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EXISTENCE RESULTS FOR A MIXED BOUNDARY VALUE PROBLEM FOR A COMPLETE STURM-LIOUVILLE EQUATION

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Abstract. This paper outlines multiple sufficient conditions for the existence of at least one generalized solution to a mixed boundary value problem for a complete Sturm-Liouville equation. Our method relies on variational techniques. We extend and enhance some recent results and also provide a specific example to illustrate an application.
Keywords. Mixed boundary value problems; Sturm-Liouville; Variational methods.

1. Introduction

In this paper, based on variational methods, we study the multiplicity of solutions for the following problem

$$\begin{cases} -z''+\gamma(\varsigma)z'+\delta(\varsigma)z=h(\varsigma,z(\varsigma)), & \varsigma\in(a,b),\\ z(a)=z'(b)=0, \end{cases} \tag{P^h}$$

where h is an L^1-Carathéodory function and $\gamma,\delta\in L^\infty([a,b])$ are such that

$$\operatorname{ess}\inf_{\varsigma\in[a,b]}\delta(\varsigma)>-\left(\frac{\pi}{2(b-a)}\right)^2. \tag{1.1}$$

In recent years, a great deal of mathematical effort has been devoted to the study of Sturm-Liouville problems with mixed boundary conditions. We refer to [3, 5, 7, 8, 9, 10, 18] for some recent pertinent results. For example, D'Aguì [7], based on variational methods, established existence results for nontrivial solutions to a mixed boundary value problem with a Sturm-Liouville equation Averna et al. [3] obtained the existence of three solutions for a Sturm-Liouville mixed boundary value problem by using multiple critical points theorems.

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In this paper, we are in line with the results obtained in [4], which, unlike other existing papers, assumed that the coefficients γ and δ can change their sign. Bonanno et al. [4], by using critical point theory, studied the existence of infinitely many distinct positive solutions for the parametric version of (P^h), which is considered later in (P^h_λ).

Inspired by the results mentioned above, this paper investigates the existence of at least one non-trivial generalized solution for problem (P^h) by focusing on the asymptotic behavior of the nonlinear datum at zero, as detailed in Theorem 3.1. In Theorem 3.2, we provide an application of Theorem 3.1 and present some remarks regarding our findings. As a particular instance of our results, we derive Theorem 3.10 for the scenario where h does not depend on ς. Finally, we present Example 3.11, which satisfies the conditions set forth in Theorem 3.10.

2. Preliminaries

The central argument supporting our results is derived from a version of Ricceri's variational principle [17, Theorem 2.1], as presented by Bonanno and Molica Bisci in [6].

Lemma 2.1. *Let X be a reflexive real Banach space, and let $\Pi, \Upsilon : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Π is sequentially weakly lower semicontinuous, strongly continuous, and coercive in* E*, and Υ is sequentially weakly upper semicontinuous in* E*. Let Θ_λ be the functional defined as $\Theta_\lambda = \Pi - \lambda\Upsilon$, $\lambda \in \mathbb{R}$, and, for every $r > \inf_X \Pi$, let φ be the function defined as*

$$\varphi(s) = \inf_{z\in\Pi^{-1}(-\infty,s)} \frac{\sup_{v\in\Pi^{-1}(-\infty,s)} \Upsilon(v) - \Upsilon(z)}{s-\Pi(z)}.$$

Then, for every $s > \inf_X \Pi$ and every $\lambda \in \left(0, \dfrac{1}{\varphi(s)}\right)$, the restriction of the functional Θ_λ to $\Pi^{-1}(-\infty, s)$ admits a global minimum, which is a critical point (precisely, a local minimum) of Θ_λ in X.

We encourage interested readers to consult the papers [1, 11, 12, 13, 14, 15], where Lemma 2.1 was effectively utilized to establish the existence of at least one non-trivial solution for boundary value problems.

In this section, we introduce the functional space, and we recall some preliminaries and basic properties in order to study problem (P^h). Take the Sobolev space

$$\mathrm{E} = \left\{z \in W^{1,2}([a,b]) : \ z(a) = 0\right\}$$

endowed with the following norm:

$$\|z\| = \left(\int_a^b |z(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_a^b |z'(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}}.$$

Moreover, for all $z \in \mathrm{E}$, put

$$\|z\|_0 = \|z'\|_{L^2} = \left(\int_a^b |z'(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}}.$$

The following proposition holds true.

