$ given by (3.11) and (3.12) has a unique solution on $(-\infty, 2]$.

4. Conclusions

In this paper, we investigated the existence and uniqueness criteria for solutions of a class of fractional IDEs with infinite delay and equipped with nonlocal antiperiodic boundary conditions. On arbitrary state spaces that satisfying the axioms $(\mathscr{C}_1)$, $(\mathscr{C}_1)$, and $(\mathscr{C}_1)$, we employed the fixed point theorem due to Krasnoselskii to establish the existence result, while the contraction mapping principle was applied for the uniqueness result. Moreover, we illustrated the applicability of our results by providing examples. The existence results established in this paper are new contributions that add a significant improvement to the theory of fractional IDEs with infinite delay, as well as some new special results that can be obtained by fixing the parameters that are given in (1.1). For example, the boundary condition in (1.1) can be replaced by the antiperiodic condition $(u(0) + u(b) = 0)$.

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References

[1] R. Tuladhar, F. Santamaria, I. Stamova, Fractional Lotka-Volterra-type cooperation models: Impulsive control on their stability behavior, Entropy, 22 (2020) 970.

[2] G. Stamov, I. Stamova, Impulsive delayed Lasota–Wazewska fractional models: Global stability of integral manifolds, Mathematics, 7 (2019) 1025.

[3] Y. Farid, F. Ruggiero, Finite-time extended state observer and fractional-order sliding mode controller for impulsive hybrid port-Hamiltonian systems with input delay and actuators saturation: Application to ball-juggler robots, Mechanism and Machine Theory, 167 (2022) 104577.

[4] G. Stamov, I. Stamova, A. Martynyuk, T. Stamov, Design and practical stability of a new class of impulsive fractional-like neural networks, Entropy 22 (2020) 337.

[5] J.J. Nieto, G. Stamov, I. Stamova, A fractional-order impulsive delay model of price fluctuations in commodity markets: Almost periodic solutions, Eur. Phys. J. Spec. Top. 226 (2017) 3811–3825.

[6] R.P. Agarwal, S. Hristova, D. O'Regan, Impulsive control of variable fractional-order multi-agent systems, Fractal and Fractional, 8 (2024) 259.

[7] M. Benchohra, J. Henderson, S. Ntouyas, Impulsive differential equations and inclusions, Contemp. Math. Appl. New York, 2006.

[8] I. Stamova, G. Stamov, Functional and impulsive differential equations of fractional order. Qualitative analysis and applications, CRC Press, Boca Raton, 2017.

[9] W. Yukunthorn, S. Suantai, S.K. Ntouyas, J. Tariboon, Boundary value problems for impulsive multi-order Hadamard fractional differential equations, Bound. Value Probl. 2015 (2015) 148.

[10] L. Zhang, J.J. Nieto, G. Wang, Extremal solutions for a nonlinear impulsive differential equations with multi-orders fractional derivatives, J. Appl. Anal. Comput. 7 (2017) 814-823.

[11] B. Ahmad, M. Alghanmi, J.J. Nieto, A. Alsaedi, On impulsive nonlocal integro-initial value problems involving multi-order Caputo-type generalized fractional derivatives and generalized fractional integrals, Adv. Differ. Equ. 2019 (2019) 1-20.

[12] A. Ullah, K. Shah, T. Abdeljawad, R.A. Khan, I. Mahariq, Study of impulsive fractional differential equation under Robin boundary conditions by topological degree method, Bound. Value Probl. 2020 (2020) 98.

[13] K. Shah, B. Abdalla, T. Abdeljawad, R. Gul, Analysis of multipoint impulsive problem of fractional-order differential equations, Bound. Value Probl. 2023 (2023), 1.

[14] H.R. Kataria, P.H. Patel, V. Shah, Existence results of noninstantaneous impulsive fractional integro-differential equation, Demonstratio Math. 53 (2020) 373-384.

[15] K. Karthikeyan, O. Ege, P. Karthikeyan, Results on impulsive fractional integro-differential equations involving Atangana-Baleanu derivative, Filomat 36 (2022) 4617-4627.

[16] K. Zhao, J. Liu, X. Lv, A unified approach to solvability and stability of multipoint BVPs for Langevin and Sturm–Liouville equations with CH–fractional derivatives and impulses via coincidence theory, Fractal and Fractional 8 (2024) 111.

[17] T. Stamov, G. Stamov, I. Stamova, Fractional-order impulsive delayed reaction-diffusion gene regulatory networks: Almost periodic solutions, Fractal and Fractional 7 (20023) 384.

[18] B. Zheng, Z. Wang, Event-based delayed impulsive control for fractional-order dynamic systems with application to synchronization of fractional-order neural networks, Neural Comput. Appl. 35 (2023) 20241–20251.

[19] X. Yang, Z. Yang, X. Nie, Exponential synchronization of discontinuous chaotic systems via delayed impulsive control and its application to secure communication, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 1529-1543.

[20] K. Hilal, Y. Allaoui, Existence of solution of neutral fractional impulsive differential equations with infinite delay, General Lett. Math. 2 (2017) 73-83

[21] Y. Zhou, S. Suganya, M. Mallika Arjunan, B. Ahmad, Approximate controllability of impulsive fractional integro-differential equation with state-dependent delay in Hilbert spaces, IMA J. Math. Control Info. 36 (2019) 603–622.

[22] B. Ahmad, M. Alghanmi, A. Alsaedi, R.P. Agarwal, Nonlinear impulsive multi-order Caputo-Type generalized fractional differential equations with infinite delay, Mathematics 7 (2019) 1108.

[23] Y. Guo, X.B. Shu, Y. Li, F. Xu, The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, Bound. Value Probl. 2019 (2019) 59.

[24] X. Ma, X.B. Xiao, J. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, Stoch. Dyn. 20 (2020) 2050003.

[25] H. Bao, J. Cao, Existence of solutions for fractional stochastic impulsive neutral functional differential equations with infinite delay, Adv. Difference Equ. 2017 (2017) 66.

[26] Y. Hino, S. Murakami, T. Naito, Functional-differential equations with infinite delay. Lecture Notes in Mathematics, 1473, Springer, Berlin, 1991.

[27] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, Funkc. Ekvacioj 21 (1978) 11-41.

[28] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[29] M.A. Krasnoselskii, Two remarks on the method of successive approximations, Uspekhi Mat. Nauk. 10 (1955) 123-127.