Proposition 2.2. [4, Proposition 2.1] *The norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent on* E.

Here, we point out the following result.

Proposition 2.3. [4, Proposition 2.2] (*Poincaré inequalities*) *For all* $z \in \mathrm{E}$, *it holds that*

(1) $\max_{\varsigma\in[a,b]} |z(\varsigma)| \le (b-a)^{\frac{1}{2}} \|z\|_0$,

(2) $\|z\|_{L^2} \le \dfrac{2(b-a)}{\pi} \|z'\|_{L^2}$.

Remark 2.4. We observe that the Poincaré inequalities hold true in the Sobolev space $W^{1,2}([a,b])$, as given in [2], with different constants.

Now, let us introduce another norm in the space E, given by

$$\|z\|_{\mathrm{E}} = \left(\int_a^b e^{-\Phi(\varsigma)} |z'(\varsigma)|^2 \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma) |z(\varsigma)|^2 \mathrm{d}\varsigma \right)^{\frac{1}{2}},$$

where $\Phi(\varsigma) = \int_a^\varsigma \gamma(\xi) d\xi, \ \forall \varsigma \in [a,b]$.

Proposition 2.5. [4, Proposition 2.3] *Let* (1.1) *hold. Then* $\|\cdot\|_{\mathrm{E}}$ *is a norm on the space* E *and it is equivalent to* $\|\cdot\|_0$. *In particular,*

$$m\|z\|_0 \le \|z\|_{\mathrm{E}} \le M\|z\|_0 \tag{2.1}$$

for all $z \in \mathrm{E}$, *where* m, M *with* $M \ge m > 0$ *are given by*

$$m = \begin{cases} \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, \\ \textit{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \ge 0, \\ \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, \\ \textit{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0 \end{cases}$$

and

$$M = \begin{cases} \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,sup}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, \\ \textit{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \ge 0, \\ \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, \\ \textit{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0. \end{cases}$$

Remark 2.6. Since $\|z\|_0$ is equivalent to $\|z\|$, as proved in Proposition 2.2, we obtain from the transitivity property the equivalence between $\|z\|_{\mathrm{E}}$ and $\|z\|$.

Remark 2.7. The space E is a Hilbert space with the dot product

$$< z - v > = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma) v'(\varsigma) \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma) z(\varsigma) v(\varsigma) \mathrm{d}\varsigma$$

that clearly induces the norm $\|z\|_{\mathrm{E}}$.

Remark 2.8. Taking into account (2.1), we have the following inequality

$$\max_{\varsigma\in[a,b]} |z(\varsigma)| \le \frac{(b-a)^{\frac{1}{2}}}{m} \|z\|_{\mathrm{E}}, \ \ \forall z \in \mathrm{E}.$$

Now, we recall the definitions of classical and generalized solutions to the problem.

Definition 2.9. We say that $z:[a,b]\to\mathbb{R}$ is a classical solution to problem (P^h) if $z\in C^2([a,b])$, $z(a)=z'(b)=0$, and

$$-z''+\gamma(\varsigma)z'+\delta(\varsigma)z=h(\varsigma,z(\varsigma)),\ \ \forall\varsigma\in[a,b].$$

Definition 2.10. We say that $z:[a,b]\to\mathbb{R}$ is a generalized solution to problem (P^h) if $z\in C^1([a,b])$, $z'\in C([a,b])$, and

$$z(a)=z'(b)=0,\ -z''+\gamma(\varsigma)z'+\delta(\varsigma)z=h(\varsigma,z(\varsigma))$$

for almost every $\varsigma\in[a,b]$.

Remark 2.11. Classical and generalized solutions of problem (P^h) coincide when h, γ and δ are continuous functions.

Definition 2.12. A function $z\in\mathrm{E}$ is called a weak solution to problem (P^h) if

$$\int_a^b e^{-\Phi(\varsigma)}z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma+\int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)(z(\varsigma))v(\varsigma)\mathrm{d}\varsigma-\int_a^b e^{-\Phi(\varsigma)}h(\varsigma,z(\varsigma))v(\varsigma)\mathrm{d}\varsigma=0$$

holds for any $v\in\mathrm{E}$.

Put

$$H(\varsigma,\zeta)=\int_0^{\zeta}h(\varsigma,x)\mathrm{d}x\ \ \text{for any } (\varsigma,\zeta)\in(a,b)\times\mathbb{R}.$$

We define the functionals Π and Υ, for each $z\in\mathrm{E}$, as follows.

$$\Pi(z)=\frac{1}{2}\|z\|_{\mathrm{E}}^2 \tag{2.2}$$

and

$$\Upsilon(z)=\int_a^b e^{-\Phi(\varsigma)}H(\varsigma,z(\varsigma))\mathrm{d}\varsigma, \tag{2.3}$$

and we put $\Theta(z)=\Pi(z)-\Upsilon(z)$, for every $z\in\mathrm{E}$.

Proposition 2.13. [4, Proposition 2.4] *Function z is a generalized solution of (P^h) if only if z is a critical point of Θ.*

3. Main Results

In this section, we outline our main results regarding the existence of at least one generalized solution for problem (P^h).

Theorem 3.1. *Assume that*

$$\sup_{\theta>0}\frac{\theta^2}{\int_a^b \max_{|x|\le\theta}H(\varsigma,x)\mathrm{d}\varsigma}>\frac{2(b-a)\max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}}{m^2}. \tag{S}$$

Then, problem (P^h) admits at least one generalized solution in E.

Proof. Our objective is to apply Theorem 2.1 to problem (P^h). We define the functionals Π and Υ as given in (2.2) and (2.3), respectively. We demonstrate that Π and Υ meet the necessary conditions outlined in Theorem 2.1. It is known (see, e.g. [2]) that they are well defined, and are Gâteaux differentiable. Note that

$$\Upsilon'(z)(v) = \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma$$

and

$$\Pi'(z)(v) = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)z(\varsigma)v(\varsigma)\mathrm{d}\varsigma,$$

for every $z, v \in \mathrm{E}$. Furthermore, Π and Υ are C^1-functions. By utilizing the definition of Π, it follows that

$$\lim_{\|z\|_{\mathrm{E}}\to+\infty} \Pi(z) = +\infty,$$

which indicates that Π is coercive. Therefore, we conclude that the regularity assumptions on Π and Υ, as specified in Theorem 2.1, are satisfied. Moreover, as we have seen in [4, Proposition 2.4], the critical points in E of the functional Θ are exactly the generalized solution of the considered problem (P^h). Furthermore, from condition (S), there exists a constant $\bar{\theta} > 0$ such that

$$\frac{\bar{\theta}^2}{\displaystyle\int_a^b \max_{|x|\le\bar{\theta}} H(\varsigma, x)\mathrm{d}\varsigma} > \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2}. \tag{3.1}$$

Put

$$s = \frac{m^2}{2(b-a)}\bar{\theta}^2.$$

In view of Remark 2.8, for each $z \in \mathrm{E}$ such that $\Pi(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 < s$, one has

$$|z(\varsigma)| \le \frac{(b-a)^{\frac{1}{2}}}{m}\|z\|_{\mathrm{E}} \le \frac{(b-a)^{\frac{1}{2}}}{m}(2s)^{\frac{1}{2}} = \left(\frac{2(b-a)}{m^2}s\right)^{\frac{1}{2}} = \bar{\theta}, \quad \forall z \in \mathrm{E}.$$

Taking into account that $\max_{\varsigma\in[a,b]}|z(\varsigma)| \le \bar{\theta}$ for all $z \in \mathrm{E}$ such that $\|z\|_{\mathrm{E}}^2 < 2s$, we have

$$\sup_{\Pi(z)<s} \Upsilon(z) \le \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_a^b \max_{|\zeta|\le\bar{\theta}} H(\varsigma, \zeta)\mathrm{d}\varsigma.$$

Taking into account the calculations mentioned above, we see that $0 \in \Pi^{-1}(-\infty, s)$. Since $\Pi(0) = \Upsilon(0) = 0$, then

$$\varphi(s) = \inf_{z\in\Pi^{-1}(-\infty,s)} \frac{\sup_{v\in\Pi^{-1}(-\infty,s)}\Upsilon(v) - \Upsilon(z)}{s - \Pi(z)} \le \frac{\sup_{z\in\Pi^{-1}(-\infty,s)}\Upsilon(z)}{s}$$

$$\le \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2}\frac{\displaystyle\int_a^b \max_{|\zeta|\le\bar{\theta}} H(\varsigma, \zeta)\mathrm{d}\varsigma}{\bar{\theta}^2}.$$

Therefore, it can be concluded that

$$\varphi(s) \leq \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2} \frac{\int_a^b \max_{|\zeta|\leq\bar{\theta}} H(\varsigma,\zeta)\mathrm{d}\varsigma}{\bar{\theta}^2}. \tag{3.2}$$

Considering equations (3.1) and (3.2), we have $\varphi(s) < 1$. Since $1 \in \left(0, \frac{1}{\varphi(s)}\right)$, Theorem 2.1 guarantees that Θ has at least one critical point (local minimum) $\tilde{z} \in \Pi^{-1}(-\infty, s)$. Moreover, since the generalized solutions to problem (P^h) correspond precisely to the critical points of Θ, we arrive at the desired conclusion immediately. ☐

We observe that Theorem 3.1 can be utilized to guarantee the existence of at least one generalized solution for the following problem

$$\begin{cases} -z'' + \gamma(\varsigma)z' + \delta(\varsigma)z = \lambda h(\varsigma, z(\varsigma)), & \varsigma \in (a,b), \\ z(a) = z'(b) = 0, \end{cases} \tag{P_λ^h}$$

where λ is a positive parameter.

Now, we can state the following existence result.

Theorem 3.2. *For every λ small enough, i.e.,*

$$\lambda \in \left(0, \frac{m^2}{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}} \sup_{\theta>0} \frac{\theta^2}{\int_a^b \max_{|x|\leq\theta} H(\varsigma,x)\mathrm{d}\varsigma}\right),$$

problem (P_λ^h) admits at least one generalized solution $z_\lambda \in$ E.

Proof. Let Π and Υ be defined as in equations (2.2) and (2.3), respectively. Define

$$\Theta_\lambda(z) = \Pi(z) - \lambda\Upsilon(z)$$

for every $z \in$ E. Pick

$$0 < \lambda < \frac{m^2}{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}} \sup_{\theta>0} \frac{\theta^2}{\int_a^b \max_{|x|\leq\theta} H(\varsigma,x)\mathrm{d}\varsigma}.$$

Therefore, there exists a positive value $\bar{\theta} > 0$ such that

$$\frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2}\lambda < \frac{\bar{\theta}^2}{\int_a^b \max_{|x|\leq\bar{\theta}} H(\varsigma,x)\mathrm{d}\varsigma}.$$

Put

$$s = \frac{m^2}{2(b-a)}\bar{\theta}^2.$$

Using the same notations as in the proof of Theorem 3.1, we can state that

$$\varphi(s) \leq \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2} \frac{\int_a^b \max_{|x|\leq\bar{\theta}} H(\varsigma,x)\mathrm{d}\varsigma}{\bar{\theta}^2} < \frac{1}{\lambda}.$$

Since $\lambda \in \left(0, \dfrac{1}{\varphi(s)}\right)$, Theorem 2.1 guarantees that Θ_λ has at least one critical point (local minimum) $z_\lambda \in \Pi^{-1}(-\infty, s)$. Given that the critical points of the functional Θ_λ correspond to the generalized solutions of the problem (P_λ^h), we arrive at the conclusion immediately. □

We would like to make a few observations pertaining to our findings.

Remark 3.3. In Theorem 3.1, we examined the critical points of the functional Θ_λ which is naturally linked to the problem in equation (P_λ^h). It is important to note that, in general, Θ_λ can be unbounded from below in E. For instance, this occurs when $H(\zeta) = |\zeta|^\alpha$ for every $\zeta \in \mathbb{R}$ with $\alpha > 2$. For any fixed $z \in \mathrm{E}\backslash\{0\}$ and $\chi \in \mathbb{R}$, we obtain

$$\begin{aligned}\Theta_\lambda(\chi z) =& \Pi(\chi z) - \lambda \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, \chi z(\varsigma)) \mathrm{d}\varsigma \\ \leq& \frac{\chi^2}{2}\|z\|_{\mathrm{E}}^2 - \lambda \chi^\alpha \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_a^b |z|^\alpha \mathrm{d}\varsigma.\end{aligned}$$

Since $\alpha > 2$, we find that $\Theta_\lambda(\chi z) \to -\infty$. Consequently, condition [16, (I_2), Theorem 2.2] is not met. Therefore, we cannot apply direct minimization to identify the critical points of the functional Θ_λ.

Remark 3.4. If, in Theorem 3.1, $h(\varsigma, x) \geq 0$ for a.e. $(\varsigma, x) \in (a,b) \times \mathbb{R}$, condition ($S$) simplifies to a more straightforward form

$$\sup_{\theta>0} \frac{\theta^2}{\displaystyle\int_a^b H(\varsigma, \theta) \mathrm{d}\varsigma} > \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2}. \tag{S_λ}$$

Additionally, if the subsequent assumption is confirmed

$$\limsup_{\theta\to+\infty} \frac{\theta^2}{\displaystyle\int_a^b H(\varsigma, \theta) \mathrm{d}\varsigma} > \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2},$$

then condition (S_λ) automatically satisfied.

Remark 3.5. For fixed $\bar{\theta} > 0$, let

$$\frac{\bar{\theta}^2}{\displaystyle\int_a^b \max_{|x|\leq\bar{\theta}} H(\varsigma, x) \mathrm{d}\varsigma} > \frac{2(b-a)\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}}{m^2}.$$

Consequently, the result of Theorem 3.2 applies, with the condition $\|z_\lambda\|_{\mathrm{E}} \leq \bar{\theta}$ ensuring a generalized solution in E.

Remark 3.6. If, in Theorem 3.2, $h(\varsigma, 0) \neq 0$ for a.e. $\varsigma \in (a,b)$, then the obtained solution is clearly non-trivial. Conversely, non-triviality can also be established in the case where $h(\varsigma, 0) = 0$ for almost every $\varsigma \in (a,b)$, by imposing an additional condition at zero. Specifically, there must exist a non-empty open set $D \subseteq [a,b]$ and a subset $B \subset D$ with positive Lebesgue measure such that

$$\limsup_{\zeta\to0^+} \frac{ess\inf_{\varsigma\in B} H(\varsigma, \zeta)}{|\zeta|^2} = +\infty$$

and

$$\liminf_{\zeta\to 0^+}\frac{ess\inf_{\varsigma\in D}H(\varsigma,\zeta)}{|\zeta|^2}>-\infty.$$

Indeed, let $0<\bar{\lambda}<\lambda^*$, where

$$\lambda^*=\frac{m^2}{2(b-a)\max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}}\sup_{\theta>0}\frac{\theta^2}{\displaystyle\int_a^b\max_{|x|\le\theta}H(\varsigma,x)\mathrm{d}\varsigma}.$$

Then, there exists $\bar{\theta}>0$ such that

$$\frac{2(b-a)\max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}}{m^2}\bar{\lambda}<\frac{\bar{\theta}^2}{\displaystyle\int_a^b\max_{|x|\le\bar{\theta}}H(\varsigma,x)\mathrm{d}\varsigma}.$$

Based on Theorem 2.1, for every $\lambda\in(0,\bar{\lambda})$, there exists a critical point of Θ_λ such that $z_\lambda\in\Pi^{-1}(-\infty,s_\lambda)$, where $s_\lambda=\dfrac{m^2}{2(b-a)}\bar{\theta}^2$. In particular, z_λ is a global minimum of the restriction of Θ_λ to $\Pi^{-1}(-\infty,s_\lambda)$. To demonstrate that z_λ cannot be trivial, we show that

$$\limsup_{\|z\|_{\mathrm{E}}\to 0^+}\frac{\Upsilon(z)}{\Pi(z)}=+\infty. \tag{3.3}$$

Based on our assumptions at zero, we can establish a sequence $\{\zeta_n\}\subset\mathbb{R}^+$ that converges to zero, along with two constants $\xi,\ \kappa$ (with $\xi>0$) such that

$$\lim_{n\to+\infty}\frac{ess\inf_{\varsigma\in B}H(\varsigma,\zeta_n)}{|\zeta_n|^2}=+\infty$$

and

$$ess\inf_{\varsigma\in D}H(\varsigma,\zeta)\ge\kappa|\zeta|^2$$

for each $\zeta\in[0,\xi]$. Now, let us consider a set $C\subset B$ with positive measure and a function $v\in\mathrm{E}$ such that

- (i) $v(\varsigma)\in[0,1]$ for each $\varsigma\in[a,b]$,
- (ii) $v(\varsigma)=1$ for each $\varsigma\in C$,
- (iii) $v(\varsigma)=0$ for each $\varsigma\in[a,b]\setminus D$.

Hence, fix $Y>0$ and consider a real positive number η with

$$Y<\frac{\eta\ meas(C)\min_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}+\min_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}\kappa\displaystyle\int_{D\setminus C}|v(\varsigma)|^2\mathrm{d}\varsigma}{\frac{1}{2}\|v\|_{\mathrm{E}}^2}.$$

Then, there exists $n_0\in\mathbb{N}$ such that $\zeta_n<\xi$ and $ess\inf_{\varsigma\in B}H(\varsigma,\zeta_n)\ge\eta|\zeta_n|^2$ for every $n>n_0$. Now, for every $n>n_0$, by using the properties of v (that is $0\le\zeta_n v(\varsigma)<\xi$ for n large enough), one has

$$\frac{\Upsilon(\zeta_n v)}{\Pi(\zeta_n v)} = \frac{\int_C e^{-\Phi(\varsigma)} H(\varsigma,\zeta_n)\mathrm{d}\varsigma + \int_{D\setminus C} e^{-\Phi(\varsigma)} H(\varsigma,\zeta_n v(\varsigma))\mathrm{d}\varsigma}{\Pi(\zeta_n v)}$$

$$> \frac{\eta\ meas(C)\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} + \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\kappa \int_{D\setminus C} |v(\varsigma)|^2 \mathrm{d}\varsigma}{\frac{1}{2}\|v\|_{\mathrm{E}}^2} > Y.$$

Since Y could be consider arbitrarily large, it is concluded that

$$\lim_{n\to\infty} \frac{\Upsilon(\zeta_n v)}{\Pi(\zeta_n v)} = +\infty,$$

from which (3.3) clearly follows. Hence, there exists a sequence $\{a_n\} \subset \mathrm{E}$ strongly converging to zero, $a_n \in \Pi^{-1}(-\infty,s)$ and

$$\Theta_\lambda(a_n) = \Pi(a_n) - \lambda\Upsilon(a_n) < 0.$$

Since z_λ is a global minimum of the restriction of Θ_λ to $\Pi^{-1}(-\infty,s)$, we conclude that $\Theta_\lambda(z_\lambda) < 0$, so that z_λ is not trivial.

Remark 3.7. From equation $\Theta_\lambda(z_\lambda) < 0$, we can readily observe that the map

$$(0,\lambda^*) \ni \lambda \mapsto \Theta_\lambda(z_\lambda) \tag{3.4}$$

is negative. Additionally, we have

$$\lim_{\lambda\to 0^+} \|z_\lambda\|_{\mathrm{E}} = 0.$$

Indeed, by recognizing that Π is coercive and that for $\lambda \in (0,\lambda^*)$ the solution $z_\lambda \in \Pi^{-1}(-\infty,s)$, we can conclude that there exists a positive constant $\mathscr{L}$ such that $\|z_\lambda\|_{\mathrm{E}} \le \mathscr{L}$ for any $\lambda \in (0,\lambda^*)$. Furthermore, it follows that there exists a positive constant $\mathscr{M}$ such that

$$\left| \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z_\lambda(\varsigma)) z_\lambda(\varsigma)\mathrm{d}\varsigma \right| \le \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \mathscr{M} \|z_\lambda\|_{\mathrm{E}}^2 \le \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \mathscr{M}\mathscr{L}, \tag{3.5}$$

for every $\lambda \in (0,\lambda^*)$. Since z_λ is a critical point of Θ_λ, we have $\Theta'_\lambda(z_\lambda)(v) = 0$ for any $v \in \mathrm{E}$ and every $\lambda \in (0,\lambda^*)$. In particular, $\Theta'_\lambda(z_\lambda)(z_\lambda) = 0$, that is,

$$\Pi'(z_\lambda)(z_\lambda) = \lambda \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z_\lambda(\varsigma)) z_\lambda(\varsigma)\mathrm{d}\varsigma, \tag{3.6}$$

for every $\lambda \in (0,\lambda^*)$. Then, $0 \le \|z_\lambda\|_{\mathrm{E}}^2 = \Pi'(z_\lambda)(z_\lambda)$. By using (3.6), it is concluded that

$$0 \le \|z_\lambda\|_{\mathrm{E}}^2 = \Pi'(z_\lambda)(z_\lambda) = \lambda \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z_\lambda(\varsigma)) z_\lambda(\varsigma)\mathrm{d}\varsigma,$$

for any $\lambda \in (0,\lambda^*)$. Letting $\lambda \to 0^+$, we find by (3.5) that $\lim_{\lambda\to 0^+} \|z_\lambda\|_{\mathrm{E}}^2 = 0$. Note that $\lim_{\lambda\to 0^+} \|z_\lambda\|_\infty = 0$.

Finally, we demonstrate that the map $\lambda \mapsto \Theta_\lambda(z_\lambda)$ is strictly decreasing in $(0,\lambda^*)$. For our goal, we observe that, for any $z \in \mathrm{E}$,

$$\Theta_\lambda(z) = \lambda\left(\frac{\Pi(z)}{\lambda} - \Upsilon(z)\right). \tag{3.7}$$

Now, let us fix $0<\lambda_1<\lambda_2<\lambda^*$ and let z_{λ_1}, z_{λ_2} be the global minimums of Θ_{λ_i} restricted to $\Pi(-\infty,s)$ for $i=1,2$. Also, let

$$m_{\lambda_i}=\left(\frac{\Pi(z_{\lambda_i})}{\lambda_i}-\Upsilon(z_{\lambda_i})\right)=\inf_{v\in\Pi^{-1}(-\infty,s)}\left(\frac{\Pi(v)}{\lambda_i}-\Upsilon(v)\right),$$

for $i=1,2$. Clearly, (3.4) together with (3.7) implies that $m_{\lambda_i}<0$ for $i=1,2$ due to $\lambda>0$. In view of $0<\lambda_1<\lambda_2$, one has $m_{\lambda_2}\le m_{\lambda_1}$. From $0<\lambda_1<\lambda_2$, we see that

$$\Theta_{\lambda_2}(z_{\lambda_2})=\lambda_2 m_{\lambda_2}\le\lambda_2 m_{\lambda_1}<\lambda_1 m_{\lambda_1}=I_{\lambda_1}(z_{\lambda_1}),$$

so that the map $\lambda\mapsto\Theta_\lambda(z_\lambda)$ is strictly decreasing in $\lambda\in(0,\lambda^*)$. Since $\lambda<\lambda^*$ is arbitrary, we show $\lambda\mapsto\Theta_\lambda(z_\lambda)$ is strictly decreasing in $(0,\lambda^*)$.

Remark 3.8. We observe that the Theorem 3.2 above is a bifurcation result in the sense that the pair $(0,0)$ belongs to the closure of the set

$$\left\{(z_\lambda,\lambda)\in \mathrm{E}\times(0,+\infty): z_\lambda \text{ is a non-trivial generalized solution of } (P_\lambda^h)\right\}$$

in $\mathrm{E}\times\mathbb{R}$. Indeed, by Remark 3.7, one has $\|z_\lambda\|_{\mathrm{E}}\to 0$ as $\lambda\to 0$. Hence, there exist two sequences $\{z_j\}$ in E and λ in $\mathbb{R}^+$ (here $z_j=z_\lambda$) such that

$$\lambda_j\to 0^+ \quad\text{and}\quad \|z_j\|_{\mathrm{E}}\to 0,$$

as $j\to+\infty$. Moreover, we need to emphasis that due to the fact that the map

$$(0,\lambda^*)\ni\lambda\mapsto\Theta_\lambda(z_\lambda)$$

is strictly decreasing, for every $\lambda_1,\lambda_2\in(0,\lambda^*)$, with $\lambda_1\neq\lambda_2$, the solutions z_{λ_1} and z_{λ_2} ensured by Remark 3.7 are different.

Remark 3.9. [4, Proposition 2.6] If h is non-negative, then the generalized solution guaranteed by Theorem 3.2 is also non-negative.

The next theorem addresses a specific case of our results.

Theorem 3.10. *Let $h:\mathbb{R}\to\mathbb{R}^+$ be a continuous function. Let $\lim_{\zeta\to 0^+}\frac{H(\zeta)}{\zeta^2}=+\infty$, where $H(\zeta)=\int_0^\zeta h(s)\mathrm{d}s$ for all $\zeta\in\mathbb{R}$. Then, for each*

$$\lambda\in\left(0,\frac{m^2}{2(b-a)^2\max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}}\sup_{\theta>0}\frac{\theta^2}{H(\theta)}\right),$$

the problem

$$\begin{cases}-z''+\gamma(\varsigma)z'+\delta(\varsigma)z=\lambda h(z(\varsigma)), & \varsigma\in(a,b),\\ z(a)=z'(b)=0\end{cases}$$

admits at least one nontrivial generalized solution $z_\lambda\in\mathrm{E}$ such that $\lim_{\lambda\to 0^+}\|z_\lambda\|_{\mathrm{E}}^2=0$ and the real function

$$\lambda\to\frac{1}{2}\|z_\lambda\|_{\mathrm{E}}^2-\lambda\int_a^b e^{-\Phi(\varsigma)}H(z_\lambda(\varsigma))\mathrm{d}\varsigma$$

is negative and strictly decreasing in $\left(0,\frac{m^2}{2(b-a)^2\max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}}\sup_{\theta>0}\frac{\theta^2}{H(\theta)}\right)$.

Finally, we present the following example to demonstrate the validity of Theorem 3.10.

Example 3.11. We consider the problem

$$\begin{cases} -z''+z'-z=h(\varsigma,z(\varsigma)), & \varsigma\in(0,1),\\ z(0)=z'(1)=0, \end{cases} \tag{3.8}$$

where

$$h(\zeta)=\begin{cases} 3\zeta^2, & \text{if } \zeta<0,\\ 2\zeta+e^{\zeta}, & \text{if } \zeta\geq 0. \end{cases}$$

Through straightforward calculations, we obtain

$$H(\zeta)=\begin{cases} \zeta^3, & \text{if } \zeta<0,\\ \zeta^2+e^{\zeta}-1, & \text{if } \zeta\geq 0. \end{cases}$$

By simple calculations, one has that $m=\sqrt{1-\frac{4}{\pi^2}}$. Since $\lim_{\zeta\to0^+}\frac{H(\zeta)}{\zeta^2}=+\infty$, then all the assumptions of Theorem 3.10 are satisfied. This implies that problem (3.8), for each $\lambda\in\left(0,\frac{\pi^2-4}{2e\pi^2}\right)$, admits at least one nontrivial generalized solution $z_\lambda\in \mathrm{E}$ such that $\lim_{\lambda\to0^+}\|z_\lambda\|_{\mathrm{E}}=0$ and the real function

$$\lambda\to\frac{1}{2}\|z_\lambda\|_{\mathrm{E}}^2-\lambda\int_0^1 e^{\varsigma}H(z_\lambda(\varsigma))\mathrm{d}\varsigma$$

is negative and strictly decreasing in $\left(0,\frac{\pi^2-4}{2e\pi^2}\right)$.

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